Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

A New Representation of Formal Power Series

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Abstract. *This paper is dedicated to the genesis arising at the boundary between the theory of formal power series (FPS) and combinatorics.*

Similarly to combinatorics where any rational sequence of natural numbers $\{r_k\}_{k\geq 0}$ *is representable for all* k *in the form*

(1.1) $$r_{k+n} = \sum_{i=1}^{n} r_{k+n-i} X_{n-i}$$

where X_j *– are, generally speaking, complex numbers (Berstel,Reutenauer,* [**BR**]*), we prove that any rational FPS* r *is representable in the form (1) where* $r_s = \sum_{|w|=s}(r,w)w$*, and* X_j *are elements of some special skew field. As a trivial consequence of such a representation were obtained: 1)truthfulness of Eilenberg's Equality Theorem* [**E**]*, decidability of the equivalence problem of finite multitape deterministic automata (Rabin,Scott* [**RS**]*) and decidability of problem of whether two given morphisms are equivalent on regular language, (Culik, Salomaa* [**CS**]*); 2) more simply formulated and proved the results from monographs on FPS (Salomaa, Soittola* [**SS**]*, Berstel, Reutenauer* [**BR**]*, Kuich, Salomaa* [**KS**]*); 3) solved partial cases of the problem of existence for an inverse element of Hadamard product and others; 4) provided 3 Conjectures and 10 Open problems.*

The conclusion contains a complete comparative analysis of the attempts to utilize linear recurrence in theory of FPS by other authors.

RÉSUMÉ.

We use the standard notations from monographs Berstel, Reuteneuer [**BR**] and Cohn [**Coh**]. In particular, it will be assumed that $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ is a finite alphabet, $\Sigma^{-1} = \{\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_t^{-1}\}$, ε is empty word and unity in semigroup Σ^* and group G, generated by Σ, $\emptyset$ is empty set and zero in semirings and fields, generated by Σ, $\underline{\varepsilon}, \underline{\sigma}_i, \underline{\sigma}_i^{-1}$ are corresponding characteristic FPS, $\boldsymbol{k}$ is commutative zero-divizor-free semiring embeddable in commutative field $\boldsymbol{K}$ (this includes the semirings $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$).

According to Salomaa, Soittola [**SS**], every FPS $r \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ can be represented as a behaviour of $\boldsymbol{k} - \Sigma^*$-automaton

$$\mathfrak{A} = < \{q_1, q_2, \ldots, q_n\}, A, q_1, F >$$

where $A \in \boldsymbol{k}^{n\times n} < \Sigma >$ - transition matrix, q_1-initial state, $F \in \boldsymbol{k}^{n\times 1} < \{\underline{\varepsilon}, \emptyset\} >$ - final states:

$$r_{\mathfrak{A}} = \sum_{i=0}^{\infty} (A^i F)_1.$$

1991 *Mathematics Subject Classification.* Primary 13F25, 11B34; Secondary 13J05, 11B39.

Key words and phrases. formal power series, representation, recurrence.

We denote $q_i^{(j)} = (A^j F)_i$, then $r_{\mathfrak{A}} = \sum_{i=0}^{\infty} q_1^{(i)}$ and

$$q_i^{(j)} = \sum_{s=1}^{n} A_{is} q_s^{(j-1)} \tag{1.2}$$

Let us consider a system of n equations with $(n+1)$ unknowns $X_0, X_1, \ldots, X_n$:

$$\begin{cases} q_i^{(n)} X_n = \sum_{j=1}^{n} q_i^{(n-j)} X_{n-j}, \ i = \overline{1,n} \end{cases} \tag{1.3}$$

and show that it always has a non-zero solution in Malcev-Neumann skew field $\boldsymbol{K}((G))$ of FPS with well-ordered support.

We solve the system (3) following the usual Gauss algorithm by successive excluding unknowns $X_0, X_1, \ldots, X_n$. On step 0 all coefficients of unknowns are $q_i^{(j)} \in \boldsymbol{k} \ll \Sigma^* \gg$ and of course are elements of $\boldsymbol{K}((G))$. Let us assume that on step i an equation for X_i has the form of

$$q_{in} X_n = q_{i(n-1)} X_{n-1} + \ldots + q_{ii} X_i, \ q_{is} \in \boldsymbol{K} \ll G \gg, \ s = \overline{n,i} \tag{1.4}$$

and we can compute a leading term for every q_{is}.

Suppose $q_{ii} \neq \emptyset$ (otherwise we can exchange i-th column with one of $n-1, ..., i+1$; if all $q_{is} = \emptyset$ for $s = \overline{i, n-1}$ then assume $X_n = \emptyset$ and go to an equation for X_{i+1}). Then for all non-zero q_{is} denote $q_{is} = \alpha_{is} + q'_{is}$ where α_{is} is a leading term in q_{is}. It follows that after multiplying both parts of the equation on α_{ii}^{-1} and solving it for X_i we obtain

$$X_i = \left(-\alpha_{ii}^{-1} q'_{ii}\right)^* \left(\alpha_{ii}^{-1} \alpha_{in} + \alpha_{ii}^{-1} q'_{in}\right) X_n - \cdots - (\ldots) X_{i+1} \tag{1.5}$$

where the bracket content $(\ldots)$ is analogous to the coefficient of X_n and is not provided for the sake of simplicity. Substitute equation (5) into the remaining equations for $X_j, \ j = \overline{1, i-1}$:

$$\begin{aligned} (\alpha_{jn} + q'_{jn} \quad &- \quad (\alpha_{ji} + q'_{ji})(-\alpha_{ii}^{-1} q'_{ii})^* (\alpha_{ii}^{-1} \alpha_{in} + \alpha_{ii}^{-1} q'_{in})) X_n = \\ &= \quad (\cdots) X_{n-1} + \cdots + (\cdots) X_{i+1} \end{aligned} \tag{1.6}$$

Since $supp(\alpha_{ii}^{-1} q'_{ii}) > \varepsilon$ then leading term of the coefficient of X_n should be searched for in $\alpha_{jn} - \alpha_{ji} \alpha_{ii}^{-1} \alpha_{in}$. If the coefficient equals $\emptyset$ then we take next in ascending order elements from $supp(q'_{jn})$, $supp(q'_{ji})$, $supp(\alpha_{ii}^{-1} q'_{in})$ and so on. This process is constructive (see Lewin [**L**], Cohn [**Coh**]), so the inductive hypothesis holds true – at the beginning of next step of Gauss algorithm all coefficients of unknowns $X_n, X_{n-1}, \ldots X_{i+1}$ will be again from $\boldsymbol{K}((G))$ with known leading terms.

At the last step for the equation $q_{nn} X_n = q_{n(n-1)} X_{n-1}$ we have:

(i) if $q_{n(n-1)} \neq \emptyset$ that is $q_{n(n-1)} = \alpha_{n(n-1)} + q'_{n(n-1)}$ then assume
$X_n = \underline{\varepsilon}, X_{n-1} = \left(-\alpha_{n(n-1)}^{-1} q'_{n(n-1)}\right)^* \alpha_{n(n-1)}^{-1} q_{nn}$;

(ii) if $q_{n(n-1)} = \emptyset$, then assume $X_n = \emptyset, X_{n-1} = \underline{\varepsilon}$. So we proved

Theorem 1.1. *A solution of the system (3) in $\boldsymbol{K}((G))$ exists always in the form:*

$$(\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_p, \underline{\varepsilon}, \emptyset, \ldots, \emptyset), \ 1 \leq p \leq n-1 \tag{1.7}$$

while some $\tilde{X}_i$ *also can be* $\emptyset$ *.* ■

We prove the main theorem of the paper.

Theorem 1.2. *For all* $k \in \mathbf{N}$ *holds*

$$q_i^{(n+k)} = \sum_{j=1}^{n} q_i^{(n+k-j)} \tilde{X}_{n-j}, \; i = \overline{1,n} \tag{1.8}$$

Proof. For $k = 0$ the statement is proved - suppose it is true for k. Then
$q_i^{(n+k+1)} \stackrel{(2)}{=} \sum_{j=1}^{n} A_{ij} q_j^{(n+k)} = \sum_{j=1}^{n} A_{ij} \sum_{l=1}^{n} q_j^{(n+k-l)} \tilde{X}_{n-l} =$
$= \sum_{l=1}^{n} \left(\sum_{j=1}^{n} A_{ij} q_j^{(n+k-l)} \right) \tilde{X}_{n-l} \stackrel{(2)}{=} \sum_{l=1}^{n} q_i^{(n+k+1-l)} \tilde{X}_{n-l}$.■

Definition 1. We call a representation of FPS q_i in the form (8) *a linear recurrence representation* (further referred shortly as LRR), vector-solution (7) - *a stencil,* $q_i^{(j)}$ - j-th *layer* of FPS q_i. ■

Example 1.4. Following the considerations about the solving of system (3) one can find that FPS $s = (\underline{a}^2(\underline{ab})^*\underline{b}^2(\underline{ab})^*)^*$ has LRR

$s_4 = \underline{a}^2\underline{b}^2, s_3 = s_2 = s_1 = \emptyset, s_0 = \underline{\varepsilon},$

$s_{n+5} = s_{n+4} \cdot \emptyset + s_{n+3}(\underline{b}^{-2}\underline{ab}^3 + \underline{ab}) + s_{n+2} \cdot \emptyset + s_{n+1}(\underline{a}^2\underline{b}^2 - \underline{ab}^{-1}\underline{ab}^3) + s_n \cdot \emptyset$.■

Having analyzed the process of solving system (3) it is not difficult to prove:

Theorem 1.3. *There exists a stencil with coefficients from the set* $\{-1, 0, 1\}$ *for a characteristic series of an arbitrary given rational languages.* ■

The opposite is intresting:

Open problem 1. *("Fatou extension") If : 1) stensil of FPS r has all the coefficients from the set* $\{-1, 0, 1\}$*, 2) layers of r have coefficients from the set* $\{0, 1\}$*, then: r is* $\boldsymbol{N}$*-rational? And* $\boldsymbol{Z}$*-rational? And* K*-algebraic?*

(we point out to the relationship of this problem with the counter-example Reutenauer [**R**]). ■

Corollary 1 (Eilenberg's Equality Theorem [**E**]). *Let* $\mathfrak{A} =< \{q_1, \ldots, q_n\}, A, q_1, F_1 >$ *and* $\mathfrak{B} =< \{p_1, \ldots, p_m\}, B, p_1, F_2 >$ *be* $\boldsymbol{k} - \Sigma^*$*-automata. Then* $r_{\mathfrak{A}} = r_{\mathfrak{B}}$ *iff* $(r_{\mathfrak{A}}, w) = (r_{\mathfrak{B}}, w)$ *for all* $w \in \Sigma^*$ *of length at most (n+m-1).*

Proof. Consider a system of equations:

$$\begin{cases} q_i^{n+m} = \sum_{j=1}^{n+m} q_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1,n} \\ p_i^{n+m} = \sum_{j=1}^{n+m} p_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1,m} \quad ■ \end{cases} \tag{9}$$

Definition 2. We call the solution of the system (9) and the system itself *a common stencil for automata* $\mathfrak{A}$ and $\mathfrak{B}$. Common stencil exists for any finite number of $\boldsymbol{k} - \Sigma^*$-automata. ■

Corollary 2. *(Equivalence Problem for Multitape Deterministic Finite Autumata, Rabin,Scott* [**RS**]*) Two automata* $\mathfrak{A}_1 =< \{q_1, \ldots, q_n\}, \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_k, \delta_1, q_1, F_1 >$ *and* $\mathfrak{A}_2 =< \{p_1, \ldots, p_m\}, \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_k, \delta_2, p_1, F_2 >$ *are equivalent iff the sets of their acceptable words of length at most* $(n + m - 1)$ *are equal.*

Proof. Consider common stencil for automata $\mathfrak{A}$ and $\mathfrak{B}$. As the direct product of fully ordered groups equipped with lexicographic order $\Sigma_1 < \Sigma_2 < \ldots < \Sigma_k$ is still fully ordered group G_k (Passman [**Pa**]), hence this common stencil exists in form of solution for system (9) in $\boldsymbol{Z}((G_k))$. ■

Remark 1. *Harju, Karhumaki* [**HK**] *result is also in checking up of all words of length at most* $(n+m-1)$*. To check the equivalence of two finite multitape deterministic automata an exponential time, therefore, is required. At the same time there exist polynomial algorithms for the checking of the equivalence of* $\boldsymbol{k} - \Sigma^*$*- automata (*$O(n^4)$ *– Tzeng* [**T**]*,* $O(n^3)$ *– Archangelsky* [**A1**]*). This provides a hint that should exist a polynomial algorithm. Indeed not every initial set of layers should be checked up because not all of them in combination with stencil would generate only 'clean' noncommutative polynomials – the ones without* σ_i^{-1}*.* ■

Open problem 2. *How many 'clean' tuples of layers there exist for a given stencil?* ■

Remark 2. *Corollary 2 could have been proven more simpler by leaving out the process of finding of stencil and the proof of its existence. According to Hebish, Weinert* [**HW**] *the semiring of FPS on partially commutative monoids over* $\mathbf{Z}$ *is zero-divisor-free and additivelly- cancellative and multiplicately-left-cancellative. This means that a solution of the system (9) exists over some partially commutative skew field.* ■

Corollary 3. *(Equivalence Problem for Morphisms on Regular Languages, Culik, Salomaa* [**CS**]*) Let* L *be a regular language, defined by the minimal deterministic automaton* $\mathfrak{A} =< \{q_1, \ldots, q_n\}$, $\Sigma, \delta, q_1, F >$, *and* $h, g : \Sigma^* \to \Delta^*$ *be morphisms. If* $h(w) = g(w)$ *for all* $w \in L$ *of length at most* $(2n-1)$, *then* $h(w) = g(w)$ *for all* $w \in L$.

Proof. One may assume that $\Sigma \cap \Delta = \emptyset$ and letters from Σ and Δ commute. We define the transition function δ_h in 2-tape automaton $\mathfrak{A}_h =< \{q_1, \ldots, q_n\}, \Sigma \cup \Delta, \delta_h, q_1, F >$ as follows $\delta_h(q_i, \sigma_j h(\sigma_j)) = q_k \Leftrightarrow \delta(q_i, \sigma_j) = q_k$. Similarly define δ_g and $\mathfrak{A}_g$. Until common stencil of $\mathfrak{A}_h$ and $\mathfrak{A}_g$ is being built we assume for convenience each $\sigma_j h(\sigma_j)$ and $\sigma_j g(\sigma_j)$ to be one unique letter. Thus the length of common stencil will be $2n$. ■

Remark 3. *Proof of Corollary 3 does not use unlike Karhumaki* [**K**] *an Eihrengeucht's conjecture. Actually we have proven a more stronger result – the decidability of morphism equivalence on regular language with multiplicities of words are taken into consideration.*■

Corollary 4. *Let* $r \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ *and* p *be number of first nonzero element in stencil of* r, *i.e.* $\tilde{X}_0 = ... = \tilde{X}_{p-1} = \emptyset, \tilde{X}_p \neq \emptyset$, $0 \le p \le n$. *Then*

(i) r *is identecically zero iff* $r_i = \emptyset$ *for all* $i = \overline{0, (n-1)}$;

(ii) r *is polynomial iff* $r_i = \emptyset$ *for all* $i = \overline{p, (n-1)}$;

(iii) r *is ultimately constant iff* $r_i = c\underline{\Sigma}^i$ *for all* $i = \overline{p, (n-1)}$;

(iv) r *is identically constant iff* $r_i = c\underline{\Sigma}^i$ *for all* $i = \overline{0, (n-1)}$.

Proof. Trivial combinatorical considerations.■

Remark 4. *Proof of Corollary 4 does not use, unlike Salomaa, Soittola* [**SS**], *Kuich, Salomaa* [**KS**] *Hadamard product and morphisms.* ■

Let us investigate more scrupulously how of the summands with negative powers of letters in $\sum r_i \tilde{X}_i$ annihilate. In the first approximation it can be done by tracing down step-by-step how only 'clean' noncommutative polynomials are left in the following examples.

Example 1.5 (Berstel, Reutenauer [**BR**]). *FPS* $s = \sum_w |w|_a w = \underline{\Sigma}^* \underline{a} \underline{\Sigma}^*$ *has the follows LRR:*

$$s_0 = \emptyset, s_1 = \underline{a}$$

$$s_{n+2} = s_{n+1}(2\underline{a} + \underline{b} + \underline{a}^{-1}\underline{b}\ \underline{a}) + s_n(-\underline{a}^2 - 2\underline{b}\ \underline{a} - \underline{b}\ \underline{a}^{-1}\ \underline{b}\ \underline{a})$$ ■

Example 1.6 (Berstel, Reutenauer [**BR**]). *FPS* $s = \sum_w (|w|_a - |w|_b) w = \underline{\Sigma}^*(\underline{a} - \underline{b})\underline{\Sigma}^*$ has the follows LRR:

$$s_0 = \emptyset, s_1 = \underline{a} - \underline{b},$$

$$s_{n+2} = s_{n+1} \cdot 2(\underline{a}^{-1}\underline{b})^*(\underline{a} - \underline{a}^{-1}\underline{b}^2) + s_n(\underline{a} + \underline{b})(\underline{a} + \underline{b} - 2(\underline{a}^{-1}\underline{b})^*(\underline{a} - \underline{a}^{-1}\underline{b}^2))$$ ■

Example 1.7 (Reutenauer [**R**]). *FPS*

$$s = \sum_w \left(\alpha^{2(|w|_x - |w|_y)} + \alpha^{2(|w|_y - |w|_x)}\right) w = \left(\alpha^2 \underline{x} + \alpha^{-2} \underline{y}\right)^* + \left(\alpha^{-2} \underline{x} + \alpha^2 \underline{y}\right)^*$$
$$\alpha = \tfrac{1}{2}(\sqrt{5} + 1),$$

has the follows LRR: $s_0 = 2\underline{\varepsilon}, s_1 = 3\underline{x} + 3\underline{y}$,

$$s_{n+2} = s_{n+1} \cdot 3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2) + s_n\left(\alpha^{-2}\underline{x} + \alpha^2\underline{y}\right)\left(\alpha^{-2}\underline{x} + \alpha^2\underline{y} - -3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2)\right) \blacksquare$$

Let us consider arbitrary sequential n-tuple of layers of $r \in \boldsymbol{k}^{rat} << \Sigma^* >>$. One can say that they are n-inert in ring $\boldsymbol{K}((G))$ in several weak sense because $r_{k+n-i} \in \boldsymbol{k} < \Sigma^* >$ (and of course, $r_{k+n-i} \in \boldsymbol{K}((G))$), $\tilde{X}_{n-i} \in \boldsymbol{K}((G))$, but $\sum_{i=1}^n r_{k+n-i}\tilde{X}_{n-i} \in \boldsymbol{k} < \Sigma^* >$. And while the inertia theorem is proved (Bergman [**Ber**], Cohn [**Coh**]) also for ring $\boldsymbol{k} < \Sigma^* >$ in ring $\boldsymbol{K} \ll \Sigma^* \gg$, but not in ring $\boldsymbol{K}((G))$, the following analogue seems to be the case.

Conjecture 1. $\boldsymbol{k} < \Sigma^* >$ *is inert in the* $\boldsymbol{K}((G))$. $\blacksquare$

Conjecture 2. *Assuming Conjecture 1 is true - would matrix-trivializer* M *exist such that* $M, M^{-1} \in \boldsymbol{K}^{n\times n} \ll \Sigma^* \gg$? $\blacksquare$

Formulae (8) implies the following formulae for computing the coefficients in LRR:

$$(r_{k+n}, w) = \sum_{\substack{(1)\ 1\le i\le n \\ (2) w_{is}\tilde{w}_{is}=w,\ w_{is}\in\Sigma^*,\ \tilde{w}_{is}\in G}} (r_{k+n-i}, w_{is})(\tilde{X}_{n-i}, \tilde{w}_{is}) \tag{10}$$

The second condition of summing means that $w_{is} = \alpha_{is}\beta_{is},\ \beta_{is}^{-1}\gamma_{is} = \tilde{w}_{is},\ \alpha_{is}, \beta_{is}, \gamma_{is} \in \Sigma^*$. Therefore $|\beta_{is}| \le |w_{is}| = k+n-i$ and number of summands in (10) is limited.

Conjecture 3. *Would the length of canceling suffixes and prefixes (like a* β_{is}*) be limited too for each LRR?*$\blacksquare$

Open problem 3 (Archangelsky [**A2**]). *For a given* $\tilde{r} \in \boldsymbol{K}((G))$ *determine whether the lengths of all negative subwords of words in supp* $(\tilde{r})$ *are limited (i.e., subwords in alphabet* Σ^{-1} *only).*$\blacksquare$

We apply rule (10) for examining coefficients in the inverse element of Hadamard product.We mean FPS p is *Hadamard inverse* of FPS r iff $r \odot p = \sum_w 1 \cdot w = \underline{\Sigma}^*$. The problem of existence of such an element is still open. All papers on the issue either study FPS on cyclic/commutative semigroups (Cori [**Cor**], Benzaghou [**Ben1, Ben2**],Benzaghou, Bezivin [**BB**], Anselmo, Bertoni [**AB**], Poorten [**Po**]) or simple samples of inversable FPS on $\Sigma^*, |\Sigma| \ge 2$ (Gerardin [**G**]).

Theorem 1.4. *Let* Σ *be alphabet,* $|\Sigma| \ge 2$, $\mathfrak{A}_r, \mathfrak{A}_p$ *be* $\boldsymbol{Q}^+ - \Sigma^*$ *– automata which behaviours are FPS* r, p *and let the coefficients in the common stencil of automata*

$$\begin{cases} \mathfrak{A}_r \\ \mathfrak{A}_p \\ q = \underline{\Sigma}q + \underline{\varepsilon} \end{cases} \tag{11}$$

are in $\mathbf{Q}^+$. *Then* $r \odot p = \underline{\Sigma}^*$ *implies both* r, p *have a finite image.*

Proof.Consider common stencil of automata (11) (t is the sum of states for automata $\mathfrak{A}_r$ and $\mathfrak{A}_p$ plus 1):

$$\begin{cases} r_{n+t} = \Sigma_{i=1}^t r_{n+t-i}\tilde{X}_{t-i} \\ p_{n+t} = \Sigma_{i=1}^t p_{n+t-i}\tilde{X}_{t-i} \\ \underline{\Sigma}^{n+t} = \Sigma_{i=1}^t \underline{\Sigma}^{n+t-i}\tilde{X}_{t-i}, \qquad n \ge 0 \end{cases} \tag{12}$$

According to (12) and (10) a coefficient of the word $w \in supp(r_{n+t})$ in r_{n+t} satisfies the follows:

$$\alpha = \sum_{s\in S} \alpha_s x_s \tag{13}$$

where α_s – coefficients of $supp(r_{n+t-i})$ and x_s – coefficients of $supp(\tilde{X}_{t-i})$, and $|S|$ is finite. Respectively,

$$\frac{1}{\alpha} = \sum_{s\in S} \frac{1}{\alpha_s} x_s \tag{14}$$

$$1 = \sum_{s\in S} 1 \cdot x_s \tag{15}$$

Multiplying (13) and (14) we obtain

$$\alpha\cdot\frac{1}{\alpha}=1=\left(\sum_{s\in S}\alpha_s x_s\right)\left(\sum_{s\in S}\frac{1}{\alpha_s}x_s\right)=$$
$$=\sum_{s\in S}x_s^2+\sum_{i\neq j;i,j\in S}\left(\frac{\alpha_i}{\alpha_j}+\frac{\alpha_j}{\alpha_i}\right)x_ix_j\geq\sum_{s\in S}x_s^2+2\sum_{i\neq j;i,j\in S}x_ix_j=$$
$$=\left(\sum_{s\in S}x_s\right)^2=1,$$

that is why all $\alpha_i=\alpha_j=\alpha$, i.e. new coefficients do not appear in r_{n+t}. ■

Open problem 4. *Positiveness of all coefficients in all stencils and layers is an essential part of the proof of Theorem 3. In general case this limitation would not exist – therefore one would require to solve (or describe the set of solutions for) the system of Diophantine equations* $\{(13),(14),(15)\}$ $(\alpha_i\in\boldsymbol{N}^+, x_i\in\boldsymbol{Q})$. *For small numbers of unknowns the system above indeed has only trivial solutions. It seems like the class of invertable by Hadamard rational FPS is very narrow.* ■

Method of Theorem 4 may be implememted for the obtaining a necessary condition for the solution of following

Open problem 5 (Restivo,Reutenauer [**RR**])**.** *Let s be a FPS with integer coefficients and p a prime number; if $\sum_w p^{(s,w)}w$ is rational, then so are s and $\sum_w p^{-(s,w)}$.*■

Corollary 5. *If $s\in\boldsymbol{Q}^{rat}\ll\Sigma^*\gg$, $p\in\boldsymbol{N}$, $s_1=\sum_w p^{(s,w)}$, $s_2=\sum_w p^{-(s,w)}$, $s_1,s_2\in(\boldsymbol{Q}^+)^{rat}\ll\Sigma^*\gg$ and the coefficients of the common stencil of automata*

$$\begin{cases}\mathfrak{A}_{s_1}\\ \mathfrak{A}_{s_2}\\ q=\underline{\Sigma}q+\underline{\varepsilon}\end{cases}$$

are in $\mathbf{R}^+$, then s,s_1,s_2 have a finite image.■

Let us try for a given LRR build a FPS, a representation of which the former is:

$$r=\sum_{i=0}^{n-1}r_i+\sum_{i=n}^{\infty}r_i=\sum_{i=0}^{n-1}r_i+\sum_{i=n}^{\infty}\sum_{j=1}^{n}r_{i-j}\tilde{X}_{n-j}=$$
$$=\sum_{i=0}^{n}r_i+\sum_{j=1}^{n}\sum_{i=j-1}^{\infty}r_i\tilde{X}_{j-1}=$$
$$=\sum_{i=0}^{n-1}-\sum_{j=1}^{n-1}\sum_{s=0}^{j-1}r_s\tilde{X}_j+\sum_{j=1}^{n}\sum_{i=0}^{\infty}r_i\tilde{X}_{j-1}=$$
$$=r_0+\sum_{i=1}^{n-1}(r_i-\sum_{s=0}^{i-1}r_s\tilde{X}_i)+r\sum_{j=1}^{n}\tilde{X}_{j-1}\tag{16}$$

Solve this equation for r:

$$r=(r_0+\sum_{i=1}^{n-1}(r_i-\sum_{s=0}^{i-1}\tilde{X}_i))(\sum_{j=1}^{n}\tilde{X}_{j-1})^*\tag{17}$$

Unarguably we took too much liberty when applying limit to both parts of identity (16). It still needs to be proved that the obtained expression is indeed the sum of r_i and only them. Because of size limit we would not do that but do illustrate using Example 2 that it is true:

$$(\underline{a}+\underline{b})^*\underline{a}(\underline{a}+\underline{b})^*=s\overset{(17)}{=}(s_1+s_0-s_0\tilde{X}_1)(\tilde{X}_1+\tilde{X}_2)^*=$$
$$=\underline{a}(2\underline{a}+\underline{b}+\underline{a}^{-1}\underline{ba}-\underline{a}^2-2\underline{ba}-\underline{ba}^{-1}\underline{ba})^*=((\underline{\varepsilon}-2\underline{a}-\underline{b}-\underline{a}^{-1}\ \underline{b}\ \underline{a}+$$
$$+\underline{a}^2+2\underline{b}\ \underline{a}+\underline{b}\ \underline{a}^{-1}\ \underline{b}\ \underline{a})\underline{a}^{-1})^{-1}=((\underline{a}^{-1}-\underline{\varepsilon}-\underline{b}\ \underline{a}^{-1})(\underline{\varepsilon}-\underline{a}-\underline{b}))^{-1}=$$
$$=((\underline{\varepsilon}-\underline{a}-\underline{b})\underline{a}^{-1}(\underline{\varepsilon}-\underline{a}-\underline{b}))^{-1}=(\underline{a}+\underline{b})^*\underline{a}(\underline{a}+\underline{b})^*.\ ■$$

Brzozowski, Cohen [**BC**] studied a decompositions of rational languages into star languages : $P=R^*S$.One may ask about such decomposition in $\boldsymbol{K}((G))$. Of course, arbitrary regular language R may be trivially decomposed into star FPS in $\boldsymbol{K}((G))$: $\underline{R}=\underline{P}^*(\underline{\varepsilon}-\underline{P})\underline{R}$, where P is arbitrary regular language too.

It is interesting to study a nontrivial case. Consider a common stencil of two arbitrary FPS in the form (17). It implies

Theorem 1.5. *Each two FPS* $r, p \in \boldsymbol{k}^{rat} \ll \Sigma^* \gg$ *have a representation in* $\boldsymbol{K}((G))$ *with nontrivial common star factor :* $r = \tilde{r}_1 \tilde{s}^*, p = \tilde{p}_1 \tilde{s}^*$. ■

Judging by appearance the regular expression (17) does not represent FPS from $\boldsymbol{k} \ll \Sigma^* \gg$, since it contains inverse elements from Σ^{-1} and $\boldsymbol{K}$. The transition matrix for the corresponding $\boldsymbol{K} - (\Sigma \cup \Sigma^{-1})^*$ - automaton would contain elements from Σ^{-1} and $\boldsymbol{K}$ too – although the behavior of this automaton would be exactly FPS r that is without Σ^{-1} and $\boldsymbol{K} \backslash \boldsymbol{k}$.

Open problem 6 ("Fatou extensions"). *Let* $A \in \boldsymbol{Z}^{n \times n} < \Sigma \cup \Sigma^{-1} >$ *but all layers of FPS* $r = \sum_{i=0}^{\infty} (A^i)_{1,n}$ *are in* $\boldsymbol{N} < \Sigma^* >$. *Would* $r \in \boldsymbol{N}^{rat} \ll \Sigma^* \gg$ *be true? And* $\boldsymbol{Z}^{rat} \ll \Sigma^* \gg$*? And* $K^{alg} \ll \Sigma^* \gg$ *?* ■

Open problem 7 (Berstel etc. [**BBCPP**]). *Does a function* $n \to r_n$ *preserve a rationality? That is if* $\{a_n\}_{n \geq 0}$ *is a rational sequence of natural numbers,* r *is rational FPS then would* $\sum_{i=0}^{\infty} r_{a_i}$ *be rational?* ■

Open problem 8. *Based on given LRR of FPS* p, q *build LRR of :* $p^*, p+q, pq,$
$p \odot q, p \sqcup\!\sqcup q$. ■

Open problem 9. *Describe the set of all stencils of given rational FPS.* ■

Open problem 10. *Stencils in their turn are rational FPS. One can be built their LRR and so on. What can be said about the process ?* ■

Conclusion

Many researchers guessed about the existence of a linear dependency between the current value of FPS and a limited number of previous ones, but have failed to express it in a convenient universal form that would allow to obtain trivially results above. Thus for example,

Restivo, Reutenauer [**RR**]: *FPS* $s \in K \ll \Sigma^* \gg$ *is rational iff for any word* x *there is a common linear recurrence relation over* K *satisfied by all the sequences* $\{(s, ux^n v)\}_{n \geq 0}, u, v \in \Sigma^*$.

The below listed authors used for stencil the same ring as for represented FPS , what undercut readability and applications:

Salomaa, Soittola [**SS**]: *Assume* $r \in K^{rat} \ll \Sigma^* \gg$ *and* N *is rank of* r. *Show that if* $|w_0| = N$ *then there are words* $w_1, \ldots, w_N$ *and elements* $c_1, \ldots, c_N$ *of* K *such that* $|w_i| < N, i = \overline{1, N}$ *and for all words* w*:*
$(r, ww_0) = c_1(r, ww_1) + \cdots + c_N(r, ww_N)$

Berstel, Reutenauer [**BR**]: *For any rational series* S *of rank* n *there exist a prefix-closed set* P *of* n *elements, with an associated prefix set* C*, and coefficients* $\alpha_{c,p}(c \in C, p \in P)$ *such that, for all words* w *and all* $c \in C$*:*

$$(S, cw) = \sum_{p \in P} \alpha_{c,p}(S, pw).$$

or limited the domain of definition of the linear relation :

Eilenberg [**E**]: $f = \sum a_n z^n$ *is rational iff the following "recursion formula" holds for all* t *sufficiently large:*

$$a_{t+m} + c_1 a_{t+m} + \ldots + c_m a_t = 0.$$

On the other hand Cohn [**Coh**] did not lost universality and convenience but to achieve that he had to 'maim' previous layers:

A series $r \in K((X; \alpha))$ *is rational iff there exist integer* m, n_0 *and elements* $c_1, \ldots, c_m \in K$ *such that for all* $n > n_0$*:*

$$r_n = r_{n-1}^{\alpha} c_1 + r_{n-2}^{\alpha^2} c_2 + \ldots + r_{n-m}^{\alpha^m} c_m$$

As for Varricchio [**V**] – he did not go beyond the statement of a linear dependency for initial interval of FPS:

Let $s \in K^{rat} \ll \Sigma^* \gg, \Sigma^{[N]}$ *be the set of words of* Σ *whose length is less then or equal to* N, μ *be matrix interpretation of* S. *Then one can effectively compute an integer* N*, depending on* S *with the property that for any* $u \in \sum^{[N+1]}$ *there exist a set* $T = \{\sigma_v\}_{v \in \sum^{[N]}} \subseteq K$ *such that* $\mu(u) = \sum_{\sigma_v \in T} \mu(v)$.

It is very strange that author failed to discover the attempts to use linear recurrence in FPS on free commutative monoid $c(\Sigma^*), |\Sigma| \geq 2$. According to Kuich, Salomaa [**KS**] $K^{alg} \ll c(\Sigma^*) \gg = K^{rat} \ll c(\Sigma^*) \gg$. Therefore many K - algebraic FPS can be studied with the help of LRR.

As one see the proposed approach contrary to the predecessors is systematic and handy. As a indirect proof of that fact is a large number of correlations between FPS and combinatorics collected by the author and left out the scope of this work.

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