# Permutation Statistics on the Alternating Group

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Feb 13, 2003

**Abstract**

Let $A_n \subseteq S_n$ denote the alternating and the symmetric groups on $1, \dots, n$. MacMahaon's theorem [11], about the equi-distribution of the length and the major indices in $S_n$, has received far reaching refinements and generalizations, by Foata [5], Carlitz [3, 4], Foata-Schützenberger [6], Garsia-Gessel [7] and followers. Our main goal is to find analogous statistics and identities for the alternating group $A_n$. A new statistic for $S_n$, *the delent number*, is introduced. This new statistic is involved with new $S_n$ equi-distribution identities, refining some of the results in [6] and [7]. By a certain covering map $f : A_{n+1} \to S_n$, such $S_n$ identities are 'lifted' to $A_{n+1}$, yielding the corresponding $A_{n+1}$ equi-distribution identities.

*Partially supported by Minerva Grant No. 8441 and by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272

†Partially supported by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities and by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272

# 1 Introduction

## 1.1 General outline

One of the most active branches in enumerative combinatorics is the study of *permutation statistics*. Let $S_n$ be the symmetric group on $1, \ldots, n$. One is interested in the refined count of permutations according to (non-negative, integer valued) combinatorial parameters. For example, the number of inversions in a permutation - namely its *length* - is such a parameter. Another parameter is MacMahon's *major index*, which is defined via the *descent* set of a permutation - see below.

Two parameters that have the same generating function are said to be *equi-distributed*. Indeed, MacMahon [11] proved the remarkable fact that the inversions and the major-index statistics are equi-distributed on $S_n$. MacMahon's classical theorem [11] has received far reaching refinements and generalizations, including: multivariate refinements which imply equi-distribution on certain subsets of permutations (done by Carlitz [3, 4], Foata-Schützenberger [6] and Garsia-Gessel [7]); analogues for other combinatorial objects, cf. [5, 9, 17]; generalizations to other classical Weyl groups, cf. [14, 2, 1].

Let $A_n \subseteq S_n$ denote the alternating group on $1, \ldots, n$. Easy examples show that the above statistics fail to be equi-distributed when restricted to $A_n$. Our main goal is to find statistics on $A_n$ which are natural generalizations of the $S_n$ statistics and are equi-distributed on $A_n$, yielding analogous identities for their generating functions. This goal is achieved by proving further refinements of the above $S_n$-identities.

It is well known that the above statistics on $S_n$ may be defined via the Coxeter generators $\{(i, i+1) \mid 1 \leq i \leq n-1\}$ of $S_n$. Mitsuhashi [12] pointed out at a certain set of generators of the alternating group $A_n$, which play a role similar to that of the above Coxeter generators of $S_n$, see Subsection 1.3 below. We use these generators to define the analogous length and descent statistics on the alternating group.

The $S_n$-Coxeter generators allow one to introduce the classical canonical presentation of the elements of $S_n$, see Subsection 3.1. Similarly, the above Mitsuhashi's 'Coxeter' generators allow us to introduce the corresponding canonical presentation of the elements of $A_{n+1}$, see Subsection 3.3. We remark that usually, $S_n$ is viewed as a double cover of $A_n$. However, the above canonical presentations enable us to introduce a covering map $f$ from the alternating group $A_{n+1}$ onto $S_n$, and thus $A_{n+1}$ can be viewed as a covering of $S_n$.

A new statistic, the *delent number*, plays a crucial role in the paper, and allows us to 'lift' $S_n$ identities to $A_{n+1}$. The delent number on $S_n$ may be defined as follows: if the transposition $(1, 2)$ appears $r$ times in the canonical presentation of $\sigma \in S_n$ then the delent number of $\sigma$, $del_S(\sigma)$, is $r$. An analogous statistic is defined for $A_{n+1}$, see Definition 4.3. We give direct combinatorial characterizations of this statistic (see Propositions 1.7 and 1.8 below) and show that this statistic is involved in new $S_n$ equi-distribution identities, refining some of the results of Foata-Schützenberger [6] and of Garsia-Gessel [7]. Identities involving the delent number are then 'lifted' by the covering map $f$, yielding $A_{n+1}$ equi-distribution identities, see Theorem 6.1, Theorem 9.1 and Corollary 9.2.

In the Appendix we present different statistics on $A_n$, and a consequent different analogue of MacMahon's equi-distribution theorem. These statistics are compatible with the usual point of view of $S_n$ as a double cover of $A_n$.

The above setting and results are connected with enumeration of other combinatorial objects, such as permutations avoiding patterns, leading to $q$-analogues of the classical $S_n$ statistics and of the Bell and Stirling numbers. A detailed study of these $q$-analogues is given in [13] (a few of these results appear in Subsection 5.3).

The paper is organized as follows : The rest of this section surveys briefly the classical background and lists our main results. Background and notations are given in detail in Section 2, while the $A$ canonical presentation is analyzed in Section 3. In Section 4 we study the length statistics, and in Section 5 we discuss the relations between various $S$- and $A$-statistics, relations given by the map $f : A_{n+1} \to S_n$. In Section 6 we study the ordinary and the reverse major indices, together with the delent statistics. Additional properties of the delent numbers are given in Section 7. In Section 8 we prove some lemmas on shuffles - lemmas that are needed for the proof of the main theorem. The main theorem (Theorem 9.1) and its proof are given in Section 9. Finally, the Appendix constitutes Section 10.

## 1.2 Classical $S_n$-Statistics

Recall that the Coxeter generators $\{(i, i+1) | \ 1 \le i \le n-1\}$ of $S_n$ give rise to various combinatorial statistics :

*The S-length*: For $\pi \in S_n$ let $\ell_S(\pi)$ be the standard length of $\pi$ with respect to $S$.

*The $S$-descent*: Given a permutation $\pi$ in the symmetric group $S_n$, the *$S$-descent set* of $\pi$ is defined by

$$Des_S(\pi) := \{i \mid \ell_S(\pi) > \ell_S(\pi s_i)\} = \{i \mid \pi(i) > \pi(i+1)\}.$$

The *descent number* of $\pi$, $des_S(\pi)$, is defined by $des_S(\pi) := |Des_S(\pi)|$. The *major index*, $maj_S(\pi)$ is

$$maj_S(\pi) := \sum_{i \in Des_S(\pi)} i.$$

The corresponding *reverse major index* does depend on $n$, and is denoted

$$rmaj_{S_n}(\pi) := \sum_{i \in Des_S(\pi)} (n-i).$$

The reverse major index $rmaj_{S_n}(\pi)$ is implicit in [6].

These statistics are involved in many combinatorial identities. First, MacMahon proved the following equi-distribution of the length and the major indices [11]:

$$\sum_{\sigma \in S_n} q^{\ell_S(\sigma)} = \sum_{\sigma \in S_n} q^{maj_S(\sigma)}.$$

Foata [5] gave a bijective proof of MacMahon's theorem, then Foata and Schützenberger [6] applied this bijection to refine MacMahon's identity by analyzing bivariate distributions. Garsia and Gessel [7] extended the analysis to multivariate distributions. Extensions of MacMahon's identity to hyperoctahedral groups appear in [1].

Combining Theorems 1 and 2 of [6] one deduces the identity

**Theorem 1.1** *For any subset $D_1 \subseteq \{1, \ldots, n-1\}$*

$$\sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) \subseteq D_1\}} q^{maj_{S_n}(\pi)} = \sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) \subseteq D_1\}} q^{rmaj_{S_n}(\pi)}$$

$$= \sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) \subseteq D_1\}} q^{\ell_S(\pi)}.$$

A bivariate equi-distribution follows.

**Corollary 1.2**

$$\sum_{\pi\in S_n} q_1^{maj_{S_n}(\pi)} q_2^{des_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{rmaj_{S_n}(\pi)} q_2^{des_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{\ell_S(\pi)} q_2^{des_S(\pi^{-1})}.$$

As already mentioned, one of the main goals in this paper is to find analogous statistics and identities for the alternating group $A_n$. In the process we first prove some further refinements of some of the above identities for $S_n$, refinements involving the new *delent* statistic, see Theorems 6.1.1 and 9.1.1.

## 1.3 Main Results

Here is a summary of the main results of this paper.

### 1.3.1 $A_n$-Statistics

Following Mitsuhashi [12] we let

$$a_i := s_1 s_{i+1} = (1,2)(i+1,i+2) \qquad (1 \le i \le n-1).$$

Thus $a_i = a_i^{-1}$ if $i \neq 1$, while $a_1^2 = a_1^{-1}$. The set $A := \{a_i \mid 1 \le i \le n-1\}$ generates the alternating group on $n+1$ letters $A_{n+1}$ (see e.g. [12]). It is the above exceptional property of $a_1$ among the elements of $A$ - which naturally leads to the 'delent' statistic (Definition 1.5 below), both for $S_n$ and for $A_{n+1}$. This new statistic enables us to deduce new refinements of the MacMahon-type identities for $S_n$, and for each such an identity to derive the analogous identity for $A_{n+1}$.

The canonical presentation in $S_n$ by the Coxeter generators is well known, and is discussed in Section 3, see Theorem 3.1. With the above generating set $A$ of $A_{n+1}$ we also have canonical presentations for the elements of $A_{n+1}$, as follows. For each $1 \le j \le n-1$ define

$$R_j^A = \{1,\ a_j,\ a_j a_{j-1},\ \ldots,\ a_j \cdots a_2,\ a_j \cdots a_2 a_1,\ a_j \cdots a_2 a_1^{-1}\}. \tag{1}$$

**Theorem 1.3** *(See Theorem 3.4) Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R_j^A$, $1 \le j \le n-1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique. Call that presentation $v = v_1 \cdots v_{n-1}$ the $A$ canonical presentation of $v$.*

The $A$ canonical presentation allow us to introduce the A-length of an element in $A_{n+1}$:

**Definition 1.4** *Let $v \in A_{n+1}$ with $v = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$ ($\epsilon_i = \pm 1$) its $A$ canonical presentation, then its $A$-length is $\ell_A(v) = r$.*

A combinatorial interpretation of the $A$-length in terms of inversions is given below, see Proposition 4.5.

The $A$-descent statistic is defined using the above generating set $A$ :

**Definition 1.5**

1. *The alternating-descent (i.e. the $A$-descent) set of $\sigma \in A_{n+1}$ is defined by:*

$$Des_A(\sigma) := \{1 \leq i \leq n-1 \mid \ell_A(\sigma) \geq \ell_A(\sigma a_i)\},$$

*and the $A$-descent number of $\sigma \in A_{n+1}$ is defined by*

$$des_A(\sigma) := |Des_A(\sigma)|.$$

*(note that the strict relation $>$, in the definition of an $S$-descent in Section 1.2, is replaced in the $A$-analogue by $\geq$).*

2. *Define the alternating reverse major index of $\sigma \in A_{n+1}$ as*

$$rmaj_{A_{n+1}}(\sigma) := \sum_{i \in Des_A(\sigma)} (n-i).$$

### 1.3.2 The Delent Number

New statistics, for the alternating group, as well as for the symmetric group, are introduced.

**Definition 1.6** *(See Definition 4.3)*

1. *Let $w \in S_n$. The $S$-delent number of $w$ is the number of times that $s_1 = (1,2)$ occurs in the $S$ canonical presentation of $w$, and is denoted by $del_S(w)$.*

2. *Let $v \in A_{n+1}$. The $A$-delent number of $v$ is the number of times that $a_1^{\pm 1}$ occur in the $A$ canonical presentation of $v$, and is denoted by $del_A(v)$.*

A combinatorial interpretation of the delent numbers, $del_S$ and $del_A$, is given in Section 7. Let $w \in S_n$, then $j$ is a *l.t.r.min* (left-to-right minimum) of $w$ if $w(i) > w(j)$ for all $1 \leq i < j$.

**Proposition 1.7** *(see Proposition 7.7) For every permutation $w \in S_n$ denote*

$$Del_S(w) = \{1 < i \leq n| \ i \text{ is a l.t.r.min}\},$$

*then*

$$del_S(w) = |Del_S(w)|.$$

Similar to l.t.r.min, we define an *almost left to right minimum* (a.l.t.r.min) of $w \in A_{n+1}$ as follows:

$j$ is an *a.l.t.r.min* of $w$ if $w(i) < w(j)$ for at most one $j$ less than $i$. Define $Del_A(w)$ as the set of the *almost left-to-right minima* of $w$. Then $del_A(v) =| Del_A(w) |$, i.e. is the number of *a.l.t.r.min* of $w$, see Proposition 7.7.

We also have

**Proposition 1.8** *(See Proposition 4.4) Let $w \in A_{n+1}$, then*

$$del_S(w) = \ell_S(w) - \ell_A(w).$$

### 1.3.3 Equi-distribution Identities

The covering map $f : A_{n+1} \to S_n$, presented in Definition 5.1, allows us to translate $S_n$-identities, which involve the delent statistic, into corresponding $A_{n+1}$-identities. This strategy is used in the proofs of part (2) of the following theorems.

Part (1) of the following theorem is a new generalization of MacMahon's classical identity, and part (2) is its $A$-analogue.

**Theorem 1.9** *(see Theorem 6.1)*

(1)
$$\sum_{\sigma \in S_n} q^{\ell_S(\sigma)} t^{del_S(\sigma)} = \sum_{\sigma \in S_n} q^{rmaj_{S_n}(\sigma)} t^{del_S(\sigma)} =$$

$$= (1 + qt)(1 + q + q^2 t) \cdots (1 + q + \ldots + q^{n-1} t);$$

*and*

(2) $$\sum_{w\in A_{n+1}} q^{\ell_A(w)}t^{del_A(w)} = \sum_{w\in A_{n+1}} q^{rmaj_{A_{n+1}}(w)}t^{del_A(w)}$$

$$= (1+2qt)(1+q+2q^2t)\cdots(1+q+\ldots+q^{n-2}+2q^{n-1}t).$$

Recall the standard notation $[m]=\{1,\ldots,m\}$. The main theorem in this paper strengthens Theorem 1.1, and also gives its $A$-analogue. This is

**Theorem 1.10** *(See Theorem 9.1) For every subsets $D_1\subseteq[n-1]$ and $D_2\subseteq[n]$*

(1) $$\sum_{\{\pi\in S_n\mid\ Des_S(\pi^{-1})\subseteq D_1,\ Del_S(\pi^{-1})\subseteq D_2\}} q^{rmaj_{S_n}(\pi)} =$$

$$\sum_{\{\pi\in S_n\mid\ Des_S(\pi^{-1})\subseteq D_1,\ Del_S(\pi^{-1})\subseteq D_2\}} q^{\ell_S(\pi)},$$

*and*

(2) $$\sum_{\{\sigma\in A_{n+1}\mid\ Des_A(\sigma^{-1})\subseteq D_1,\ Del_A(\sigma^{-1})\subseteq D_2\}} q^{rmaj_{A_{n+1}}(\sigma)} =$$

$$\sum_{\{\sigma\in A_{n+1}\mid\ Des_A(\sigma^{-1})\subseteq D_1,\ Del_A(\sigma^{-1})\subseteq D_2\}} q^{\ell_A(\sigma)}.$$

This shows that the delent set and the descent set play a similar role in these identities.

The $A$-analogue of Corollary 1.2 follows. It is obtained as a special case of Corollary 9.2(2) (by substituting $q_3=1$).

**Corollary 1.11** *(See Corollary 9.2)*

$$\sum_{\sigma\in A_{n+1}} q_1^{rmaj_{A_{n+1}}(\sigma)} q_2^{des_A(\sigma^{-1})} = \sum_{\sigma\in A_{n+1}} q_1^{\ell_A(\sigma)} q_2^{des_A(\sigma^{-1})}.$$

Note that, while the $S$-identity holds for $maj_{S_n}$ as well as for $rmaj_{S_n}$, it is not possible to replace $rmaj_{A_{n+1}}$ by $maj_{A_{n+1}}$ in the $A$-analogue.

## 2 Preliminaries

### 2.1 Notation

For an integer $a$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Let $n_1, \ldots, n_r$ be non-negative integers such that $\sum_{i=1}^{r} n_i = n$. Recall that the *$q$-multinomial coefficient* $\left[{n \atop n_1,\ldots,n_r}\right]_q$ is defined by:

$$[0]!_q := 1,$$

$$[n]!_q := [n-1]!_q \cdot (1 + q + \ldots + q^{n-1}) \qquad (n \geq 1),$$

$$\begin{bmatrix} n \\ n_1 \ldots n_r \end{bmatrix}_q := \frac{[n]!_q}{[n_1]!_q \cdots [n_r]!_q}.$$

Represent $\sigma \in S_n$ by 'its second row' $\sigma = [\sigma(1), \ldots, \sigma(n)]$. We also use the cycle-notation; in particular, we denote $s_i := (i, i+1)$, the transposition of $i$ and $i+1$. Thus

$$[\ldots, \sigma(r), \sigma(r+1), \ldots]s_r = [\ldots, \sigma(r+1), \sigma(r), \ldots] \tag{2}$$

(i.e. only $\sigma(r), \sigma(r+1)$ switch places).

### 2.2 The Coxeter System of the Symmetric Group

The symmetric group on $n$ letters, denoted by $S_n$, is generated by the set of adjacent transpositions $S := \{(i, i+1) |\ 1 \leq i < n\}$.

The defining relations of $S$ are the Moore-Coxeter relations :

$$(s_i s_{i+1})^3 = 1 \qquad (1 \leq i < n),$$

$$(s_i s_j)^2 = 1 \qquad (|i-j| > 1)$$

$$s_i^2 = 1 \qquad (\forall i).$$

This set of generators is called the *Coxeter system* of $S_n$.

For $\pi \in S_n$ let $\ell_S(\pi)$ be the standard length of $\pi$ with respect to $S$ (i.e. the length of the canonical presentation of $\pi$, see Subsection 3 ). Let $w$ be a word on the letters $S$. A *commuting move* on $w$ switches the positions of consequent letters $s_i s_j$ where $|i-j| > 1$. A *braid move* replaces $s_i s_{i+1} s_i$ by $s_{i+1} s_i s_{i+1}$ or vice versa. The following is a well known fact, but we shall not use it in this paper.

**Fact 2.1** *All irreducible expressions of $\pi \in S_n$ are of length $\ell_S(\pi)$. For every pair of irreducible words of $\pi \in S_n$, it is possible to move from one to another along commuting and braid moves.*

## 2.3 Permutation Statistics

There are various statistics on the symmetric groups $S_n$, like the *descent* number and the *major* index. We introduce and study analogous statistics on the alternating groups $A_n$. To distinguish, we add 'sub S' and 'sub A' accordingly.

Given a permutation $\pi = [\pi(1), \ldots, \pi(n)]$ in the symmetric group $S_n$, we say that a pair $(i, j)$, $1 \le i < j \le n$ is an *inversion* of $\pi$ if $\pi(i) > \pi(j)$. The set of inversions of $\pi$ is denoted by $Inv_S(\pi)$ and its cardinality is denoted by $inv_S(\pi)$. Also $1 \le i < n$ is a *descent* of $\pi$ if $\pi(i) > \pi(i+1)$. For the definitions of the descent set $Des_S(\pi)$, the descent number $des_S(\pi)$, the major index $maj_S(\pi)$ and the reverse major index $rmaj_{S_n}(\pi)$, see Subsection 1.2.

Note that $i$ is a descent of $\pi$ if and only if $\ell_S(\pi s_i) < \ell_S(\pi)$. Thus (as already mentioned in Subsection 1.2), the descent set, and consequently the other statistics, have an algebraic interpretation in terms of the Coxeter system. Also, for every $\pi \in S_n$

$$inv_S(\pi) = \ell_S(\pi). \tag{3}$$

The following well known identity is due to MacMahon [11]. See, e.g. [5] and [16, Corollaries 1.3.10 and 4.5.9].

**Theorem 2.2**

$$\sum_{\pi \in S_n} q^{inv_S(\pi)} = \sum_{\pi \in S_n} q^{maj_S(\pi)} =$$

$$= [n]!_q = (1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-2}+q^{n-1}).$$

The following theorem is a reformulation of [6, Theorem 1].

**Theorem 2.3** *For every $B \subseteq [n-1]$,*

$$\sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) = B\}} q^{inv_S(\pi)} = \sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) = B\}} q^{maj_S(\pi)}.$$

**Note.** Let $\sigma \in S_n$, $\sigma = [\sigma(1), \ldots, \sigma(n)]$. Then $\sigma = [\ldots, k, \ldots, \ell, \ldots]$ (i.e. $k$ is left of $\ell$ in $\sigma$ ) if and only if $\sigma^{-1}(k) < \sigma^{-1}(\ell)$.

**Shuffles.** Let $1 \le i \le n-1$, then $w \in S_n$ is an $\{i\}$-shuffle if it shuffles $\{1,\ldots,i\}$ with $\{i+1,\ldots,n\}$; in other words, if $1 \le a < b \le i$ then $w^{-1}(a) < w^{-1}(b)$, and similarly, if $i+1 \le k < \ell \le n$, then $w^{-1}(k) < w^{-1}(\ell)$.
**Example.** Let $n = 4$ and $B = \{2\}$, then $\{1,2\}$ and $\{3,4\}$ are being shuffled, hence

$$[1,2,3,4],\ \ [1,3,2,4],\ \ [1,3,4,2],\ \ [3,1,2,4],\ \ [3,1,4,2],\ \ [3,4,1,2]$$

are all the $\{2\}$-shuffles.

More generally, let $B = \{i_1,\ldots,i_k\} \subseteq [n-1]$, where $i_1 < \ldots < i_k$. Denote $i_0 := 0$ and $i_{k+1} := n$. A *$B$-shuffle* is a permutation which shuffles $\{1,\ldots,i_1\}, \{i_1+1,\ldots,i_2\},\ldots$ Thus $\pi \in S_n$ is a $B$-shuffle if it satisfies: if $i_j \le a < b \le i_{j+1}$ for some $0 \le j \le k$, then $\pi = [\ldots,a,\ldots,b,\ldots]$ (i.e. $a$ is left of $b$ in $\pi$). Notice that in particular there can be no descent for $\pi^{-1}$ on any $a$, $i_j < a < i_{j+1}$, hence $Des_S(\pi^{-1}) \subseteq B$. The opposite is also clear, hence

**Fact 2.4** *For every $B \subseteq [n-1]$*

$$\{\pi \in S_n \mid Des_S(\pi^{-1}) \subseteq B\} = \{\pi \in S_n \mid \pi \textit{ is a } B\textit{-shuffle}\}.$$

For a permutation $\pi \in S_n$ let

$$supp(\pi) := \{1 \le i \le n \mid \pi(i) \ne i\}$$

be the *support* of $\pi$.

Let $k \in [n-1]$, and let $\pi_1, \pi_2$ be permutations in $S_n$, such that $supp(\pi_1) \subseteq [k]$ and $supp(\pi_2) \subseteq [k+1,n]$. A permutation $\sigma \in S_n$ is called a shuffle of $\pi_1$ and $\pi_2$ if $\sigma = \pi_1\pi_2 r$ for some $\{k\}$-shuffle $r$. Equivalently, $\sigma$ is a shuffle of $\pi_1$ and $\pi_2$ if and only if the letters of $[k]$ appear in $\sigma$ in the same order as they appear in $\pi_1$ and the letters of $[k+1,n]$ appear in $\sigma$ in the same order as they appear in $\pi_2$. The following is a special case of [16, Prop. 1.3.17].

**Fact 2.5** *Let $k \in [n]$, and let $\pi_1, \pi_2$ be permutations in $S_n$, such that $supp(\pi_1) \subseteq [k]$ and $supp(\pi_2) \subseteq [k+1,n]$. Then*

$$\sum_{Des(r^{-1}) \subseteq \{k\}} q^{inv_S(\pi_1\pi_2 r) - inv_S(\pi_1) - inv_S(\pi_2)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

The following analogue is a special case of a well known theorem of Garsia and Gessel. It should be noted that, while Garsia-Gessel's Theorem is stated in terms of sequences, our reformulation is in terms of permutations.

**Theorem 2.6** *[7, Theorem 3.1] Let $k \in [n-1]$, and let $\pi_1, \pi_2$ be permutations in $S_n$, such that $supp(\pi_1) \subseteq [k]$ and $supp(\pi_2) \subseteq [k+1, n]$. Let $\nu_k := (1, k+1)(2, k+2)\cdots(n-k, n) \in S_n$. Then*

$$\sum_{Des_S(r^{-1}) \subseteq \{k\}} q^{maj_S(\pi_1\pi_2 r) - maj_S(\pi_1) - maj_S(\nu_k^{-1}\pi_2\nu_k)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

In order to translate Theorem 2.6 into Garsia-Gessel's terminology, note that $\pi_1\pi_2 r$ are shuffles of $\pi_1$ and $\pi_2$ (as mentioned above); thus the sum runs over all shuffles of $\pi_1$ and $\pi_2$. Also, $maj_S(\nu_k^{-1}\pi_2\nu_k)$ is the major index of $\pi_2$, when it is considered as a sequence on the letters $[k+1, n]$.

**Remark 2.7** *In general, it is possible to replace a statement involving maj by a corresponding statement involving rmaj, using the following automorphism $\sigma \to \hat{\sigma}$ :*
*Let $\rho_n \in S_n$ denote the involution*

$$\rho_n := (1, n)(2, n-1)\cdots([n/2], [(n+3)/2]),$$

*where for a real number $\alpha$, $[\alpha]$ is the 'integer part' of $\alpha$. Define*

$$\hat{\sigma} := \rho_n \sigma \rho_n.$$

*Then $\sigma \to \hat{\sigma}$ is an automorphism of $S_n$ with the following properties :*

1. *Let $i < j$ and $\sigma(i) > \sigma(j)$, then $n+1-j < n+1-i$ and $\hat{\sigma}(n+1-j) > \hat{\sigma}(n+1-i)$. In particular, $i \in Des_S(\sigma)$ if and only if $n-i \in Des_S(\hat{\sigma})$, hence*

$$rmaj_{S_n}(\sigma) = maj_S(\hat{\sigma}). \tag{4}$$

2. *There is a bijection between $Inv_S(\sigma)$ and $Inv_S(\hat{\sigma})$ given by $(i, j) \leftrightarrow (n+1-j, n+1-i)$, hence*

$$inv_S(\sigma) = inv_S(\hat{\sigma}). \tag{5}$$

3. *Part 1 implies that $\sigma$ is an $\{i\}$-shuffle if and only if $\hat{\sigma}$ is an $\{n-i\}$-shuffle, i.e.*

$$Des_S(\sigma^{-1}) \subseteq \{i\} \Longleftrightarrow Des_S(\hat{\sigma}^{-1}) \subseteq \{n-i\}. \tag{6}$$

*This easily generalizes to $B$-shuffles.*

Note that by (4) and [15, Claim 0.4], for every $\pi \in S_n$

$$rmaj_{S_n}(\pi) = charge(\pi^{-1}),$$

where the charge is defined as in [10, p. 242].

# 3 The $S$ and $A$ Canonical Presentations

In this section we consider canonical presentations of elements in $S_n$ and in $A_n$ by the corresponding Coxeter generators. This presentation for $S_n$ is well known, see for example [8, pp. 61-62]. The analogous presentation for $A_n$ follows from the properties of the Mitsuhashi's Coxeter generators.

## 3.1 The $S_n$ Case

The $S_n$ canonical presentation is proved below, using the $S$-procedure, which is also applied later.

Recall that $s_i = (i, i+1)$, $1 \le i < n$, are the Coxeter generators of $S_n$. For each $1 \le j \le n-1$ define

$$R_j^S = \{1, s_j, s_j s_{j-1}, \dots, s_j s_{j-1} \cdots s_1\} \tag{7}$$

and note that $R_1^S, \dots, R_{n-1}^S \subseteq S_n$. .

**Theorem 3.1** *(see [8, pp. 61-62]) Let $w \in S_n$, then there exist unique elements $w_j \in R_j^S$, $1 \le j \le n-1$, such that $w = w_1 \cdots w_{n-1}$. Thus, the presentation $w = w_1 \cdots w_{n-1}$ is unique.*

**Definition 3.2** *Call the above $w = w_1 \cdots w_{n-1}$ in Theorem 3.1 the $S$ canonical presentation of $w \in S_n$.*

**A proof** of Theorem 3.1 follows from the following S-Procedure.

**The S-Procedure.** The following is a simple procedure for calculating the $S$ canonical presentation of a given $w \in S_n$. It can also be used to prove Theorem 3.1, as well as various other facts. Let $\sigma \in S_n$, $\sigma(r) = n$, $\sigma = [\ldots, n, \ldots]$, then apply Equation (2) to 'pull $n$ to its place on the right': $\sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots\ldots, n]$. This gives $w_{n-1} = s_{n-1} \cdots s_{r+1} s_r$. Next, in

$$\sigma w_{n-1}^{-1} = \sigma s_r s_{r+1} \cdots s_{n-1} = [\ldots, n-1, \ldots, n],$$

pull $n-1$ to its right place (second from right) by a similar product $s_t s_{t+1} \cdots s_{n-2}$. This yields $w_{n-2} = s_{n-2} \cdots s_t$. Continue! Finally, $\sigma = w_1 \cdots w_{n-1}$.

For example, let $\sigma = [2,5,4,1,3]$, then $w_{n-1} = w_4 = s_4 s_3 s_2$; $\sigma w_4^{-1} = [2,4,1,3,5]$, therefore $w_3^{-1} = s_2 s_3$. Check that $w_2 = 1$ and, finally, $w_1 = s_1$. Thus $\sigma = w_1 \cdots w_4 = (s_1)(1)(s_3 s_2)(s_4 s_3 s_2)$.

The uniqueness in Theorem 3.1 follows by cardinality, since the number of canonical words in $S_n$ is at most

$$\prod_{j=1}^{n-1} card(R_j^S) = \mid S_n \mid .$$

This proves Theorem 3.1. $\square$

## 3.2 A Generating Set for $A_n$

We turn now to $A_n$. As was already mentioned in 1.3.1, we let

$$a_i := s_1 s_{i+1} \qquad (1 \le i \le n-1).$$

The set

$$A := \{a_i \mid 1 \le i \le n-1\}$$

generates the alternating group on $n$ letters $A_{n+1}$. This generating set and its following properties appear in [12].

**Proposition 3.3** *[12, Proposition 2.5] The defining relations of $A$ are*

$$(a_i a_j)^2 = 1 \qquad (|i-j| > 1);$$

$$(a_i a_{i+1})^3 = 1 \qquad (1 \le i < n-1);$$

$$a_1^3 = 1 \qquad \textit{and} \qquad a_i^2 = 1 \qquad (1 < i \le n-1).$$

The general braid-relation $(a_i a_{i+1})^3 = 1$ implies the following braid-relations.

1. $a_2 a_1 a_2 = a_1^{-1} a_2 a_1^{-1}$ and

2. $a_2 a_1^{-1} a_2 = a_1 a_2 a_1$.

3. $a_{i+1} a_i a_{i+1} = a_i a_{i+1} a_i$ if $i \geq 2$ (since $a_i^{-1} = a_i$).

Let

$$\overline{A} := A \cup \{a_1^{-1}\},$$

where $A$ is defined as above. Clearly, $\overline{A}$ is a generating set for $A_{n+1}$.

### 3.3 The $A$ Canonical Presentation

Mitsuhashi's Coxeter generators are now applied to obtain a unique canonical presentation for elements in the alternating group.

For each $1 \leq j \leq n-1$ define

$$R_j^A = \{1,\ a_j,\ a_j a_{j-1},\ \ldots,\ a_j \cdots a_2,\ a_j \cdots a_2 a_1,\ a_j \cdots a_2 a_1^{-1}\} \tag{8}$$

and note that $R_1^A, \ldots, R_{n-1}^A \subseteq A_{n+1}$.

**Theorem 3.4** *Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R_j^A$, $1 \leq j \leq n-1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique.*

**Definition 3.5** *Call the above $v = v_1 \cdots v_{n-1}$ in Theorem 3.4 the $A$ canonical presentation of $v$.*

**Proof** of Theorem 3.4. Let $v = w_1 \cdots w_n$, $w_j \in R_j^S$, be the $S$ canonical presentation of $v$. Rewrite that presentation explicitly as

$$v = (s_{i_1} s_{i_2}) \cdots (s_{i_{2r-1}} s_{i_{2r}}). \tag{9}$$

Note that $s_i s_j = (s_i s_1)(s_1 s_j) = a_{i-1}^{-1} a_{j-1}$ (denote $a_0 = 1$). Thus each $s_i$ in (9) is replaced by a corresponding $a_{i-1}^{\pm 1}$. It follows that for each $2 \leq j \leq n$, $w_j$ is replaced by $v_{j-1} \in R_{j-1}^A$ and $v = v_1 \cdots v_{n-1}$. This proves the existence of such a presentation.

A second proof of the existence follows from the following A-procedure.

**The A-Procedure** is similar to the S-procedure. We describe its first step, which is also its inductive step.
Let $\sigma \in A_{n+1}$, $\sigma = [\ldots, n+1, \ldots]$. As in the S-procedure, pull $n+1$ to the

right: $\sigma s_r s_{r+1} \cdots s_n = [b_1, b_2, \ldots, n+1]$. The (S-) length of $s_r s_{r+1} \cdots s_n$ is $n - r + 1$; if it is odd, use $\sigma s_r s_{r+1} \cdots s_n s_1 = [b_2, b_1, \ldots, n+1]$. Thus

$$v_{n-1} = \begin{cases} s_n s_{n-1} \cdots s_r, & \text{if } n - r + 1 \text{ is even;} \\ s_1 s_n s_{n-1} \cdots s_r, & \text{if } n - r + 1 \text{ is odd.} \end{cases}$$

*The case* $r \geq 2$. Then $s_1 s_j = s_j s_1$ for all $j \geq r + 1$, hence

$$v_{n-1} = \begin{cases} (s_1 s_n)(s_1 s_{n-1}) \cdots (s_1 s_r) = a_{n-1} \cdots a_{r-1}, & \text{if } n - r + 1 \text{ is even;} \\ (s_1 s_1)(s_1 s_n) \cdots (s_1 s_r) = a_{n-1} \cdots a_{r-1}, & \text{if } n - r + 1 \text{ is odd.} \end{cases}$$

*The case* $r = 1$. If $n - r + 1 = n$ is even,

$$v_{n-1} = s_n \cdots s_2 s_1 = (s_1 s_n) \cdots (s_1 s_3)(s_2 s_1) = a_{n-1} \cdots a_2 a_1^{-1},$$

and similarly if $n - r + 1$ is odd.
This completes the first step. In the next step, pull $n$ to the $n$-th position (i.e. second from the right), etc. This proves the existence of such a presentation $v = v_1 \cdots v_{n-1}$.

*Example.* Let $\sigma = [3, 5, 4, 2, 1]$, so $n + 1 = 5$. Now $\sigma s_2 s_3 s_4 = [3, 4, 2, 1, 5]$ and since $s_2 s_3 s_4$ is of odd length (=3), permute 3 and 4: $\sigma s_2 s_3 s_4 s_1 = [4, 3, 2, 1, 5]$. Thus $v_3 = s_1 s_4 s_3 s_2 = (s_1 s_1)(s_1 s_4)(s_1 s_3)(s_1 s_2) = a_3 a_2 a_1$. Similarly, $v_2 = a_2 a_1^{-1}$ and $v_1 = a_1$, hence $[3, 5, 4, 2, 1] = (a_1)(a_2 a_1^{-1})(a_3 a_2 a_1)$.

**Uniqueness** follows by cardinality: note that for all $1 \leq j \leq n - 1$, $| R_j^A | = j + 2$, hence the number of such words $v_1 \cdots v_{n-1}$ in $A_{n+1}$ is at most

$$\prod_{j=1}^{n-1} (j + 2) = | A_{n+1} | .$$

Since each element in $A_{n+1}$ does have such a presentation, this implies the uniqueness - and the proof of Theorem 3.4 is complete. $\square$

Given $w \in S_n$, we say that $s_i$ occurs $\ell$ times in $w$ if it occurs $\ell$ times in the canonical presentation of $w$. Similarly for the number of occurrences of $a_i$, or of $a_1^{-1}$, in $v \in A_{n+1}$. The number of occurrences of $s_1$, as well as those of $a_1^{\pm 1}$, are of particular importance in this paper.

**Lemma 3.6** *Let $w \in S_n$, then the number of occurrences of $s_i$ in $w$ equals the number of occurrences of $s_i$ in $w^{-1}$. Similarly for $A_{n+1}$ and $a_1^{\pm 1}$.*

This is an obvious corollary of

**Lemma 3.7** *Let $w = s_{i_1} \cdots s_{i_p}$ be the canonical presentation of $w \in S_n$. Then the canonical presentation of $w^{-1}$ is obtained from the presentation $w^{-1} = s_{i_p} \cdots s_{i_1}$ by commuting moves only - without any braid moves. Similarly for $v$, $v^{-1} \in A_{n+1}$.*

*Proof.* We prove for $S_n$. The proof is by induction on $n$. Write $w = w_1 \cdots w_{n-1}$ , $w_j \in R_j^S$. If $w_{n-1} = 1$ then $w \in S_{n-1}$ and the proof follows by induction.

Let $w_{n-1} = s_{n-1}s_{n-2} \cdots s_k$ where $1 \le k \le n-1$. Now either $w_{n-2} = 1$ or $w_{n-2} = s_{n-2}s_{n-3} \cdots s_\ell$ for some $1 \le \ell \le n-2$, and similarly for $w_{n-3}$, $w_{n-4}$ etc. The case $w_{n-2} = 1$ is similar to the case $w_{n-2} \neq 1$ and is left to the reader, so let $w_{n-2} \neq 1$ and

$$w^{-1} = w_{n-1}^{-1} w_{n-2}^{-1} \cdots = (s_k \cdots s_{n-1})(s_\ell \cdots s_{n-2}) w_{n-3}^{-1} w_{n-4}^{-1} \cdots$$

Notice that $s_{n-1}(s_\ell \cdots s_{n-3}) = (s_\ell \cdots s_{n-3})s_{n-1}$, hence

$$w^{-1} = (s_k \cdots s_{n-2})(s_\ell \cdots s_{n-3})(s_{n-1}s_{n-2}) w_{n-3}^{-1} w_{n-4}^{-1} \cdots$$

Next, move $s_{n-1}s_{n-2}$ to the right, similarly, by commuting moves. Continue by similarly pulling $s_{n-3}$ - in $w_{n-3}^{-1}$ - to the right, etc. It follows that by such commuting moves we obtain

$$w^{-1} = \overline{w}^{\,-1}(s_{n-1}s_{n-2} \cdots s_d)$$

for some $d$, where $\overline{w} = s_{j_r} \cdots s_{j_1} \in S_{n-1}$, and is in canonical form. By induction, transform $\overline{w}^{\,-1}$ to its canonical form by commuting moves - and the proof is complete. $\square$

# 4 The Lengths Statistics

The canonical presentations of the previous sections allow us to introduce the S and the A lengths.

**Definition 4.1** *(The length statistics).*

1. *Let $w \in S_n$ with $w = s_{i_1} \cdots s_{i_r}$ its $S$ canonical presentation, then its S-length is $\ell_S(w) = r$.*

2. *Let $v \in A_{n+1}$ with $v = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$ $(\epsilon_i = \pm 1)$ its A canonical presentation, then its A-length is $\ell_A(v) = r$.*

For example, $\ell_A(a_1) = 1$ and $\ell_S(a_1) = \ell_S(s_1 s_2) = 2$.

**Remark 4.2** *An analogue of Fact 2.1 holds : All irreducible expressions of $v \in A_{n-1}$ are of length $\ell_A(v)$. This fact will not be used in the paper.*

**Definition 4.3** *1. Let $w \in S_n$. The number of times that $s_1$ occurs in the S canonical presentation of $w$ is denoted by $del_S(w)$.*

2. *Let $v \in A_{n-1}$. The number of times that $a_1^{\pm 1}$ occurs in the A canonical presentation of $v$ is denoted by $del_A(v)$.*

For example, $del_S(s_1 s_2 s_1 s_3) = 2$ and $del_A(a_1^{-1} a_2 a_1 a_3 a_2 a_1^{-1})) = 3$.

A combinatorial characterization of $del_S$ ($del_A$) is given in section 7.
Relations between $del_S$ and the S and the A lengths of $v \in A_{n+1}$ are given by the following proposition.

**Proposition 4.4** *Let $w \in A_{n+1}$, then*

$$\ell_A(w) = \ell_S(w) - del_S(w).$$

*Moreover, let*

$$w = s_{i_1} \cdots s_{i_{2r}} = w_1 \cdots w_n, \qquad w_i \in R_i^S, \tag{10}$$

*be its S canonical presentation and*

$$w = a_{j_1}^{\epsilon_1} \cdots a_{j_t}^{\epsilon_t} = v_1 \cdots v_{n-1}, \qquad v_i \in R_i^A, \tag{11}$$

*its A canonical presentation. Then*

$$\ell_A(v_i) = \begin{cases} \ell_S(w_{i+1}) & \text{if } s_1 \text{ does not occur in } w_{i+1}; \\ \ell_S(w_{i+1}) - 1 & \text{if } s_1 \text{ occurs in } w_{i+1}. \end{cases} \tag{12}$$

*Proof.* As in the proof of Theorem 3.4, the proof easily follows from (9) by replacing $s_i s_j$ by $(s_i s_1)(s_1 s_j)$.

□

The S-lengths $\ell_S(w_{i+1})$ and the A–lengths $\ell_A(v_i)$ in (12) can be calculated directly from $w = [b_1, \ldots, b_{n+1}]$ as follows.

**Proposition 4.5** *Let $w \in S_{n+1}$ as above. For each $2 \le j \le n$ let $T_j(w)$ denote the set of indices $i$ such that $i < j$ and $w = [\ldots, j, \ldots, i, \ldots]$ (i.e. $w^{-1}(i) > w^{-1}(j)$); denote $t_j(w) =\mid T_j(w) \mid$. Keeping the notations of Proposition 4.4 we have:*

1. *$\ell_S(w_j) = t_{j+1}(w)$. Moreover, $T_{j+1}(w)$ is the full set $\{1, \ldots, j\}$ (i.e. $t_{j+1}(w) = j$) if and only if $s_1$ occurs in $w_j$.*

2. *$\ell_A(v_k)$ equals $\mid T_k(w) \mid$, provided that $T_k(w)$ is not the full set $\{1, \ldots, k-1\}$, and it equals $\mid T_k(w) \mid -1$ otherwise.*

*Proof.* By an easy induction on $n$, prove that

$$(\ell_S(w_1), \ldots, \ell_S(w_n)) = (t_2(w), \ldots, t_{n+1}(w)).$$

This follows since
$[b_1, \ldots, b_n, n+1]s_n s_{n-1} \cdots s_r = [b_1, \ldots, b_{r-1}, n+1, b_r, \ldots, b_n]$.
Here are the details: Write $w = w_1 \cdots w_n$, let $\sigma = w_1 \cdots w_{n-1}$, so $\sigma = [d_1, \ldots, d_n, n+1]$. If $w_n = 1$, the claim follows by induction. Let $w_n = s_n s_{n-1} \cdots s_r$ for some $r \ge 1$. Then $w = \sigma w_n = [d_1, \ldots, d_{r-1}, n+1, d_r, \ldots d_n]$. Thus $t_{n+1}(w) = n - r + 1 = \ell_S(w_n)$. Also, for $2 \le j \le n$, $t_j(w) = t_j(\sigma)$, and the proof of part 1 follows by induction. Part 2 now follows from (12). $\square$

# 5 $f$-Pairs of Statistics

## 5.1 The Covering Map

Theorems 3.1 and 3.4 allow us to introduce the following definition.

**Definition 5.1** *Define $f : A_{n+1} \to S_n$ as follows.*

$$f(a_1) = f(a_1^{-1}) = s_1 \qquad \textit{and} \qquad f(a_i) = s_i, \qquad 2 \le i \le n-1.$$

*Now extend $f : R_j^A \to R_j^S$ via*

$$f(a_j a_{j-1} \cdots a_\ell) = s_j s_{j-1} \cdots s_\ell, \qquad f(a_j \cdots a_1) = f(a_j \cdots a_1^{-1}) = s_j \cdots s_1.$$

*Finally, let $v \in A_{n+1}$, $\quad v = v_1 \cdots v_{n-1}$ its unique $A$ canonical presentation, then*

$$f(v) = f(v_1) \cdots f(v_{n-1})$$

*which is clearly the $S$ canonical presentation of $f(v)$.*

Notice that for $v \in A_{n+1}$, $\ell_A(v) = \ell_S(f(v))$. We therefore say that the pair of the length statistics $(\ell_S, \ell_A)$ is an $f$-pair. More generally, we have

**Definition 5.2** *Let $m_S$ be a statistic on the symmetric groups and $m_A$ a statistic on the alternating groups. We say that $(m_S, m_A)$ is an $f$-pair (of statistics) if for any $n$ and $v \in A_{n+1}$, $m_A(v) = m_S(f(v))$.*

Examples of $f$-pairs are given in Proposition 5.4 below.

**Proposition 5.3** *Recall Definition 1.5.1. For every $\pi \in A_{n+1}$*

$$Des_A(\pi) = Des_S(f(\pi)).$$

**Proof -** is left to the reader. $\square$

It follows that the descent statistics are $f$-pairs. By Definition 4.3, $(del_S, del_A)$ is an $f$-pair. We summarize:

**Proposition 5.4** *The following pairs*

$$(\ell_S, \ell_A),$$

$$(des_S, des_A),$$

$$(maj_S, maj_A),$$

$$(rmaj_{S_n}, rmaj_{A_{n+1}})$$

*and*

$$(del_S, del_A)$$

*are $f$-pairs.*

## 5.2 The 'del' Statistics

The following basic properties of $del_S$ play an important role in this paper.

**Proposition 5.5** *1. For each $w \in S_n$, $\mid f^{-1}(w) \mid = 2^{del_S(w)}$.*

*2. For each $w \in S_n$ and $v \in A_{n+1}$*

$$del_S(w) = del_S(w^{-1}) \quad \text{and} \quad del_A(v) = del_A(v^{-1}). \tag{13}$$

**Proof.** Part 1 follows since each occurrence of $s_1$ can be replaced by an occurrence of either $a_1$ or $a_1^{-1}$. Part 2 follows from Lemma 3.6.

□

We have the following general proposition.

**Proposition 5.6** *Let $(m_S, m_A)$ be an $f$-pair of statistics, then for all $n$*

$$\sum_{v\in A_{n+1}} q^{m_A(v)}t^{del_A(v)} = \sum_{w\in S_n} q^{m_S(w)}(2t)^{del_S(w)}.$$

*Proof.* Since $A_{n+1} = \cup_{w\in S_n} f^{-1}(w)$, a disjoint union, we have:

$$\sum_{v\in A_{n+1}} q^{m_A(v)}t^{del_A(v)} = \sum_{w\in S_n}\sum_{v\in f^{-1}(w)} q^{m_A(v)}t^{del_A(v)} =$$

$$\sum_{w\in S_n}\sum_{v\in f^{-1}(w)} q^{m_S(f(v))}t^{del_S(f(v))} = \sum_{w\in S_n}\sum_{v\in f^{-1}(w)} q^{m_S(w)}t^{del_S(w))} =$$

$$\sum_{w\in S_n} 2^{del_S(w)}q^{m_S(w)}t^{del_S(w))}.$$

□

A refinement of Proposition 5.6 is given in Proposition 5.10

**Proposition 5.7** *With the above notations we have:*

*1.*

$$\sum_{\sigma\in S_n} q^{\ell_S(\sigma)}t^{del_S(\sigma)} = (1+qt)(1+q+q^2t)\cdots(1+q+\ldots+q^{n-1}t).$$

*2.*

$$\sum_{w\in A_{n+1}} q^{\ell_A(w)}t^{del_A(w)} = (1+2qt)(1+q+2q^2t)\cdots(1+q+\ldots+q^{n-2}+2q^{n-1}t).$$

*Proof.*

1. The proof of part 1 is similar to the proof of Corollary 1.3.10 in [16]. Let $w_j \in R_j^S$, then $del_S(w_j) = 1$ if $w_j = s_j \ldots s_1$ and $= 0$ otherwise. Let $w \in S_n$ and let $w = w_1 \cdots w_{n-1}$ be its $S$ canonical presentation,

then $del_S(w) = del_S(w_1) + \ldots + del_S(w_{n-1})$ and $\ell_S(w) = \ell_S(w_1) + \cdots + \ell_S(w_{n-1})$. Thus

$$\sum_{w \in S_n} q^{\ell_S(w)} t^{del_S(w)} = \prod_{j=1}^{n-1} \left( \sum_{w_j \in R_j^S} q^{\ell_S(w_j)} t^{del_S(w_j)} \right).$$

The proof now follows since

$$\sum_{w_j \in R_j^S} q^{\ell_S(w_j)} t^{del_S(w_j)} = 1 + q + q^2 + \ldots + q^{j-2} + q^{j-1} t.$$

2. By Proposition 5.6, part 2 follows from part 1.

□

### 5.3 Connection with the Stirling Numbers

Recall that $c(n,k)$ is the number of permutations in $S_n$ with exactly $k$ cycles, $1 \leq k \leq n$: $c(n,k)$ are *the sign-less Stirling numbers of the first kind.* Let $w_S(n,\ell)$ denote the number of $S$ canonical words in $S_n$ with $\ell$ appearances of $s_1$. Similarly, let $w_A(n+1,\ell)$ denote the number of $A$ canonical words in $A_{n+1}$ with $\ell$ appearances of $a_1^{\pm 1}$.
We prove

**Proposition 5.8** *Let* $0 \leq \ell \leq n-1$*, then*

*1.*

$$\sum_{\ell \geq 0} w_S(n,\ell) t^\ell = (t+1)(t+2)\cdots(t+n-1),$$

*hence* $w_S(n,\ell) = c(n,\ell+1)$.

*2.*

$$\sum_{\ell \geq 0} w_A(n,\ell) t^\ell = (2t+1)(2t+2)\cdots(2t+n-1),$$

*hence* $w_A(n+1,\ell) = 2^\ell \cdot c(n,\ell+1)$.

*Proof.* Substitute $q = 1$ in Proposition 5.7 and, in part 1, apply Proposition 1.3.4 of [16], which states that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)(\cdots(x+n-1).$$

$\square$

further connections with the Stirling numbers are given below (Propositions 5.11, 5.12 and 7.10) and in [13].

## 5.4 A Multivariate Refinement

**Definition 5.9** *Let $w \in S_n$, $w = w_1 \cdots w_{n-1}$ its $S$ canonical presentation and let $1 \leq j \leq n-1$. Denote $\epsilon_{S,j}(w) = 1$ if $s_1$ occurs in $w_j$, and $= 0$ otherwise; also denote*

$$\bar{\epsilon}_S(w) = (\epsilon_{S,1}(w), \ldots, \epsilon_{S,n-1}(w))$$

*and*

$$t^{\bar{\epsilon}_S(w)} = t_1^{\epsilon_{S,1}(w)} \cdots t_{n-1}^{\epsilon_{S,n-1}(w)}.$$

*Similarly for $v = v_1 \cdots v_{n-1} \in A_{n+1}$: $\epsilon_{A,j}(v) = 1$ if $a_1^{\pm 1}$ occurs in $v_j$, and $= 0$ otherwise, and define $\bar{\epsilon}_A(w)$ similarly. Clearly, $del_S(w) = \sum_j \epsilon_{S,j}(w)$ and $del_A(v) = \sum_j \epsilon_{A,j}(v)$.*

Proposition 5.6 admits the following generalization.

**Proposition 5.10** *Let $(m_S, m_A)$ be an $f$-pair of statistics, then for all $n$*

$$\sum_{v \in A_{n+1}} q^{m_A(v)} \prod_{j=1}^{n-1} t_j^{\epsilon_{A,j}(v)} = \sum_{w \in S_n} q^{m_S(w)} \prod_{j=1}^{n-1} (2t_j)^{\epsilon_{S,j}(w)}.$$

**The proof** is a slight generalization of the proof of Proposition 5.6 - and is left to the reader.

We end this section with another two multivariate generalizations, which will not be used in the rest of the paper. Proposition 5.7 generalizes as follows.

**Proposition 5.11** *Let $\ell_S$, $\ell_A$ be the length statistics, then*

*1.*

$$\sum_{w\in S_n} q^{\ell_S(w)} \prod_{j=1}^{n-1} (t_j)^{\epsilon_{S,j}(w)} = (1+qt_1)(1+q+q^2t_2)\cdots(1+q+\ldots+q^{n-1}t_{n-1}).$$

*2.*

$$\sum_{v\in A_{n+1}} q^{\ell_A(v)} \prod_{j=1}^{n-1} t_j^{\epsilon_{A,j}(v)} = (1+2qt_1)\cdots(1+q+\ldots+q^{n-2}+2q^{n-1}t_{n-1}).$$

One can generalize Proposition 5.8 as follows. Let $w = w_1 \cdots w_{n-1} \in S_n$, a canonical presentation, with $\epsilon_{S,j}(w)$ and $\overline{\epsilon}_S(w)$ as in Definition 5.9. Given $\overline{\epsilon} = (\epsilon_1, \cdots, \epsilon_{n-1})$ with all $\epsilon_i \in \{0,1\}$, denote $w_S(n,\overline{\epsilon}) = card\{w \in S_n \mid \overline{\epsilon}_S(w) = \overline{\epsilon}\}$. Also denote $|\overline{\epsilon}| = \sum_j \epsilon_j$ and $t^{\overline{\epsilon}} = \prod_j t_j^{\epsilon_j}$. Note that

$$\sum_{|\overline{\epsilon}|=\ell} w_S(n,\overline{\epsilon}) = w_\ell(n,\ell) = c(n,\ell+1).$$

Similarly, introduce the analogous notations for $A_{n+1}$.
Proposition 5.8 now generalizes as follows.

**Proposition 5.12** *With the above notations*

*1.*

$$\sum_{\overline{\epsilon}} w_S(n,\overline{\epsilon})\, t^{\overline{\epsilon}} = (t_1+1)\cdots(t_{n-1}+n-1).$$

*2.*

$$\sum_{\overline{\epsilon}} w_A(n,\overline{\epsilon})\, t^{\overline{\epsilon}} = (2t_1+1)\cdots(2t_{n-1}+n-1).$$

# 6 The Major Index and the Delent Number

Recall the definitions of $rmaj_{S_n}$ and $rmaj_{A_{n+1}}$ from Subsections 1.2 and 1.3. In this section we prove

**Theorem 6.1**

$$(1) \qquad \sum_{\sigma\in S_n} q^{\ell_S(\sigma)} t^{del_S(\sigma)} = \sum_{\sigma\in S_n} q^{rmaj_{S_n}(\sigma)} t^{del_S(\sigma)} =$$

$$= (1+qt)(1+q+q^2t)\cdots(1+q+\ldots+q^{n-1}t);$$

*and*

(2)
$$\sum_{w\in A_{n+1}} q^{\ell_A(w)}t^{del_A(w)} = \sum_{w\in A_{n+1}} q^{rmaj_{A_{n+1}}(w)}t^{del_A(w)}$$

$$= (1+2qt)(1+q+2q^2t)\cdots(1+q+\ldots+q^{n-2}+2q^{n-1}t).$$

Note that Theorem 6.1 follows from our main theorem 9.1. However, the proof of Theorem 9.1 applies the machinery required for the proof of Theorem 6.1 combined with additional, more elaborate arguments - therefore we prove it here.

Comparing the coefficients of $t^k$ in both parts, we obtain

**Theorem 6.2** *Let* $B^S_{n,k} := \{\sigma \in S_n \mid del_S(\sigma) = k\}$ *and* $B^A_{n+1,k} := \{\sigma \in A_{n+1} \mid del_A(\sigma) = k\}$. *Then for each* $0 \le k \le n-1$,

(1)
$$\sum_{\sigma\in B^S_{n,k}} q^{\ell_S(\sigma)} = \sum_{\sigma\in B^S_{n,k}} q^{rmaj_{S_n}(\sigma)};$$

*and*

(2)
$$\sum_{\sigma\in B^A_{n+1,k}} q^{\ell_A(\sigma)} = \sum_{\sigma\in B^A_{n,k}} q^{rmaj_{A_{n+1}}(\sigma)}.$$

Note that part 1 is a refinement of MacMahon's equi-distribution theorem.

The proof of Theorem 6.1 follows from the lemmas below. Recall that the descent set - hence also the major-indices $maj_S$ and $rmaj_{S_n}$ - are defined for any sequence of integers, not necessarily distinct. Here $n$ denotes the number of letters in the sequence.

**Lemma 6.3** *Let* $x_1,\ldots,x_n$ *and* $y$ *be integers, not necessarily distinct, such that* $x_i < y$ *for* $1 \le i \le n$. *Let* $u$ *be the* $n$*-tuple* $u = [x_1,\ldots,x_n]$, *and let*

$$v_i = [x_1,\ldots,x_{i-1},y,x_i,\ldots,x_n], \qquad 1 \le i \le n+1$$

*(thus* $v_1 = [y,x_1,\ldots,x_n]$ *and* $v_{n+1} = [x_1,\ldots,x_n,y]$*). Then*

*1.*

$$\sum_{i=1}^{n+1} q^{maj_S(v_i)} = q^{maj_S(u)}(1+q+\ldots+q^n) \tag{14}$$

*and*

$$\sum_{i=1}^{n} q^{maj_S(v_i)} = q^{maj_S(u)}(q+q^2+\ldots+q^n). \tag{15}$$

*2.*

$$\sum_{i=1}^{n+1} q^{rmaj_{S_{n+1}}(v_i)} = q^{rmaj_{S_n}(u)}(1+q+\ldots+q^n) \tag{16}$$

*and*

$$\sum_{i=2}^{n+1} q^{rmaj_{S_{n+1}}(v_i)} = q^{rmaj_{S_n}(u)}(1+q+\ldots+q^{n-1}). \tag{17}$$

Part 1 of Lemma 6.3 is well known. The proof of part 2 is similar. For the sake of completeness the proof is included.

*Proof.* Denote $u' = [x_1, \ldots, x_{n-1}]$ and $u'' = [x_2, \ldots, x_n]$. Similarly, denote $v'_i = [x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}]$, $\quad 1 \le i \le n$, and
$v''_i = [x_2, \ldots, x_{i-1}, y, x_i, \ldots, x_n]$, $\quad 2 \le i \le n+1$
(thus $v'_n = [x_1, \ldots, x_{n-1}, y]$ and $v''_2 = [y, x_2, \ldots, x_n]$).

1. We argue by induction on $n$, proving (14) and (15) first.

   (a) Assume $x_{n-1} \le x_n$, then $maj_S(v_{n+1}) = maj_S(u) = maj_S(u') = maj_S(v'_n)$, $maj_S(v_n) = maj_S(u) + n$ and $maj_S(v_i) = maj_S(v'_i)$ for $1 \le i \le n-1$. It follows that

$$\sum_{i=1}^{n+1} q^{maj_S(v_i)} = q^{maj_S(v_{n+1})} + q^{maj_S(v_n)} + \sum_{i=1}^{n-1} q^{maj_S(v_i)} =$$

$$q^{maj_S(u)} + q^n q^{maj_S(u)} + \sum_{i=1}^{n-1} q^{maj_S(v'_i)} =$$

$$q^n q^{maj_S(u)} + \sum_{i=1}^{n} q^{maj_S(v'_i)} = \quad \text{(by induction)}$$

$$q^{maj_S(u)}(1+q+\ldots+q^{n-1}) + q^n q^{maj_S(u)}.$$

(b) Assume $x_{n-1} > x_n$, then $maj_S(v_{n+1}) = maj_S(u) = maj_S(u') + n - 1$ and $maj_S(v_i) = maj_S(v'_i) + n$ for $1 \leq i \leq n$. Thus

$$\sum_{i=1}^{n+1} q^{maj_S(v_i)} = q^{maj_S(u)} + \sum_{i=1}^{n} q^{n+maj_S(v'_i)} =$$

$$q^{maj_S(u)} + q^n \sum_{i=1}^{n} q^{maj_S(v'_i)} = \quad \text{(by induction)}$$

$$q^{maj_S(u)} + q^n q^{maj_S(u')}(1 + q + \ldots + q^{n-1}) =$$

$$q^{maj_S(u)} + q^{maj_S(u)+1}(1 + q + \ldots + q^{n-1}) =$$

$$q^{maj_S(u)}(1 + q + \ldots + q^n).$$

Together, (a) and (b) prove Equation (14). Note that Equation (15) follows from Equation (14) since, in both cases above, $maj_S(v_{n+1}) = maj_S(u)$.

2. We prove now (16) and (17).

(a) Assume $x_1 \leq x_2$. First, note that $1 \in Des_S(v_1)$ and it contributes $n + 1 - 1 = n$ to $rmaj_{S_{n+1}}(v_1)$. Let $2 \leq k \leq n - 1$, then $k \in Des_S(u)$ if and only if $k + 1 \in Des_S(v_1)$; such $k$ contributes $n - k = (n+1) - (k+1)$ to both $rmaj_{S_{n+1}}(v_1)$ and to $rmaj_{S_n}(u)$. It follows that $rmaj_{S_{n+1}}(v_1) = rmaj_{S_n}(u) + n$.
By similar arguments $rmaj_{S_{n+1}}(v_i) = rmaj_{S_n}(v''_i)$ for $2 \leq i \leq n + 1$ and also $rmaj_{S_n}(u) = rmaj_{S_{n-1}}(u'')$. Thus

$$\sum_{i=1}^{n+1} q^{rmaj_{S_{n+1}}}(v_i) = q^{rmaj_{S_n}(u)+n} + \sum_{i=2}^{n+1} q^{rmaj_{S_n}(v''_i)} \quad \text{(by induction)}$$

$$= q^{rmaj_{S_n}(u)+n} + q^{rmaj_{S_{n-1}}(u'')}(1 + q + \ldots + q^{n-1})$$

and the proof follows.

(b) Assume $x_1 > x_2$. Here (again) $rmaj_{S_{n+1}}(v_1) = rmaj_{S_n}(u) + n$ while $rmaj_{S_{n+1}}(v_2) = rmaj_{S_n}(v''_2) = rmaj_{S_n}(u)$. By similar arguments as above, $rmaj_{S_{n+1}}(v_i) = rmaj_{S_n}(v''_i) + n$ for $3 \leq i \leq n + 1$. Also $rmaj_{S_n}(u) = rmaj_{S_{n-1}}(u'') + n - 1$. Thus

$$\sum_{i=1}^{n+1} q^{rmaj_{S_{n+1}}(v_i)} = q^{rmaj_{S_n}(u)+n} + q^{rmaj_{S_n}(u)} + q^n \sum_{i=3}^{n+1} q^{rmaj_{S_n}(v''_i)} =$$

$$q^{rmaj_{S_n}(u)} + q^n \sum_{j=2}^{n+1} q^{rmaj_{S_n}(v_j'')} = \text{(by induction)}$$
$$= q^{rmaj_{S_n}(u)} + q^{rmaj_{S_{n-1}}(u'')+n}(1+q+\ldots+q^{n-1}) =$$
$$= q^{rmaj_{S_n}(u)} + q^{rmaj_{S_n}(u)+1}(1+q+\ldots+q^{n-1}),$$

which implies the proof. Together, (a) and (b) prove Equation (16). Now Equation (17) follows from Equation (16) since, in both cases (a) and (b) above, $rmaj_{S_{n+1}}(v_1) = rmaj_{S_n}(u) + n$.

□

**Lemma 6.4** *Recall that* $R_n^S = \{1, s_n, \ldots, s_n s_{n-1} \cdots s_1\} \subseteq S_{n+1}$ *and let* $w \in S_n$ *(so* $w \in S_{n+1}$*, where* $w(n+1) = n+1$*). Then*

$$\sum_{\tau \in R_n^S} q^{maj_S(w\tau)} = q^{maj_S(w)}(1+q+\ldots+q^n),$$

*and*

$$\sum_{\tau \in R_n^S} q^{rmaj_{S_{n+1}}(w\tau)} = q^{rmaj_{S_n}(w)}(1+q+\ldots+q^n).$$

*Proof.* Write $w \in S_n$ as $w = [w(1), \ldots, w(n)]$ $(= u$ in 6.3). Similarly write $w \in S_n \subseteq S_{n+1}$ as $w = [w(1), \ldots, w(n), n+1]$ $(= v_{n+1}$, in 6.3, where $y = n+1)$. Thus

$$ws_n = [w(1), \ldots, n+1, w(n)] \quad (= v_n),$$

$$ws_n s_{n-1} = [w(1), \ldots, n+1, w(n-1), w(n)] \quad (= v_{n-1}),$$

etc, and the proof follows by the previous lemma. □

**Remark 6.5** *Let* $\tilde{R}_n^S = R_n^S - \{s_n s_{n-1} \cdots s_1\} \subseteq S_{n+1}$*, and let* $\sigma \in S_n$*. It follows from Equation (17) that*

$$\sum_{\tau \in \tilde{R}_n^S} q^{rmaj_{S_{n+1}}(\sigma\tau)} = q^{rmaj_{S_n}(\sigma)}(1+q+\ldots+q^{n-1}).$$

**Lemma 6.6** *For every $\sigma \in S_n$*

$$\sum_{\tau \in R_n^S} q^{rmaj_{S_{n+1}}(\sigma\tau)} t^{del_S(\sigma\tau)} = q^{rmaj_{S_n}(\sigma)} t^{del_S(\sigma)} (1 + q + \ldots + q^{n-1} + tq^n).$$

*Proof.* By Lemma 6.4

$$\{rmaj_{S_{n+1}}(\sigma\tau) \mid \tau \in R_n^S\} = \{rmaj_{S_n}(\sigma) + i \mid 0 \leq i \leq n\}.$$

Let $\eta = s_n s_{n-1} \cdots s_1$ and note that $rmaj_{S_n}(\sigma) + n = rmaj_{S_{n+1}}(\sigma\eta)$ (this is the statement "$rmaj_{S_{n+1}}(v_1) = rmaj_{S_n}(u) + n$" in the proof of Lemma 6.3).

Let $\tau \in R_n^S$.

If $\tau \neq \eta$ then $del_S(\sigma\tau) = del_S(\sigma)$ since both $\sigma$ and $\sigma\tau$ have the same number of occurrences of $s_1$. By a similar reason $del_S(\sigma\eta) = del_S(\sigma) + 1$. Thus

$$\begin{aligned}
&\{rmaj_{S_{n+1}}(\sigma\tau) del_S(\sigma\tau) \mid \tau \in R_n^S\} = \\
&\quad \{rmaj_{S_{n+1}}(\sigma\tau) del_S(\sigma\tau) \mid \tau \in R_n^S,\ \tau \neq \eta\} \cup \{rmaj_{S_{n+1}}(\sigma\eta) del_S(\sigma\eta)\} = \\
&\quad \{(rmaj_{S_n}(\sigma) + i) del_S(\sigma) \mid 0 \leq i \leq n-1\} \cup \{(rmaj_{S_n}(\sigma) + n)(del_S(\sigma) + 1)\}
\end{aligned}$$

(disjoint unions with no repetitions in the sets) which translates to

$$\sum_{\tau \in R_n^S} q^{rmaj_{S_{n+1}}(\sigma\tau)} t^{del_S(\sigma\tau)} = q^{rmaj_{S_n}(\sigma)} t^{del_S(\sigma)} (1 + q + \ldots + q^{n-1} + tq^n).$$

□

**Proposition 6.7** *For all $n$*

$$\sum_{\sigma \in S_n} q^{rmaj_{S_n}(\sigma)} t^{del_S(\sigma)} = (1 + tq)(1 + q + tq^2) \cdots (1 + q + \ldots + q^{n-2} + tq^{n-1}).$$

*Proof.* Follows from Lemma 6.6 by induction on $n$, since

$$S_{n+1} = \cup_{\tau \in R_n^S} S_n \tau.$$

□

**The proof of Theorem 6.1.**
Part (1) clearly follows by comparing part 1 of Proposition 5.7 with Proposition 6.7.
Part (2) follows from part (1) by Proposition 5.6. □

## 7 Additional Properties of the Delent Number

We show first that $del_S(w)$ is the number of left-to-right minima of $w$.

**Definition 7.1** *Let $w \in S_n$, then $j$ is a l.t.r.min (left-to-right minimum) of $w$ if $w(i) > w(j)$ for all $1 \leq i < j$. Write $w = [b_1, \ldots, b_n]$, then $i = 1$ is a l.t.r.min and so is $i$ such that $b_i = 1$. We slightly modify the definition, so that the identity $e = [1, \ldots, n]$ has no l.t.r.min. This can be done in one of the following two ways.*
*Either:*
*1. Do not count $i = 1$ as a l.t.r.min (which is Definition 7.1.1 of l.t.r.min), or:*
*2. Do not count $i$ such that $b_i = 1$ as a l.t.r.min (which is Definition 7.1.2 of l.t.r.min).*
*3. Define $Del_S(w)$ as the l.t.r.min according to Definition 7.1.1:*

$$Del_S(w) := \{1 < i \leq n \mid \forall j < i \;\; w(i) < w(j)\}.$$

For example let $w = [3, 2, 7, 8, 4, 6, 1, 5]$, then $\{2, 7\}$ are the l.t.r.min according to 7.1.1, and $\{1, 2\}$ according to 7.1.2.

With either definition we have

**Proposition 7.2** *Let $w \in S_n$, then $del_S(w)$ equals the number of l.t.r.min of $w^{-1}$ according to either Definition 7.1.1 or 7.1.2. Since by Lemma 3.6 $del_S(w) = del_S(w^{-1})$, this also equals the number of l.t.r.min of $w$. In particular,*

$$\mid Del_S(w) \mid = del_S(w) = del_S(w^{-1}).$$

*Proof.* By induction on $n \geq 2$. First, $S_2 = \{1, s_1\}$ and $s_1 = [2, 1]$ has one l.t.r.min - according to either 7.1.1 or 7.1.2. Proceed now with the inductive step, which is essentially the same for both definitions. Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of $w$, let $\sigma = w_1 \cdots w_{n-2}$ (so $\sigma \in S_{n-1}$) and assume true for $\sigma$. Write $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$. If $w_{n-1} = 1$, the proof is given by the induction hypothesis. Otherwise $w_{n-1}^{-1} = s_k s_{k+1} \cdots s_{n-1}$ for some $1 \leq k \leq n-1$. Denoting $s_{[k,n-1]} = s_k s_{k+1} \cdots s_{n-1}$ we see that $w^{-1} = s_{[k,n-1]}\sigma^{-1}$. Comparing $\sigma^{-1}$ with $w^{-1} = s_{[k,n-1]}\sigma^{-1}$, we see that

1. the (position with) $n$ in $\sigma^{-1}$ is replaced in $w^{-1}$ by $k$;
2. each $j$ in $\sigma^{-1}$, $k \leq j \leq n-1$, is replaced by $j+1$ in $w^{-1}$;

3. each $j$, $1 \leq j \leq k-1$ is unchanged.

Thus $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$, $w^{-1} = [c_1, \ldots, c_{n-1}, k]$, and the two tuples $(b_1, \ldots, b_{n-1})$ and $(c_1, \ldots, c_{n-1})$ are order-isomorphic. This implies that if $k > 1$ then $\sigma^{-1}$ and $w^{-1}$ have the same left-to-right minima. Let $k = 1$ and adopt Definition 7.1.1 first, then $w^{-1}$ has $i = n$ as an additional left-to-right minimum, which completes the proof in that case. In the case of Definition 7.1.2, compare $k = 2$ with $k = 1$ to deduce that $i$ such that $c_i = 2$ is an additional l.t.r.min, and the proof follows. $\square$

**Remark 7.3** *The above proof implies a bit more: Note that the above case $k = 1$ is equivalent to both $n \in Del_S(w^{-1})$ and to $\epsilon_{S,n-1}(w) = 1$, where $\epsilon_{S,i}(w)$ are given by Definition 5.9. By induction on $n$, the above proof implies that $Del_S(w^{-1}) = \{i+1 \mid \epsilon_{S,i}(w) = 1\}$. Let now $D \subseteq [n-1]$ and let $\pi \in S_n$. The condition $D = Del_S(\pi^{-1})$ implies that $D = \{i+1 \mid \epsilon_{S,i}(\pi) = 1\}$; this uniquely determines $\bar{\epsilon}_S(\pi)$, and hence determines a unique value $t^{\epsilon_D} := t^{\bar{\epsilon}_S(\pi)}$: if $D \neq H$ then $t^{\epsilon_D} \neq t^{\epsilon_H}$. We shall apply this observation in the proof of Theorem 9.1.*

Each of the two definitions of l.t.r.min can be extended as follows.

**Definition 7.4** *Let $w = [b_1, \ldots, b_n] \in S_n$. Then $1 \leq i \leq n$ is an a.l.t.r.min (almost-left-to-right minimum) if, first of all, there is at most one $b_j$ smaller than $b_i$ and left of $b_i$: card$\{1 \leq j \leq i \mid b_j < b_i\} \leq 1$.*
*The second condition is one of the following:*
*Either*
*1. Do not count $i = 1$ and $i = 2$ as a.l.t.r.min (which is Definition 7.4.1 of a.l.t.r.min),*
*or:*
*2. Do not count $i$ such that $b_i = 1, 2$ as an a.l.t.r.min (which is Definition 7.4.2 of a.l.t.r.min).*
*3. For $w \in A_{n+1}$ define $Del_A(w)$ to be the set of a.l.t.r.min of $w$ according to Definition 7.4.1.*

**Remark 7.5**

1. *Without the above restrictions 1 and 2 in Definition 7.4, $i$ such that $b_i \in \{1, 2\}$ is an a.l.t.r.min; similarly, if $i \in \{1, 2\}$ then $i$ is an a.l.t.r.min.*

2. *If $b_i = 1$ and $b_j = 2$ are interchanged in $w = [b_1, \ldots, b_n]$, this does not change the set of a.l.t.r.min indices. Also, if $b_1$ and $b_2$ are interchanged this would not change the set of a.l.t.r.min indices. Thus with either*

*definition 7.4.1 or 7.4.2, $s_1 w$ and $w s_1$ have the same a.l.t.r.min as $w$ itself.*

**Proposition 7.6** *Let $w \in S_n$, then the number of occurrences of $s_2$ in (the canonical presentation of) $w$ equals the number of a.l.t.r.min of $w^{-1}$ according to either Definition 7.4.1 or 7.4.2. Lemma 3.6 implies that this is also the number of a.l.t.r.min of $w$.*

*Proof.* By induction on $n$. This is easily verified for $n = 2$, and we proceed with the inductive step.
Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of $w$, and denote $\sigma = w_1 \cdots w_{n-2}$, so that $w^{-1} = w_{n-1}^{-1}\sigma^{-1}$. If $w_{n-1} = 1$ we are done by induction. Otherwise, by the S-procedure, $w_{n-1} = s_{n-1} \cdots s_k x$ where $k \geq 2$ and $x \in \{1, s_1\}$.
Write $w^{-1} = x s_k \cdots s_{n-1}\sigma^{-1} = x s_{[k,n-1]}\sigma^{-1}$. By Remark 7.5, $s_{[k,n-1]}\sigma^{-1}$ and $x s_{[k,n-1]}\sigma^{-1}$ have the same number of a.l.t.r.min. Therefore it suffices to show:

1. If $k \geq 3$ then $\sigma^{-1}$ has equal number of a.l.t.r.min as $s_{[k,n-1]}\sigma^{-1}$.

2. If $k = 2$, $s_{[2,n-1]}\sigma^{-1}$ has one more a.l.t.r.min than $\sigma^{-1}$.

Let $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$, then $s_{[k,n-1]}\sigma^{-1} = [c_1, \ldots, c_{n-1}, k]$, and as in the proof of 7.2, $(b_1, \ldots, b_{n-1})$ and $(c_1, \ldots, c_{n-1})$ are order isomorphic . If $k \geq 3$, the last position (with $k$) is not an a.l.t.r.min, while if $k = 2$, it is an additional a.l.t.r.min, and this implies the proof in the case of 7.4.1. In case of 7.4.2, compare $k = 3$ with $k = 2$: a 2 is changed into a 3, which is an additional a.l.t.r.min. □

By essentially the same argument we have

**Proposition 7.7** *Let $v \in A_{n+1}$ then, with either Definition 7.4.1 or 7.4.2 of a.l.t.r.min, $del_A(v)$ equals the number of a.l.t.r.min of $v^{-1}$. In particular $\mid Del_A(v) \mid = del_A(v) = del_A(v^{-1})$.*

*Proof.* Again, by induction on $n$. This is easily verified for $n + 1 = 3$, so proceed with the inductive step.
Let $v = v_1 \cdots v_{n-1}$ be the A- canonical presentation of $v$, and denote $\sigma = v_1 \cdots v_{n-2}$, so that $v^{-1} = v_{n-1}^{-1}\sigma^{-1}$. If $v_{n-1} = 1$ we are done by induction. Otherwise, by the A-procedure, $v_{n-1} = x s_n \cdots s_k y$ where $k \geq 2$ and $x, y \in$

$\{1, s_1\}$; moreover, $k = 2$ if and only if either $a_1$ or $a_1^{-1}$ occurs in $v_{n-1}$. Write $v^{-1} = ys_k \cdots s_n x\sigma^{-1} = ys_{[k,n]}x\sigma^{-1}$, and proceed as in the proof of 7.6, applying 7.5.2. □

**Remark 7.8** *Given $w \in S_n$, one can define a.a.l.t.r.min, a.a.a.l.t.r.min, etc, then prove the corresponding propositions, analogue of Proposition 7.6. For example, we have*

***Definition 7.9*** *Let $w = [b_1, \ldots, b_n] \in S_n$. Then $1 \le i \le n$ is an a.a.l.t.r.min (almost-almost-left-to-right minimum) if card$\{1 \le j \le i \mid b_j < b_i\} \le 2$ and*
*1. $i \ne 1, 2, 3$ (which is Definition 7.9.1 of a.a.l.t.r.min),*
*or*
*2. $b_i \ne 1, 2, 3$. (which is Definition 7.9.2 of a.a.l.t.r.min).*

*One can then prove that, with either definition of a.a.l.t.r.min, the number of a.a.l.t.r.min of $w \in S_n$ equals the number of occurrences of $s_3$ in $w$. Similarly for the occurrences of the other $s_i$'s.*

Similar to Proposition 5.8, we define $w_S(n, \ell, k)$ to be the number of $S$ canonical words in $S_n$ with $\ell$ occurrences of $s_k$ (define $w_A(n+1, \ell, k)$ similarly), and we have

**Proposition 7.10** *Let $k \le n-1$, then*

$$\sum_{\ell=0}^{n-k} w_S(n, \ell, k) t^\ell = k!(kt+1)(kt+2)\cdots(kt+n-k),$$

*hence $w_S(n, \ell, k) = k!k^\ell c(n-k+1, \ell+1)$, and similarly for $w_A(n+1, \ell, k)$.*

*Proof* - is omitted.

# 8 Lemmas on Shuffles

In this section we prove lemmas, which will be used in the next section to prove the main theorem.

## 8.1 Equi-distribution on Shuffles

The following result follows from Theorem 2.6.

**Proposition 8.1** *Let $i \in [n-1]$, and let $\pi \in S_n$ with $supp(\pi) \subseteq [i]$. Then*

$$\sum_{Des_S(r^{-1}) \subseteq \{i\}} q^{rmaj_{S_n}(\pi r) - rmaj_{S_i}(\pi)} = \sum_{Des(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi r) - \ell_S(\pi)} = \begin{bmatrix} n \\ i \end{bmatrix}_q .$$

**Proof.** Let $\rho_n := (1,n)(2,n-1)\cdots \in S_n$ and $\rho_i := (1,i)(2,i-1)\cdots \in S_i$. By (4)

$$\sum_{Des_S(r^{-1}) \subseteq \{i\}} q^{rmaj_{S_n}(\pi r) - rmaj_{S_i}(\pi)} = \sum_{Des_S(r^{-1}) \subseteq \{i\}} q^{maj_S(\rho_n \pi r \rho_n) - maj_S(\rho_i \pi \rho_i)} =$$

$$\sum_{Des_S(r^{-1}) \subseteq \{i\}} q^{maj_S(\rho_n \pi \rho_n \rho_n r \rho_n) - maj_S(\rho_i \pi \rho_i)} = \sum_{Des_S(\hat{r}^{-1}) \subseteq \{n-i\}} q^{maj_S(\rho_n \pi \rho_n \hat{r}) - maj_S(\rho_i \pi \rho_i)}.$$

The last equality follows from (6).
Note that $supp(\rho_n \pi \rho_n) \subseteq [n-i+1, n]$ and verify that $\nu_{n-i}^{-1} \rho_n \pi \rho_n \nu_{n-i} = \rho_i \pi \rho_i$, where $\nu_{n-i} := (1, n-i+1)(2, n-i+2)\cdots$. Indeed, let $j \le i$, then $\nu_{n-i}(j) = j+n-i$, hence $\rho_n \nu_{n-i}(j) = \rho_n(j+n-i) = n-(j+n-i)+1 = i-j+1 = \rho_i(j)$. Similarly, if $k \le i$, also $\nu_{n-i}^{-1} \rho_n(k) = \rho_i(k)$. This implies the above equality. Now, obviously $supp(1) \subseteq [n-i]$ and $maj_S(1) = 0$. Thus by Garsia-Gessel's Theorem (Theorem 2.6) (taking $\pi_1 = 1$ and $\pi_2 = \rho_n \pi \rho_n$) the right-hand-side is equal to

$$\sum_{Des_S(\hat{r}^{-1}) \subseteq \{n-i\}} q^{maj_S(1 \cdot \rho_n \pi \rho_n \cdot \hat{r}) - maj_S(1) - maj_S(\nu_{n-i}^{-1} \rho_n \pi \rho_n \nu_{n-i})} = \begin{bmatrix} n \\ i \end{bmatrix}_q .$$

The equality

$$\sum_{Des(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi r) - \ell_S(\pi)} = \sum_{Des(r^{-1}) \subseteq \{i\}} q^{\ell_S(\pi \cdot 1 \cdot r) - \ell_S(\pi) - \ell_S(1)} = \begin{bmatrix} n \\ i \end{bmatrix}_q$$

is an immediate consequence of Fact 2.5, combined with (3).

□

**Note 8.2** *Let $r$ be an $\{i\}$-shuffle and let $supp(\pi) \subseteq [i]$ as above. If $r(1) \neq 1$, necessarily $r(1) = i+1$, hence also $\pi r(1) = i+1$. It follows that*

$$\pi r(1) \in \{\pi(1), i+1\}.$$

The next lemma requires some preparations.

Fix $1 \le i \le n-1$ and define $g_i : S_n \to S_{n-1}$ as follows: Let $\sigma = [a_1, \ldots, a_n] \in S_n$, then $g_i(\sigma) = [a'_1, \ldots, a'_{n-1}]$ is defined as follows: delete $a_j = i+1$, leave $a'_k = a_k$ unchanged if $a_k \le i$, and change $a'_t = a_t - 1$ if $a_t \ge i+2$. Denote $g_i(\sigma) = \sigma'$. For example, let $\sigma = [5,2,3,6,1,4]$ and $i = 2$, then $g_2(\sigma) = \sigma' = [4,2,5,1,3]$. Let $supp(\pi) \subseteq [i]$, then $g_i(\pi) = \pi$: $\pi' = \pi$. Moreover, since $\pi$ only permutes $1, \ldots, i$, the following basic property of $g_i$ is rather obvious, since $supp(\pi) \subseteq [i]$:

**Fact 8.3** *1. Let $\sigma \in S_n$, then $\pi(g_i\sigma) = g_i(\pi\sigma)$, namely, $(\pi\sigma)' = \pi'\sigma' = \pi\sigma'$.*

*2. $g_i$ is a bijection between the $\{i\}$-shuffles $r \in S_n$ satisfying $r(1) = i+1$, and all the $\{i\}$-shuffles $r' \in S_{n-1}$:*

$$g_i : \{r \in S_n \mid Des_S(r^{-1}) \subseteq \{i\},\ r(1) = i+1\} \to \{r' \in S_{n-1} \mid Des_S(r^{-1}) \subseteq \{i\}\}$$

*is a bijection.*

We need

**Lemma 8.4** *Let $1 \le i \le n-2$, $supp(\pi) \subseteq [i]$ and $r(1) = i+1$. Also let $g_i(\pi) = \pi'$ and $g_i(r) = r'$.*

1. *If $r(2) = i+2$ then $rmaj_{S_n}(\pi r) = rmaj_{S_{n-1}}(\pi' r')$.*
2. *If $r(2) = 1$ then $rmaj_{S_n}(\pi r) = n - 1 + rmaj_{S_{n-1}}(\pi' r')$.*

*Proof.* By Note 8.2, $\pi r = [i+1, a_2, \ldots, a_n]$ then, applying $g_i$, we have $\pi' r' = [a'_2, \ldots, a'_n]$, and it is easy to check that for all $2 \le k \le n-1$, $a_k > a_{k+1}$ if and only if $a'_k > a'_{k+1}$. Thus, for $2 \le k \le n-1$, $k \in Des(\pi r)$ if and only if $k - 1 \in Des(\pi' r')$; note also that such $k$ contributes $n - k = (n-1) - (k-1)$ to both $rmaj_{S_n}(\pi r)$ and to $rmaj_{S_{n-1}}(\pi' r')$.

1. If $r(2) = i+2$ then $a_2 = \pi r(2) = i+2$, hence $1 \notin Des(\pi r)$, and the descents of $\pi r$ occur only for (some) $2 \le k \le n-1$, and the above argument implies the proof.
2. If $r(2) = 1$ then $a_2 = \pi r(2) = \pi(1) < i+1$, hence 1 is a descent of $\pi r$, contributing $n-1$ to $rmaj_{S_n}(\pi r)$, and again, the above argument completes the proof. □

**Lemma 8.5** *With the notations of Proposition 8.1*

$$(1) \qquad \sum_{Des_S(r^{-1})\subseteq\{i\} \text{ and } \pi r(1)=i+1} q^{rmaj_{S_n}(\pi r)-rmaj_{S_i}(\pi)} = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q .$$

*and*

$$(2) \qquad \sum_{Des_S(r^{-1})\subseteq\{i\} \text{ and } \pi r(1)=\pi(1)} q^{rmaj_{S_n}(\pi r)-rmaj_{S_i}(\pi)} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q .$$

**Proof.** By induction on $n-i$. For $n-i=1$, the $\{n-1\}$-shuffles are $[1,\ldots,j-1,n,j,\ldots,n-1]=[1,\ldots,n]s_{n-1}s_{n-2}\cdots s_j,\ 1\le j\le n-1$. Thus the summation in (2) is over $r\in R^S_{n-1}-\{s_{n-1}s_{n-2}\cdots s_1\}$ and Equation (2) follows from Remark 6.5 (with $n-1$ replacing $n$). Now,

$$sum(1)+sum(2) = \sum_{Des_S(r^{-1})\subseteq\{n-1\}} q^{rmaj_{S_n}(\pi r)-rmaj_{S_{n-1}}(\pi)},$$

Hence, by Proposition 8.1

$$sum(1)+sum(2) = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q ,$$

so

$$sum(1) = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q - \begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q = q^{n-1},$$

which verifies (1) in that case.

Let now $n-i\ge 2$ and assume the lemma holds for $n-1-i$.

**(1)** Since $Des(r^{-1})\subseteq\{i\}$ and $r(1)=i+1$, either $r(2)=i+2$ (then $\pi r(2)=i+2$), or $r(2)=1$ (then $\pi r(2)=\pi(1)$). Thus, the sum in (1) equals $sum[r(2)=i{+}2]+sum[r(2)=1]$. Apply $g_i$ to the permutations in these sums, and apply Lemma 8.4.1 and Fact 8.3; then, by induction on $n$,

$$\begin{aligned} sum[r(2)=i{+}2] &= \sum_{Des_S(r'^{-1})\subseteq\{i\} \text{ and } \pi'r'(1)=i+1} q^{rmaj_{S_{n-1}}(\pi'r')-rmaj_{S_i}(\pi')} \\ &= q^i \begin{bmatrix} n-2 \\ i \end{bmatrix}_q . \end{aligned}$$

Similarly, by Lemma 8.4.2 and Fact 8.3,

$$sum[r(2)=1] = \sum_{Des_S(r'^{-1})\subseteq\{i\} \text{ and } \pi' r'(1)=\pi'(1)} q^{n-1+rmaj_{S_{n-1}}(\pi' r')-rmaj_{S_i}(\pi')}$$

$$= q^{n-1}\begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q .$$

Adding the last two sums, we conclude:

$$\sum_{Des_S(r^{-1})\subseteq\{i\} \text{ and } \pi r(1)=i+1} q^{rmaj_{S_{n-1}}(\pi r)-rmaj_{S_i}(\pi)} = q^i \begin{bmatrix} n-2 \\ i \end{bmatrix}_q + q^{n-1}\begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q =$$

$$= q^i \left( \begin{bmatrix} n-2 \\ i \end{bmatrix}_q + q^{n-1-i}\begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_q \right) = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q .$$

**(2)** is an immediate consequence of Proposition 8.1 and part (1), since

$$\begin{bmatrix} n \\ i \end{bmatrix}_q - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q .$$

$\square$

We have an analogous lemma for length.

**Lemma 8.6** *With the notation of Proposition 8.1*

(1) $$\sum_{Des_S(r^{-1})\subseteq\{i\} \text{ and } \pi r(1)=i+1} q^{\ell_S(\pi r)-\ell_S(\pi)} = q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q .$$

*and*

(2) $$\sum_{Des_S(r^{-1})\subseteq\{i\} \text{ and } \pi r(1)=\pi(1)} q^{\ell_S(\pi r)-\ell_S(\pi)} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q .$$

**Proof**. The case $n-i=0$ is obvious (the sum in (1) is empty while in (2), $r=1$), so assume $i \leq n-1$. Recall that in general, $\ell_S(\sigma)$ equals the number $inv_S(\sigma)$ of inversions of $\sigma$.

We prove (1) first, so let $\pi r(1) = i+1$. As in Lemma 8.4, write

$$\pi r = [i+1, a_2, \ldots, a_n] \quad \text{and} \quad g_i(\pi r) = \pi' r' = [a'_2, \ldots, a'_n],$$

and compare their inversions. Clearly, $i+1$ contributes $i$ inversions to $inv_S(\pi r)$. Also, as in the proof of Lemma 8.4, there is a bijection between the inversions among $\{a_2, \ldots, a_n\}$ and those among $\{a'_2, \ldots, a'_n\}$. Thus $inv_S(\pi r) = i + inv_S(\pi' r')$. Also, since $supp(\pi) \subseteq [i]$ , $inv_S(\pi) = inv_S(\pi')$. Induction, Fact 8.3 and Proposition 8.1 imply the proof of (1). Now, by Proposition 8.1, (1) implies the proof of (2). $\square$

## 8.2 Canonical Presentation of Shuffles

**Observation 8.7** *Let $1 \le i < n$. Every $\{i\}$-shuffle has a unique canonical presentation of the form $w_i w_{i+1} \cdots w_{n-1}$, where $\ell(w_j) \ge \ell(w_{j+1})$ for all $j \ge i$.*

*Proof.* Apply the 'S-Procedure' that follows Theorem 3.1. Note that after pulling $n, n-1, \ldots, i+1$ to the right, an $\{i\}$-shuffle is transformed into the identity permutation. $\square$

Let $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1})$, then denote $t^{\bar{\epsilon}} = t_1^{\epsilon_1} \cdots t_{n-1}^{\epsilon_{n-1}}$.

**Corollary 8.8** *Recall Definition 5.9. For an $\{i\}$-shuffle $w$,*

$$del_S(w) = \begin{cases} 1, & \text{if } w(1) = i+1; \\ 0, & \text{otherwise,} \end{cases}$$

*and therefore*

$$t^{\bar{\epsilon}_S(w)} = t_i^{del_S(w)} = \begin{cases} t_i, & \text{if } w(1) = i+1; \\ 1, & \text{otherwise} \end{cases}$$

*Proof.* Write $w = w_i w_{i+1} \cdots w_{n-1}$ (canonical presentation) with $\ell_S(w_i) \ge \cdots \ge \ell_S(w_{n-1})$, then $\epsilon_{S,j}(w) = 0$ for $j > i$. Thus $del_S(w)$ is either 1 or 0, and is 1 exactly when $w_i = s_i \cdots s_1$, in which case $w(1) = i+1$. $\square$

**Remark 8.9** *Let $r, \pi \in S_n$, $r$ an $\{i\}$-shuffle and $supp(\pi) \subseteq [i]$. Then the corresponding canonical presentations are: $\pi = w_1 \cdots w_i$, $\quad r = w_{i+1} \cdots w_{n-1}$, hence also $\pi r = w_1 \cdots w_{n-1}$ is canonical presentation. In particular, $\bar{\epsilon}_S(\pi r) = \bar{\epsilon}_S(\pi) + \bar{\epsilon}_S(r)$.*

We generalize: Let $B = \{i_1, i_2\}$ and let $w \in S_n$ be a $B$-shuffle. Then $w$ shuffles the three subsets $\{1, \ldots, i_1\}$, $\{i_1+1, \ldots, i_2\}$ and $\{i_2+1, \ldots, n\}$. Clearly $w$ has a unique presentation as a product $w = \tau_1\tau_2$ where $\tau_2 \in S_n$ shuffles $\{1, \ldots, i_2\}$ with $\{i_2+1, \ldots, n\}$, and $\tau_1 \in S_{i_2}$ shuffles $\{1, \ldots, i_1\}$ with $\{i_1+1, \ldots, i_2\}$. By Observation 8.7, $\tau_1 = w_{i_1}w_{i_1+1}\cdots w_{i_2-1}$ and $\tau_2 = w_{i_2}w_{i_2+1}\cdots w_{n-1}$, where each $w_j \in R_j^S$. Thus

$$w = w_{i_1}\cdots w_{i_2-1}w_{i_2}\cdots w_{n-1}$$

is the $S$ canonical presentation of $w$,

$$del_S(w) = del_S(\tau_1) + del_S(\tau_2) \qquad \text{and} \qquad t^{\bar{\epsilon}_S(w)} = t_{i_1}^{del_S(\tau_1)} t_{i_2}^{del_S(\tau_2)}.$$

This easily generalizes to an arbitrary $B = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n-1\}$, which proves the following proposition.

**Proposition 8.10** *Let $B = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n-1\}$ and let $i_{k+1} := n$. Every $B$-shuffle $\pi \in S_n$ has a unique presentation*

$$\pi = \tau_1 \cdots \tau_k$$

*where $\tau_j$ is an $\{i_j\}$-shuffle in $S_{i_{j+1}}$ (for $1 \leq j \leq k$). Moreover,*

$$del_S(\pi) = \sum_{j=1}^{k} del_S(\tau_j) \qquad \textit{and} \qquad t^{\bar{\epsilon}_S(\pi)} = t_{i_1}^{del_S(\tau_1)} \cdots t_{i_k}^{del_S(\tau_k)}.$$

## 9 The Main Theorem

Recall the definitions of the $A$-descent set $Des_A$ and the $A$-descent number $des_A$ (Definition 1.5). Let $B \subseteq [n-1]$ and $\pi \in S_n$. Recall from Fact 2.4 that $Des_S(\pi^{-1}) \subseteq B$ if and only if $\pi$ is a $B$-shuffle.

The following is our main theorem, which we now prove.

**Theorem 9.1** *For every subsets $D_1 \subseteq [n-1]$ and $D_2 \subseteq [n-1]$*

(1)
$$\sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) \subseteq D_1, \ Del_S(\pi^{-1}) \subseteq D_2\}} q^{rmaj_{S_n}(\pi)} =$$

$$\sum_{\{\pi \in S_n \mid \ Des_S(\pi^{-1}) \subseteq D_1, \ Del_S(\pi^{-1}) \subseteq D_2\}} q^{\ell_S(\pi)},$$

*and*

$$(2) \qquad \sum_{\{\sigma\in A_{n+1}|\ Des_A(\sigma^{-1})\subseteq D_1,\ Del_A(\sigma^{-1})\subseteq D_2\}} q^{rmaj_{A_{n+1}}(\sigma)} = \sum_{\{\sigma\in A_{n+1}|\ Des_A(\sigma^{-1})\subseteq D_1,\ Del_A(\sigma^{-1})\subseteq D_2\}} q^{\ell_A(\sigma)}.$$

An immediate consequence of Theorem 9.1 is

**Corollary 9.2**

$$(1) \qquad \sum_{\pi\in S_n} q_1^{rmaj_{S_n}(\pi)} q_2^{des_S(\pi^{-1})} q_3^{del_S(\pi^{-1})} = \sum_{\pi\in S_n} q_1^{\ell_S(\pi)} q_2^{des_S(\pi^{-1})} q_3^{del_S(\pi^{-1})}.$$

$$(2) \quad \sum_{\sigma\in A_n} q_1^{rmaj_{A_{n+1}}(\sigma)} q_2^{des_A(\sigma^{-1})} q_3^{del_A(\sigma^{-1})} = \sum_{\sigma\in A_n} q_1^{\ell_A(\sigma)} q_2^{des_A(\sigma^{-1})} q_3^{del_A(\sigma^{-1})}.$$

Note that in Corollary 9.2(1) both definitions 7.1.1 and 7.1.2 for calculating $del_S$ could be used. This follows from Proposition 7.2. Similarly, in Corollary 9.2(2) both definitions 7.4.1 and 7.4.2 for calculating $del_A$ could be used (by Proposition 7.7).

## 9.1 A Lemma

**Lemma 9.3** *Let $i \in [n]$, and let $\sigma$ be a permutation in $S_n$, such that $supp(\sigma) \subseteq [i]$. Then*

$$(1) \qquad \sum_{Des(r^{-1})\subseteq\{i\}} q^{\ell_S(\sigma r)} t^{\bar{\epsilon}_S(\sigma r)} = q^{\ell_S(\sigma)} t^{\bar{\epsilon}_S(\sigma)} \cdot \left( \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right)$$

*and*

(2)

$$\sum_{Des(r^{-1})\subseteq\{i\}} q^{rmaj_{S_n}(\sigma r)} t^{\bar{\epsilon}_S(\sigma r)} = q^{rmaj_{S_i}(\sigma)} t^{\bar{\epsilon}_S(\sigma)} \cdot \left( \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right).$$

**Proof.** By Definition 5.9 and Remark 8.9

$$t^{\bar{\epsilon}_S(\sigma r)} = t^{\bar{\epsilon}_S(\sigma)+\bar{\epsilon}_S(r)},$$

and by Corollary 8.8

$$t^{\bar{\epsilon}_S(r)} = \begin{cases} t_i, & \text{if } r(1) = i+1; \\ 1, & \text{otherwise} \end{cases}$$

Noting that $r(1) = i+1$ if and only if $\sigma r(1) = i+1$, and recalling that $\sigma r(1) \in \{\sigma(1), i+1\}$, we obtain

$$t^{\bar{\epsilon}_S(\sigma r)} = \begin{cases} t^{\bar{\epsilon}_S(\sigma)} t_i, & \text{if } \sigma r(1) = i+1; \\ t^{\bar{\epsilon}_S(\sigma)}, & \text{if } \sigma r(1) = \sigma(1) \end{cases}$$

Combining this with Lemmas 8.5 and 8.6 gives the desired result. For example, regarding length,

$$\sum_{Des(r^{-1})\subseteq\{i\}} q^{\ell_S(\sigma r)} t^{\bar{\epsilon}_S(\sigma r)} =$$

$$= \sum_{Des(r^{-1})\subseteq\{i\} \text{ and } \sigma r(1)=\sigma(1)} q^{\ell_S(\sigma r)} t^{\bar{\epsilon}_S(\sigma r)} + \sum_{Des(r^{-1})\subseteq\{i\} \text{ and } \sigma r(1)=i+1} q^{\ell_S(\sigma r)} t^{\bar{\epsilon}_S(\sigma r)} =$$

$$= q^{\ell_S(\sigma)} t^{\bar{\epsilon}_S(\sigma)} \cdot \left( \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q + t_i q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}_q \right).$$

This proves part (1). A similar argument proves (2).

□

## 9.2 Proof of Main Theorem

**Proof of Theorem 9.1(1).**

By the principle of inclusion and exclusion, we may replace $Del_S(\pi^{-1}) \subseteq D_2$ by $Del_S(\pi^{-1}) = D_2$ in both hand-sides of Theorem 9.1(1). By Remark 7.3, $\{\pi \in S_n \mid Del_S(\pi^{-1}) = D_2\}$ (i.e. the set $D_2$) determines the unique value $t^{\epsilon_{D_2}} := t^{\bar{\epsilon}_S(\pi)}$.

Hence, Theorem 9.1(1) is equivalent to the following statement :

For every subset $B \subseteq [n-1]$

$$\sum_{\{\pi\in S_n \mid \; Des_S(\pi^{-1})\subseteq B\}} q^{rmaj_{S_n}(\pi)} t^{\bar{\epsilon}_S(\pi)} =$$

$$\sum_{\{\pi\in S_n \mid \ Des_S(\pi^{-1})\subseteq B\}} q^{\ell_S(\pi)} t^{\bar{\epsilon}_S(\pi)},$$

This statement is proved by induction on the cardinality of $B$. If $|B|=1$ then $B=\{i\}$ for some $i\in[n-1]$ and Theorem 9.1(1) is given by Lemma 9.3 (with $\sigma=1$). Assume that the theorem holds for every $B\subseteq[n-1]$ of cardinality less than $k$. Let $B=\{i_1,\dots,i_k\}\subseteq[n-1]$ and denote $\bar{B}:=\{i_1,\dots,i_{k-1}\}$. By Proposition 8.10, for every $\pi\in S_n$ with $Des_S(\pi^{-1})\subseteq B$ there is a unique presentation

$$\pi=\bar{\pi}\tau_k,$$

where $\bar{\pi}$ is a $\bar{B}$-shuffle in $S_{i_k}$ and $\tau_k$ is an $\{i_k\}$-shuffle in $S_n$. Moreover, $Des_S(\pi^{-1})\subseteq B$ if and only if $\pi$ has such a presentation. Hence

$$\sum_{\{\pi\in S_n \mid \ Des_S(\pi^{-1})\subseteq B\}} q^{rmaj_{S_n}(\pi)} t^{\bar{\epsilon}_S(\pi)} =$$

$$\sum_{\{\bar{\pi}\in S_{i_k},\tau_k\in S_n \mid \ Des_S(\bar{\pi}^{-1})\subseteq\bar{B}, Des_S(\tau_k^{-1})\subseteq\{i_k\}\}} q^{rmaj_{S_n}(\bar{\pi}\tau_k)} t^{\bar{\epsilon}_S(\bar{\pi}\tau_k)} =$$

$$\sum_{\{\bar{\pi}\in S_{i_k} \mid \ Des_S(\bar{\pi})\subseteq\bar{B}\}} \quad \sum_{\{\tau_k\in S_n \mid \ Des_S(\tau_k^{-1})\subseteq\{i_k\}\}} q^{rmaj_{S_n}(\bar{\pi}\tau_k)} t^{\bar{\epsilon}_S(\bar{\pi}\tau_k)}.$$

By Lemma 9.3(2) this equals to

$$\sum_{\{\bar{\pi}\in S_{i_k} \mid \ Des_S(\bar{\pi}^{-1})\subseteq\bar{B}\}} q^{rmaj_{S_{i_{k-1}}}(\bar{\pi})} t^{\bar{\epsilon}_S(\bar{\pi})} \cdot \left( \begin{bmatrix} n-1 \\ i_k-1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right)$$

which, by induction, equals

$$\sum_{\{\bar{\pi}\in S_{i_k} \mid \ Des_S(\bar{\pi}^{-1})\subseteq\bar{B}\}} q^{\ell_S(\bar{\pi})} t^{\bar{\epsilon}_S(\bar{\pi})} \cdot \left( \begin{bmatrix} n-1 \\ i_k-1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right).$$

Now by a similar argument, this time applying Lemma 9.3(1),

$$\sum_{\{\pi\in S_n \mid \ Des_S(\pi^{-1})\subseteq B\}} q^{\ell_S(\pi)} t^{\bar{\epsilon}_S(\pi)} =$$

$$\sum_{\{\bar\pi\in S_{i_k} \mid \ Des_S(\bar\pi^{-1})\subseteq \bar B\}} q^{\ell_S(\bar\pi)} t^{\bar\epsilon_S(\bar\pi)} \cdot \left( \begin{bmatrix} n-1 \\ i_k-1 \end{bmatrix}_q + t_{i_k} q^i \begin{bmatrix} n-1 \\ i_k \end{bmatrix}_q \right),$$

and the proof follows. □

**Proof of Theorem 9.1(2).** By the principle of inclusion and exclusion and Remark 7.3, Theorem 9.1(2) is equivalent to the following statement :

For every subset $B \subseteq [n-1]$

$$\sum_{\{\sigma\in A_{n+1} \mid \ Des_A(\sigma^{-1})\subseteq B\}} q^{rmaj_{A_{n+1}}(\sigma)} t^{\bar\epsilon_A(\sigma)} =$$

$$\sum_{\{\sigma\in A_{n+1} \mid \ Des_A(\sigma^{-1})\subseteq B\}} q^{\ell_A(\sigma)} t^{\bar\epsilon_A(\sigma)},$$

By Proposition 5.10 this part is reduced to Theorem 9.1(1).

□

# 10 Appendix

In this section we present another pair of statistics, leading to a different analogue of MacMahon's Theorem.

For $1 \le i < n$ define a map $h_i : S_n \longmapsto S_n$ as follows:

$$h_i(\pi) := \begin{cases} s_i\pi, & \text{if } i \in Des_S(\pi^{-1}); \\ \pi, & \text{if } i \notin Des_S(\pi^{-1}). \end{cases}$$

For every permutation $\pi \in S_n$ define

$$\hat\ell_i(\pi) := \ell_S(h_i(\pi)),$$

and

$$\hat{maj}_i(\pi) := maj_S(h_i(\pi)).$$

Then $\hat\ell_i$ and $\hat{maj}_i$ are equi-distributed over the even permutations in $S_n$ (i.e. over the alternating group $A_n$).

**Theorem 10.1** *Let $n \ge 2$, then*

$$\sum_{\pi\in A_n} q^{\hat\ell_i(\pi)} = \sum_{\pi\in A_n} q^{\hat{maj}_i(\pi)} = \prod_{i=3}^{n}(1+q+\ldots+q^{i-1}).$$

**Proof.** By definition,

$$Image(h_i) = \{\pi \in S_n \mid i \notin Des_S(\pi^{-1})\} = \{\pi \in S_n \mid \pi^{-1} \text{ is an } [n] \setminus \{i\}\text{-shuffle}\}.$$

Also, for each $\sigma \in Image(h_i)$, $h_i^{-1}(\sigma) = \{\sigma, s_i\sigma\}$, and exactly one element in the set $\{\sigma, s_i\sigma\}$ is even.

Thus, by Garsia-Gessel's Theorem (Theorem 2.6),

$$\sum_{\pi \in A_n} q^{\hat{maj}_i(\pi)} =$$

$$\sum_{\{\pi \in S_n \mid \pi^{-1} \text{ is an } [n] \setminus \{i\}\text{-shuffle}\}} q^{maj(\pi)} = \begin{bmatrix} n \\ 2, 1, \ldots, 1 \end{bmatrix}_q = \prod_{i=3}^{n} (1+q+\ldots+q^{i-1}),$$

and similarly for $\hat{\ell}_i$.

□

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