



# The Bergman Complex of a Matroid and Phylogenetic Trees

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**Abstract.** *We study the Bergman complex  $\mathcal{B}(M)$  of a matroid  $M$ : a polyhedral complex which arises in algebraic geometry, but which we describe purely combinatorially. We prove that a natural subdivision of the Bergman complex of  $M$  is a geometric realization of the order complex of its lattice of flats. In addition, we show that the Bergman fan  $\tilde{\mathcal{B}}(K_n)$  of the graphical matroid of the complete graph  $K_n$  is homeomorphic to the space of phylogenetic trees  $T_n$ .*

## 1. Introduction

In [1], Bergman defined the *logarithmic limit-set* of an algebraic variety in order to study its exponential behavior at infinity. We follow [15] in calling this set the *Bergman complex* of the variety. Bergman conjectured that this set is a finite, pure polyhedral complex. He also posed the question of studying the geometric structure of this set; *e.g.*, its connectedness, homotopy, homology and cohomology. Bieri and Groves first proved the conjecture in [2] using valuation theory.

Recently, Bergman complexes have received considerable attention in several areas, such as tropical algebraic geometry and dynamical systems. They are the *non-archimedean amoebas* of [7] and the *tropical varieties* of [15, 13]. In particular, Sturmfels [15] gave a new description of the Bergman complex and an alternative proof of Bergman's conjecture in the context of Gröbner basis theory. Moreover, when the variety is a linear space, so the defining ideal  $I$  is generated by linear forms, he showed that the Bergman complex can be described solely in terms of the matroid associated to the linear ideal.

Sturmfels used this description to define the Bergman complex of an arbitrary matroid, and suggested studying its combinatorial, geometric and topological properties [15]. The goal of the paper is to undertake this study.

In Section 2 we study the collection of bases of minimum weight of a matroid. We show that this collection is itself the collection of bases of a matroid, and we give several descriptions of it.

In Section 3 we prove the main result of the paper. We show that, appropriately subdivided, the Bergman complex of a matroid  $M$  is the order complex of the proper part of the lattice of flats  $L_M$  of the matroid. These order complexes are well-understood objects [4], and an immediate corollary of our result is an answer to Bergman's questions about the geometry of  $\mathcal{B}(M)$  in this special case. The Bergman complex of an arbitrary matroid  $M$  is a finite, pure polyhedral complex. In fact, it is homotopy equivalent to a wedge of  $(r - 2)$ -dimensional spheres, where  $r$  is the rank of  $M$ .

In Section 4, we take a closer look at the Bergman complex of the graphical matroid of the complete graph  $K_n$ . We show that the Bergman fan  $\tilde{\mathcal{B}}(K_n)$  is exactly the space of ultrametrics on  $[n]$ , which is homeomorphic to the space of phylogenetic trees as in [3]. As a consequence, we show that the order complex of the partition

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lattice  $\Pi_n$  is a subdivision of the link of the origin of this space. This provides a new explanation for the known result that these two simplicial complexes are homotopy equivalent [11, 16, 17, 18, 19].

Finally, in the appendix, we review some matroid theory which we will use throughout the paper. For a more thorough introduction, we refer the reader to [9].

## 2. The bases of minimum weight of a matroid

Let  $M$  be a matroid of rank  $r$  on the ground set  $[n] = \{1, 2, \dots, n\}$ , and let  $\omega \in \mathbb{R}^n$ . Regard  $\omega$  as a weight function on  $M$ , so that the weight of a basis  $B = \{b_1, \dots, b_r\}$  of  $M$  is given by  $\omega_B = \omega_{b_1} + \omega_{b_2} + \dots + \omega_{b_r}$ .

Let  $M_\omega$  be the collection of bases of  $M$  having minimum  $\omega$ -weight. This is one of the central objects of our study, and we wish to understand it from three different points of view: geometric, algorithmic and matroid theoretic.

Geometrically, we can understand  $M_\omega$  in terms of the matroid polytope. We will use the following characterization of matroid polytopes, due to Gelfand and Serganova:

**Theorem 2.1.** [6, Theorem 1.11.1] *Let  $S$  be a collection of  $r$ -subsets of  $[n]$ . Let  $P_S$  be the polytope in  $\mathbb{R}^n$  with vertex set  $\{e_{b_1} + \dots + e_{b_r} \mid \{b_1, \dots, b_r\} \in S\}$ , where  $e_i$  is the  $i$ -th unit vector. Then  $S$  is the collection of bases of a matroid if and only if every edge of  $P_S$  is a translate of the vector  $e_i - e_j$  for some  $i, j \in [n]$ .*

Let  $P_M$  be the matroid polytope of  $M$ . We can now think of  $\omega$  as a linear functional in  $\mathbb{R}^n$ . The bases in  $M_\omega$  correspond to the vertices of  $P_M$  which minimize the linear functional  $\omega$ . Their convex hull is  $P_{M_\omega}$ , the face of  $P_M$  where  $\omega$  is minimized. It follows that the edges of  $P_{M_\omega}$ , being edges of  $P_M$  also, are parallel to vectors of the form  $e_i - e_j$ . Therefore  $M_\omega$  is the collection of bases of a matroid.

Algorithmically, matroids have the property that their  $\omega$ -minimum bases are precisely the possible outputs of the greedy algorithm: Start with  $B = \emptyset$ . At each stage, look for an  $\omega$ -minimum element of  $[n]$  which can be added to  $B$  without making it dependent, and add it. After  $r$  steps, output the basis  $B$ . [9, Theorem 1.8.5]

**Definition 2.2.** Given  $\omega \in \mathbb{R}^n$ , let  $\mathcal{F}(\omega)$  denote the unique flag of subsets

$$\emptyset =: F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} := E$$

for which  $\omega$  is constant on each set  $F_i - F_{i-1}$  and has  $\omega|_{F_i - F_{i-1}} < \omega|_{F_{i+1} - F_i}$ . The *weight class* of a flag  $\mathcal{F}$  is the set of vectors  $\omega$  such that  $\mathcal{F}(\omega) = \mathcal{F}$ .

We can describe weight classes by their defining equalities and inequalities. For example, one of the weight classes in  $\mathbb{R}^5$  is the set of vectors  $\omega$  such that  $\omega_1 = \omega_4 < \omega_2 < \omega_3 = \omega_5$ . It corresponds to the flag  $\{\emptyset \subset \{1, 4\} \subset \{1, 2, 4\} \subset \{1, 2, 3, 4, 5\}\}$ .

**Proposition 2.3.** *If  $\omega$  is in the weight class of  $\mathcal{F} = \{\emptyset =: F_0 \subset \dots \subset F_{k+1} := E\}$ , then the  $\omega$ -minimum bases of  $M$  are exactly those containing  $r(F_i) - r(F_{i-1})$  elements of  $F_i - F_{i-1}$ , for each  $i$ . Consequently,  $M_\omega$  depends only on  $\mathcal{F}$ , and we call it  $M_{\mathcal{F}}$ .*

**PROOF.** The greedy algorithm picks  $r(F_1)$  elements of the lowest weight, until it reaches a basis of  $F_1$ ; then it picks  $r(F_2) - r(F_1)$  elements of the second lowest weight, until it reaches a basis of  $F_2$ , and so on. Therefore, the possible outputs of the algorithm are precisely the ones described.  $\square$

Matroid theoretically,  $M_\omega$  can be constructed as a direct sum of minors of  $M$ , and its lattice of flats  $L_{M_\omega}$  can be constructed from intervals of  $L_M$ , as follows:

**Proposition 2.4.** *If  $\mathcal{F} = \{\emptyset =: F_0 \subset \dots \subset F_{k+1} := E\}$ , then*

$$M_{\mathcal{F}} = \bigoplus_{i=1}^{k+1} (M|_{F_i})/F_{i-1} \quad \text{and} \quad L_{M_{\mathcal{F}}} \cong \prod_{i=1}^{k+1} [F_{i-1}, F_i].$$

PROOF. After  $r(F_{i-1})$  steps, the greedy algorithm has chosen a basis of  $F_{i-1}$ . In the following  $r(F_i) - r(F_{i-1})$  steps, it needs to choose elements which, when added to  $F_{i-1}$ , give a basis of  $F_i$ . The possible choices are, precisely, the bases of  $(M|F_i)/F_{i-1}$ . The first equality follows, and the second one follows from it.  $\square$

### 3. The Bergman complex

We now define the two main objects of study of this paper.

**Definition 3.1.** The *Bergman fan* of a matroid  $M$  with ground set  $[n]$  is the set

$$\tilde{\mathcal{B}}(M) := \{\omega \in \mathbb{R}^n : M_\omega \text{ has no loops}\}.$$

The *Bergman complex* of  $M$  is

$$\mathcal{B}(M) := \{\omega \in S^{n-2} : M_\omega \text{ has no loops}\},$$

where  $S^{n-2}$  is the sphere  $\{\omega \in \mathbb{R}^n : \omega_1 + \cdots + \omega_n = 0, \omega_1^2 + \cdots + \omega_n^2 = 1\}$ .

For the moment, we are slightly abusing notation by calling these two objects a *fan* and a *complex*. We will very soon see that they are a polyhedral fan and a spherical polyhedral complex, respectively; this justifies their name. We will concentrate on the Bergman complex, but the same arguments apply to the Bergman fan.

**Definition 3.2.** The weight class of a flag  $\mathcal{F}$  is *valid* for  $M$  if  $M_{\mathcal{F}}$  has no loops.

Since the matroid  $M_\omega$  only depends on the weight class that  $\omega$  is in, the Bergman complex of  $M$  is a disjoint union of the following weight classes:

**Definition 3.3.** The weight class of a flag  $\mathcal{F}$  is *valid* for  $M$  if  $M_{\mathcal{F}}$  has no loops.

We will study two polyhedral subdivisions of  $\mathcal{B}(M)$ , one of which is clearly finer than the other.

**Definition 3.4.** The *fine subdivision* of  $\mathcal{B}(M)$  is the subdivision of  $\mathcal{B}(M)$  into valid weight classes: two vectors  $u$  and  $v$  of  $\mathcal{B}(M)$  are in the same class if and only if  $\mathcal{F}(u) = \mathcal{F}(v)$ .

The *coarse subdivision* of  $\mathcal{B}(M)$  is the subdivision of  $\mathcal{B}(M)$  into  $M_\omega$ -equivalence classes: two vectors  $u$  and  $v$  of  $\mathcal{B}(M)$  are in the same class if and only if  $M_u = M_v$ .

**Theorem 3.5.** *The weight class of a flag  $\mathcal{F}$  is valid for  $M$  if and only if  $\mathcal{F}$  is a flag of flats of  $M$ . Therefore, the fine subdivision of the Bergman complex  $\mathcal{B}(M)$  is a geometric realization of  $\Delta(L_M - \{\hat{0}, \hat{1}\})$ , the order complex of the proper part of the lattice of flats of  $M$ .*

PROOF. Assume  $F_i$  in  $\mathcal{F}$  is not a flat of  $M$ , so there exists some  $e \in \overline{F_i} - F_i$ . By Proposition 2.3, any basis  $B$  in  $M_{\mathcal{F}}$  contains  $r(F_i)$  elements of  $F_i$ ; since  $e$  is dependent on them, it cannot be in  $B$ . Hence  $e$  is a loop in  $M_{\mathcal{F}}$ , so the weight class of  $\mathcal{F}$  is not valid.

Conversely, assume every  $F_i$  in  $\mathcal{F}$  is a flat of  $M$ . Consider any  $e \in E$ , and find the value of  $i$  such that  $e \in F_i - F_{i-1}$ . After  $r(F_{i-1})$  steps the greedy algorithm produces a basis of  $F_{i-1}$ . Since  $F_{i-1}$  is a flat,  $e$  is not dependent on it, and in the next step of the algorithm we can choose  $e$ . In the end, this produces an  $\omega$ -minimum basis of  $M$  containing  $e$ . Therefore the weight class of  $\mathcal{F}$  is valid.  $\square$

The order complex  $\Delta(L_M - \{\hat{0}, \hat{1}\})$  is a well understood object [4]. As an immediate consequence of Theorem 3.5, we get the following result.

**Corollary 3.6.** *The Bergman complex  $\mathcal{B}(M)$  is homotopy equivalent to a wedge of  $\hat{\mu}(L_M)$   $(r-2)$ -dimensional spheres. Its subdivision into weight classes is a pure, shellable simplicial complex.*

Here  $\hat{\mu}(L_M) = (-1)^{r(M)} \mu_{L_M}(\hat{0}, \hat{1})$  is an evaluation of the *Möbius function*  $\mu_{L_M}$  of the lattice  $L_M$ . The Möbius function is an extremely useful combinatorial invariant of a poset; for more information, see [14, Chapter 3].

**Example:** Let  $M$  be the graphical matroid of the complete graph on four nodes. The bases of this matroid are given by spanning trees. The flats are complete subgraphs and vertex disjoint unions of complete subgraphs (see Figure 1). Note that in this case, the flats are in correspondence with the partitions of the

set  $\{A, B, C, D\}$ . In general, the flats of the graphical matroid of  $K_n$  are in bijection with partitions of the set  $[n]$ . Furthermore, the lattice of flats is the partition lattice  $\Pi_n$ , which orders partitions by refinement.

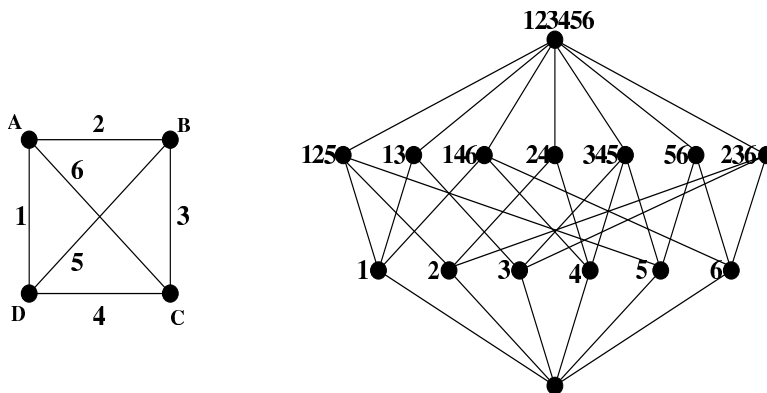


FIGURE 1. The graph  $K_4$  and the lattice of flats of the corresponding matroid.

The Bergman complex  $\mathcal{B}(K_4)$  is shown in Figure 2. It is a wedge of six 1-spheres. More generally,  $\mathcal{B}(K_n)$  is a wedge of  $\hat{\rho}(\Pi_n) = (n - 1)!$  spheres of dimension  $n - 3$ . The vertices of  $\mathcal{B}(K_4)$  are labeled with the corresponding flats, and a few of the corresponding weight classes are shown. Notice that the ground set of a matroid is always a flat, which corresponds to the weight class in which all weights are equal. We removed this weight class when normalizing the Bergman complex to the sphere.

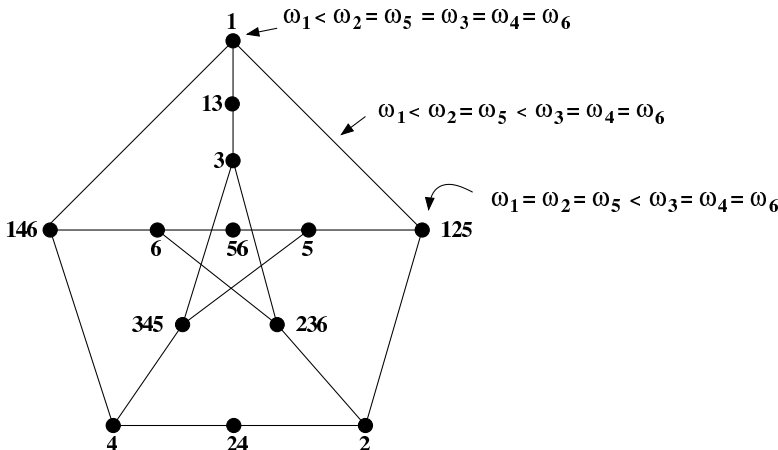


FIGURE 2. The Bergman complex of the graphical matroid of  $K_4$ .

The fine subdivision of the Bergman complex is almost the Petersen graph. The only difference is the presence of the three extra vertices, 13, 24 and 56. In the coarse subdivision into  $M_\omega$ -equivalence classes, these three vertices do not appear. For example, the weight class  $\omega_1 < \omega_3 < \omega_2 = \omega_4 = \omega_5 = \omega_6$  induces the same matroid  $M_\omega$  as  $\omega_1 = \omega_3 < \omega_2 = \omega_4 = \omega_5 = \omega_6$  and  $\omega_3 < \omega_1 < \omega_2 = \omega_4 = \omega_5 = \omega_6$ . Next we describe the relationship between these two subdivisions in general.

The coarse decomposition of  $\mathcal{B}(M)$  into cells which induce the same  $M_\omega$  is also a pure, polyhedral complex: it is a subcomplex of the spherical polar to the matroid polytope of  $M$ . To describe this decomposition, it is enough to describe its full-dimensional cells.

Therefore, we only need to determine when two full-dimensional weight classes give the same matroid  $M_\omega$ . It is clearly enough to answer this question when the two weight classes are *adjacent*; *i.e.*, the intersection of their closures is a facet of both. This happens when the two corresponding flags, which have one flat in each rank, are equal in all but one rank.

Let  $\diamond$  be the *diamond poset*; *i.e.*, the rank 2 poset consisting of a minimum element, a maximum element, and two rank 1 elements.

**Theorem 3.7.** *Suppose that the weight classes of two maximal flags  $\mathcal{F}$  and  $\mathcal{F}'$  are adjacent. Say  $\mathcal{F}$  and  $\mathcal{F}'$  only differ in rank  $i$ ; that is,  $\mathcal{F} - F_i = \mathcal{F}' - F'_i$ . Then the following conditions are equivalent:*

- (i)  $M_{\mathcal{F}} = M_{\mathcal{F}'}$ .
- (ii)  $M_{\mathcal{F}} = M_{\mathcal{F} - F_i}$ .
- (iii)  $F_i \cup F'_i = F_{i+1}$ .
- (iv) *The interval  $[F_{i-1}, F_{i+1}]$  of  $L_M$  is a diamond poset.*

PROOF. Let  $M_j = (M|F_j)/F_{j-1}$ ,  $M'_j = (M|F'_j)/F'_{j-1}$ ,  $N_i = (M|F_{i+1})/F_{i-1}$ , and  $N = M_1 \oplus \cdots \oplus M_{i-1} \oplus M_{i+2} \oplus \cdots \oplus M_{k+1}$ . By Proposition 2.4,

$$M_{\mathcal{F}} = N \oplus M_i \oplus M_{i+1}, \quad M_{\mathcal{F}'} = N \oplus M'_i \oplus M'_{i+1}, \quad M_{\mathcal{F} - F_i} = N \oplus N_i.$$

Since  $M_i, M_{i+1}, M'_i$  and  $M'_{i+1}$  have rank 1 and  $N_i$  has rank 2,

$$\begin{aligned} L_{M_i \oplus M_{i+1}} &= \{\emptyset, F_i - F_{i-1}, F_{i+1} - F_i, F_{i+1} - F_{i-1}\} \cong \diamond, \\ L_{M'_i \oplus M'_{i+1}} &= \{\emptyset, F'_i - F_{i-1}, F_{i+1} - F'_i, F_{i+1} - F_{i-1}\} \cong \diamond, \\ L_{N_i} &= \{F - F_{i-1} : F \in [F_{i-1}, F_{i+1}]\} \cong [F_{i-1}, F_{i+1}]. \end{aligned}$$

If (iv) does not hold, then we know immediately that  $L_{N_i} \neq L_{M_i \oplus M_{i+1}}$ . Also  $F_i \cup F'_i \neq F_{i+1}$ , and therefore  $L_{M_i \oplus M_{i+1}} \neq L_{M'_i \oplus M'_{i+1}}$ .

If (iv) holds, then  $F_i$  and  $F'_i$  are the only rank  $i$  flats of  $M$  in  $[F_{i-1}, F_{i+1}]$ . Since  $N_i$  has no loops, (iii) holds; and therefore  $L_{M_i \oplus M_{i+1}} = L_{M'_i \oplus M'_{i+1}} = L_{N_i}$ .  $\square$

#### 4. The space of phylogenetic trees

In this section, we show that the Bergman fan  $\tilde{\mathcal{B}}(K_n)$  of the matroid of the complete graph  $K_n$  is homeomorphic to the space of phylogenetic trees  $\mathcal{T}_n$ , as defined in [3]. To do so, we start by reviewing the connection between phylogenetic trees and ultrametrics.

**Definition 4.1.** A *dissimilarity map* on  $[n]$  is a map  $\delta : [n] \times [n] \rightarrow \mathbb{R}$  such that  $\delta(i, i) = 0$  for all  $i \in [n]$ , and  $\delta(i, j) = \delta(j, i)$  for all  $i, j \in [n]$ .

**Definition 4.2.** A dissimilarity map is an *ultrametric* if, for all  $i, j, k \in [n]$ , two of the values  $\delta(i, j), \delta(j, k)$  and  $\delta(i, k)$  are equal and not less than the third.

Let  $T$  be a rooted metric  $n$ -tree; that is, a tree with  $n$  leaves labeled  $1, 2, \dots, n$ , together with a length assigned to each one of its edges. For each pair of leaves  $u, v$  of the tree, we define the *distance*  $d_T(u, v)$  to be the length of the unique path joining leaves  $u$  and  $v$  in  $T$ . This gives us a distance function  $d_T : [n] \times [n] \rightarrow \mathbb{R}$ . We will only consider *equidistant  $n$ -trees*. These are the rooted metric  $n$ -trees such that the distance between the root and any leaf is equal to 1, and the lengths of the interior edges are positive. (For technical reasons, the edges incident to a leaf are allowed to have negative lengths.)

We can think of equidistant trees as a model for the evolutionary relationships between a certain set of species. The various species, represented by the leaves, descend from a single root. The descent from the root to a leaf tells us the history of how a particular species branched off from the others. For more information on the applications of this and other similar models, see for example [3] and [12].

The connection between equidistant trees and ultrametrics is given by the following theorem.

**Theorem 4.3.** [12, Theorem 7.2.5] *A map  $\delta : [n] \times [n] \rightarrow \mathbb{R}$  is an ultrametric if and only if it is the distance function of an equidistant  $n$ -tree.*

We can think of a dissimilarity map  $\delta : [n] \times [n] \rightarrow \mathbb{R}$  as a weight function  $\omega_\delta$  on the edges of the complete graph  $K_n$ . This leads us to the following result, which connects these ideas to the Bergman fan.

**Theorem 4.4.** *A dissimilarity map  $\delta : [n] \times [n] \rightarrow \mathbb{R}$  is an ultrametric if and only if  $\omega_\delta$  is in the Bergman fan  $\tilde{\mathcal{B}}(K_n)$ .*

PROOF. We claim that the following three statements about a weight function on the edges of  $K_n$  are equivalent.

- (i) In any triangle, the largest weight is achieved (at least) twice.
- (ii) In any cycle, the largest weight is achieved (at least) twice.
- (iii) Every edge is in a spanning tree of minimum weight.

The theorem will follow from this claim, because ultrametries are characterized by (i) and weight functions in the Bergman complex are characterized by (iii).

The implication (ii)  $\Rightarrow$  (i) is trivial. Conversely, assume that (i) holds and (ii) does not. Without loss of generality, assume that the cycle  $v_1v_2 \dots v_k$  has  $v_1v_2$  as its unique edge of largest weight. The largest weight in triangle  $v_1v_2v_3$  must be achieved at  $\omega(v_1v_2) = \omega(v_1v_3)$ . The largest weight in triangle  $v_1v_3v_4$  must then be achieved at  $\omega(v_1v_3) = \omega(v_1v_4)$ . Continuing in this way we get that  $\omega(v_1v_2) = \omega(v_1v_3) = \dots = \omega(v_1v_k)$ , and (ii) follows.

Now we prove (ii)  $\Rightarrow$  (iii). Consider an arbitrary edge  $f$ . Let  $T$  be a spanning tree of minimum weight. If  $f \in T$  we are done; otherwise,  $T \cup f$  has a unique cycle. There is at least one edge  $e$  in this cycle with  $\omega(e) \geq \omega(f)$ . Therefore, the weight of the spanning tree  $T \setminus e \cup f$  is not larger than the weight of  $T$ . This is then a spanning tree of minimum weight containing  $f$ .

Finally, assume that (iii) holds and (i) does not. Assume that the triangle with edges  $e, f, g$  has  $\omega(e) > \omega(f), \omega(g)$ , and consider a spanning tree  $T$  of minimum weight which contains edge  $e$ . If  $f$  is in  $T$ , then  $g$  cannot be in  $T$ , and replacing  $e$  with  $g$  will give a spanning tree of smaller weight. If neither  $f$  nor  $g$  is in  $T$ , we can still replace  $e$  with one of them to obtain a spanning tree of smaller weight. If we could not, that would imply that both  $f$  and  $g$  form a cycle when added to  $T \setminus e$ . Call these cycles  $C_f$  and  $C_g$ . But then  $(C_f \setminus f) \cup (C_g \setminus g) \cup e$  would contain a cycle in  $T$ , a contradiction.  $\square$

The previous two theorems give us a one-to-one correspondence between the vectors in the Bergman fan  $\tilde{\mathcal{B}}(K_n)$  and the equidistant  $n$ -trees:  $\tilde{\mathcal{B}}(K_n)$  parameterizes equidistant  $n$ -trees by the distances between their leaves. This leads us to consider the space of trees  $\mathcal{T}_n$  of [3]. This space parameterizes equidistant  $n$ -trees in a different way: it keeps track of their combinatorial type, and the lengths of their internal edges. We recall the construction of the space  $\mathcal{T}_n$ . Each maximal cell corresponds to a combinatorial type of rooted binary tree on  $n$  labeled leaves; *i.e.*, a rooted tree where each internal vertex has two descendants. Such trees have  $n - 2$  internal edges, and are parameterized by vectors in  $\mathbb{R}_{>0}^{n-2}$  recording these edge lengths. Moving to a lower dimensional face of a maximal cell corresponds to setting some of these edge lengths to 0, which gives non-binary degenerate cases of the original tree. Maximal cells are glued along these lower-dimensional cells when two trees specialize to the same degenerate tree.

Given a fixed combinatorial type of tree and the vector of internal edge lengths, we can recover the pairwise distances of leaves as linear functions on the internal edge lengths. For example, consider the tree type of Figure 3. We obtain  $(\delta(A, B), \delta(A, C), \delta(A, D), \delta(B, C), \delta(B, D), \delta(C, D)) \in \mathcal{B}(K_4)$  from  $(x, y)$  by the map  $f : (x, y) \mapsto (2(1 - x - y), 2(1 - y), 2, 2(1 - y), 2, 2)$ . The converse is also true; given the pairwise distances of leaves we can recover the internal edge lengths via linear relations on these distances [12].

In general, doing this for each type of tree, we get a map  $f : \mathcal{T}_n \rightarrow \tilde{\mathcal{B}}(K_n)$ . It follows from the previous two theorems that  $f$  is a one-to-one correspondence between  $\mathcal{T}_n$  and  $\tilde{\mathcal{B}}(K_n)$ . We will now see that, in fact,  $\mathcal{T}_n$  and  $\tilde{\mathcal{B}}(K_n)$  have the same combinatorial structure.

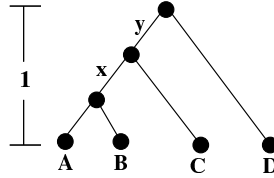


FIGURE 3. Combinatorial type of tree with 4 leaves.

**Proposition 4.5.** *The map  $f : \mathcal{T}_n \rightarrow \tilde{\mathcal{B}}(K_n)$  is a piecewise linear homeomorphism. It identifies the decomposition of the space of trees  $\mathcal{T}_n$  into combinatorial tree types with the coarse decomposition of the Bergman fan  $\tilde{\mathcal{B}}(K_n)$ .*

PROOF. Restricting to a maximal cell of  $\mathcal{T}_n$ , corresponding to a fixed tree type,  $f$  is a linear map from the lengths of internal edges (in the space of trees) to the pairwise distances of the leaves (in the space of ultrametrics). Also, it is clear that when two maximal cells of  $\mathcal{T}_n$  intersect, the linear restrictions of  $f$  to these two cells agree on their intersection. The first claim follows.

Suppose we are given a combinatorial type of equidistant  $n$ -tree. From the branching order of each triple of leaves (*i.e.*, which, if any, of the three branched off first), we can recover which edges of each triangle of  $K_n$  are maximum in the corresponding weight vector. In turn, this allows us to recover which edges of any cycle are maximum: one can check that an edge is maximum in a cycle  $C$  if and only if it is maximum in each triangle that it forms with a vertex of  $C$ . Knowing the maximum edges of each cycle of the graph, we can determine  $M_\omega$  using the following version of the greedy algorithm. Start with the complete graph  $K_n$  and break its cycles successively: at each step pick an existing cycle, and remove one of its maximum edges. The trees which can result by applying this procedure are precisely the  $\omega$ -minimum spanning trees [8]. Therefore  $f$  maps a fixed tree type class of  $\mathcal{T}_n$  to a fixed  $M_\omega$ -equivalence class; *i.e.*, a fixed cell in the coarse subdivision of  $\tilde{\mathcal{B}}(K_n)$ .

Conversely, suppose we are given  $M_\omega$  (which has no loops) and we want to determine the combinatorial tree type of  $f^{-1}(\omega)$ . Consider the edges  $\{e, f, g\}$  of any triangle in  $K_n$ ; we can find out whether  $e$  is maximum in this triangle as follows. Take a minimum spanning tree  $T$  containing  $e$ . Either  $T \setminus e \cup f$  or  $T \setminus e \cup g$  is a spanning tree; assume it is the first. If  $T \setminus e \cup f$  is a minimum spanning tree, then  $\omega(e) = \omega(f)$ , and  $e$  is maximum in the triangle. Otherwise  $\omega(e) < \omega(f)$  and  $e$  is not maximum in the triangle. Determining this information for each triangle tells us, for each triple of leaves, which one (if any) branched off first in the corresponding tree. It is easy to reconstruct the combinatorial type of the tree from this data, in the same way that one recovers an equidistant tree from its corresponding ultrametric [12, Theorem 7.2.5].  $\square$

The link of the origin in the coarse subdivision of  $\mathcal{T}_n$ , which we call  $T_n$ , is a simplicial complex which has appeared in many different contexts. It was first considered by Boardman [5], and also studied by Readdy [10], Robinson and Whitehouse [11], Sundaram [16], Trappmann and Ziegler [17], Vogtmann [18], and Wachs [19], among others. By Theorem 3.5, the link of the origin in the fine subdivision of  $\tilde{\mathcal{B}}(K_n)$  is the order complex of the partition lattice  $\Pi_n$ . We conclude the following result.

**Corollary 4.6.** *The order complex of the partition lattice  $\Pi_n$  is a subdivision of the complex  $T_n$ .*

This provides a new explanation of the known result [10, 11, 16, 17, 18, 19] that these two simplicial complexes are homotopy equivalent; namely, they have the homotopy type of a wedge of  $(n-1)!$   $(n-3)$ -dimensional spheres.

Let us now revisit the example of the last section. In Figure 4 we show the Bergman complex  $\mathcal{B}(K_4)$ , with some of the corresponding trees. We now know that this is a subdivision of  $T_4$ , the link of the origin in the space of phylogenetic trees with 4 leaves, which is the Petersen graph. The three extra vertices in the fine

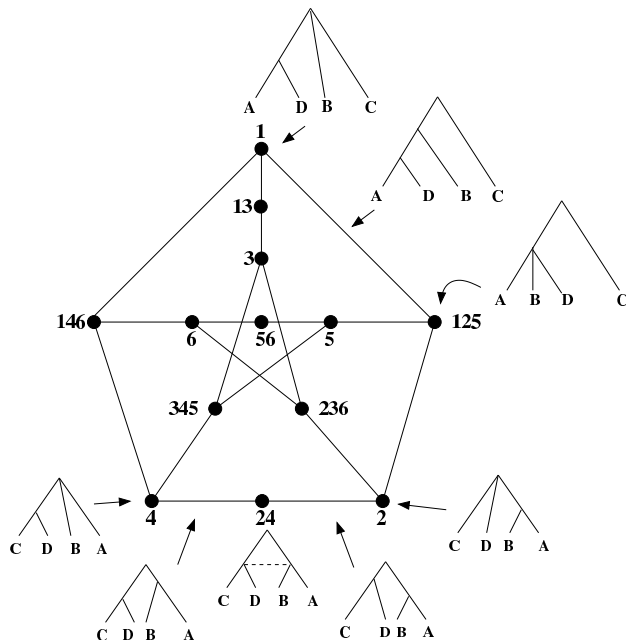


FIGURE 4. The fine subdivision of  $\mathcal{B}(K_4)$ .

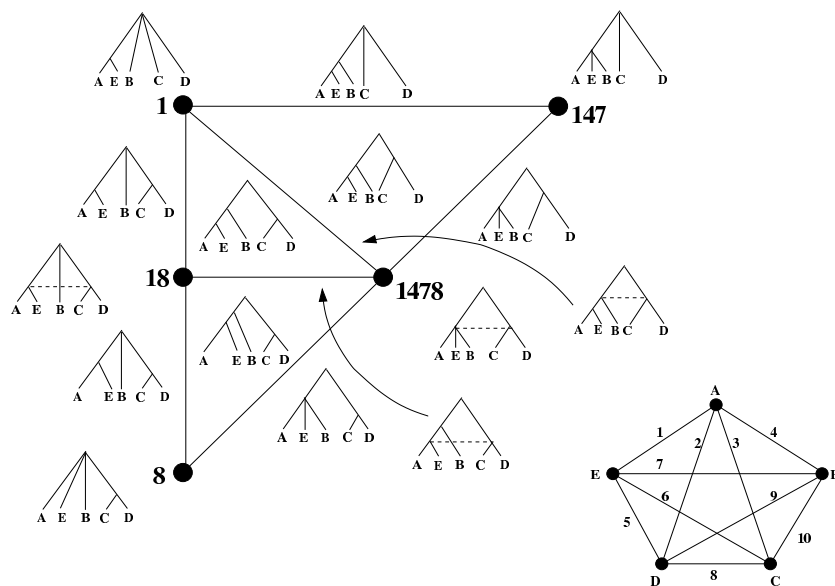


FIGURE 5. A piece of the fine subdivision of  $\mathcal{B}(K_5)$ .

subdivision are 13, 24 and 56. The tree corresponding to vertex 24 of the fine subdivision has the property that the vertex joining the leaves  $C$  and  $D$  is at the same height as the vertex joining the leaves  $A$  and  $B$ . This information is not captured by the combinatorial type of the tree; *i.e.*, by the coarse subdivision.



In Figure 5, we show a representative piece of the fine subdivision of the space of trees with 5 leaves, with  $K_5$  labeled as shown.

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## 6. Appendix: Matroids

**Definition 6.1.** A *matroid*  $M$  on a finite ground set  $E$  is a collection of subsets  $\mathcal{I}$  such that:

- (1)  $\emptyset \in \mathcal{I}$
- (2) If  $I_1 \in \mathcal{I}$  and  $I_2 \subseteq I_1$  then  $I_2 \in \mathcal{I}$ .
- (3) If  $I_1, I_2 \in \mathcal{I}$  and  $|I_1| < |I_2|$  then there exists an element  $x \in I_2 - I_1$  such that  $I_1 \cup x \in \mathcal{I}$ .

Elements of  $\mathcal{I}$  are referred to as the *independent sets* of  $M$

For example, let  $V$  be a finite set of vectors in  $\mathbb{R}^n$ . Let  $\mathcal{I}$  be the collection of all linearly independent subsets of  $V$ . Then  $\mathcal{I}$  is the collection of independent sets of a matroid on the ground set  $V$ . This fundamental example motivates many concepts and results in matroid theory.

First we review some standard matroid concepts. A *basis* is a maximal independent set. A *circuit* is a minimal dependent set and we call a one element dependent set a *loop*.

**Definition 6.2.** The *rank* of a subset  $X \subset E$ ,  $r(X)$ , is the size of the largest independent set in  $X$ . A set  $X$  is a *flat* if for all elements  $e \in E - X$ ,  $r(X \cup e) > r(X)$ . The *closure*  $\overline{X}$  of a subset  $X$  of  $E$  is the minimal flat that contains it. The poset of flats ordered by containment is a lattice, which we call the *lattice of flats*,  $L_M$ .

**Definition 6.3.** The *order complex* of a finite poset  $P$  is the simplicial complex  $\Delta(P) = \{C \subset P \mid C \text{ is a chain of } P\}$ .

**Definition 6.4.** The *matroid polytope* of  $M$  is the polytope  $P_M$  in  $\mathbb{R}^E$  with vertex set  $\{e_{b_1} + \dots + e_{b_r} \mid \{b_1, \dots, b_r\} \text{ is a basis of } M\}$ , where  $e_i$  is the  $i$ -th unit vector.

**Definition 6.5.** For a subset  $T \subset E$ , the *restriction of  $M$  to  $T$* , or *deletion of  $E - T$  from  $M$* , is the matroid on the ground set  $T$ , whose rank function is  $r_{M|T}(X) = r_M(X)$  for  $X \subseteq T$ . This matroid is denoted  $M|T$  or  $M \setminus (E - T)$ .

**Definition 6.6.** For a subset  $T \subset E$ , the *contraction of  $T$  from  $M$*  is the matroid on the ground set  $E - T$ , whose rank function is  $r_{M/T}(X) = r_M(X \cup T) - r_M(T)$ . This matroid is denoted  $M/T$ .

**Definition 6.7.** Given two matroids  $M_1$  and  $M_2$  on disjoint sets  $E_1$  and  $E_2$ , there is a matroid  $M_1 \oplus M_2$  on the ground set  $E_1 \cup E_2$ , called the *direct sum* of  $M_1$  and  $M_2$ . Its bases are the sets of the form  $B_1 \cup B_2$ , where  $B_1$  and  $B_2$  are bases of  $M_1$  and  $M_2$  respectively. Its flats are the sets of the form  $F_1 \cup F_2$ , where  $F_1$  and  $F_2$  are flats of  $M_1$  and  $M_2$  respectively.

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