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Sign balance for finite groups of Lie type (Extended Abstract)

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Abstract. *A product formula for the parity generating function of the number of $1'$s in invertible matrices over $\mathbb{Z}_2$ is given. The computation is based on algebraic tools such as the Bruhat decomposition. The same technique is used to obtain a parity generating function also for symplectic matrices over $\mathbb{Z}_2$. We present also a generating function for the sum of entries modulo p of matrices over $\mathbb{Z}_p$. This formula is a new appearance of the Mahonian distribution.*

1. Introduction

Let G be a subgroup of $GL_n(\mathbb{Z}_2)$. For every $K \in G$ define $o(K)$ to be the number of 1's in K. A natural problem is to find the number of matrices with a given number of 1's, or in other words, to compute the following generating function:

$$O(G,t) = \sum_{K \in G} t^{o(K)}.$$

It is not hard to see that in the case $G = GL_n(\mathbb{Z}_2)$, $O(G,t)$ has $n!$ as a factor but the complete generating function can be rather hard to compute. A weaker variation of this problem is to evaluate $O(G,-1)$. This is equivalent to determining the difference between the numbers of even and odd matrices, where a matrix is called *even* if it has an even number of 1's and *odd* otherwise. The number $O(G,-1)$ will be called the *parity difference* or the *imbalance* of G. A set S is called *sign-balanced* if $O(S,-1) = 0$.

The notion of sign-balance has recently reappeared in a number of contexts. Simion and Schmidt [**9**] proved that the number of 321-avoiding even permutations is equal to the number of such odd permutations if n is even, and exceeds it by the Catalan number $C_{\frac{1}{2}(n-1)}$ otherwise. Adin and Roichman [**1**] refined this result by taking into account the maximum descent. In a recent paper [**11**], Stanley established the importance of the sign-balance.

In this work we calculate the parity difference for $GL_n(\mathbb{Z}_2)$ as well as for the symplectic groups $Sp_{2n}(\mathbb{Z}_2)$.

We also generalize the problem of sign-balance to matrix groups over prime fields other than $\mathbb{Z}_2$. It turns out that the appropriate parameter for these fields is the sum of non zero entries of the matrix (mod p) rather than just the number of nonzero elements. A generalization of these results to groups over arbitrary finite fields has also been done. It will be published in a future publication.

Another aspect of this work is the occurrence of the Mahonian distribution in our results. Recall that a permutation statistic over S_n is called Mahonian if it has the same distribution as the number of inversions. MacMahon proved that major index has such distribution, explicitly:

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!$$

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where $[n]_q = \frac{1-q^n}{1-q}$.

Our results can also be seen as an example of the *cyclic sieving phenomenon* (See [**8**] for details). They also hint at the existence of permutation statistics theory for finite groups of Lie type. For a pioneering results in this direction see [**5**].

We finally note that another approach to the case of type A was proposed to us by Alex Samorodnitzky [**10**], and will be explained elsewhere.

2. Preliminaries

2.1. The groups of Lie Type A. Let $\mathbb{F}$ be any field and let $G = GL_n(\mathbb{F})$ be the group of invertible $n \times n$ matrices over $\mathbb{F}$. Let H be the subgroup of G consisting of the diagonal matrices. This is a choice of a torus in G. It is easy to show that the normalizer of H, $N(H)$, is the group of monomial matrices (where each row and column contains exactly one non-zero element). The quotient $N(H)/H$ is called the *Weyl group of type A*, and is isomorphic to S_n, the group of permutations on n letters. The *Borel subgroup* $\mathbb{B}^+$ of the group G consists of the upper triangular matrices in G. The *opposite Borel subgroup*, consisting of the lower triangular matrices, is denoted by $\mathbb{B}^-$. We denote by $\mathbb{U}^+$ and $\mathbb{U}^-$ the groups of upper and lower triangular matrices (respectively,) with 1's along the diagonal.

We finish this section with the following:

Proposition 2.1. *(See for example* [**6**, p.20]*) For every finite field $\mathbb{F}$ with q elements the order of $GL_n(\mathbb{F})$ is*

$$q^{\binom{n}{2}}(q-1)^n[n]_q!$$

2.2. Lie Type C. Let J denote the $n \times n$ matrix

$$\begin{pmatrix} 0 & \cdot & \cdot & 1 \\ 0 & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 0 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}.$$

The Lie group of type C, or the symplectic group, is defined over the field $\mathbb{F}$ by:

$$Sp_{2n}(\mathbb{F}) = \{AT \in SL_{2n}(\mathbb{F}) \mid A^T M A = M\}.$$

This is the set of fixed points of the automorphism $\varphi : SL_{2n}(\mathbb{F}) \longrightarrow SL_{2n}(\mathbb{F})$ given by: $\varphi(A) = M^{-1}(A^T)^{-1}M$.

An alternative way to present the symplectic group is the following: We define first a bilinear form on $\mathbb{F}^{2n}$:

Definition 2.1. For every $x = (x_1, ..., x_{2n}), y = (y_1, ..., y_{2n}) \in \mathbb{F}^{2n}$

$$B(x, y) = \sum_{i=1}^{n} x_i \cdot y_{2n+1-i} - \sum_{i=n+1}^{2n} x_i \cdot y_{2n+1-i}.$$

Denoting by $\{x_1, ..., x_{2n}\}$ the set of columns of X it is easy to see that $X \in Sp_{2n}(\mathbb{F})$ if and only if the columns satisfy the following set of equations:

$$B(v_i, v_j) = \begin{cases} (-1)^{i-j} & i+j = 2n+1 \\ 0 & i+j \neq 2n+1 \end{cases}$$

We end this section with the following well known fact:

Proposition 2.2. *(See for example* [**6**, p.35]*)*
For every finite field $\mathbb{F}$ with q elements the order of $Sp_{2n}(\mathbb{F})$ is:

$$q^{n^2}(q-1)^n[2]_q\cdots[2n]_q$$

2.3. The Bruhat Decomposition for type A. The Bruhat decomposition is a way to write every invertible matrix as a product of two triangular matrices and a permutation matrix. We start with the following definitions:

Recall from Section 2.1 the definition of the Borel subgroup $\mathbb{B}^+$ and the unipotent subgroup $\mathbb{U}^-$. For every permutation $\pi \in S_n$ we identify π with the matrix:

$$[\pi]_{i,j} = \begin{cases} 1 & i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

Define for every $\pi \in S_n$:

$$\mathbb{U}_\pi = \mathbb{U}^- \cap (\pi \mathbb{U}^- \pi^{-1}).$$

$\mathbb{U}_\pi$ consists of the matrices with 1-s along the diagonal and zeros in place (i,j) whenever $i < j$ or $\pi^{-1}(i) < \pi^{-1}(j)$. This is an affine space of dimension $\binom{n}{2} - \ell(\pi)$ over $\mathbb{F}$. ($\ell(\pi)$ is the length of π with respect to the Coxeter generators).

Now, given $g \in G$, we can column reduce g by multiplying on the right by Borel matrices in order to get an element gb^{-1} satisfying the following condition:

$$(*) \qquad \begin{array}{l} \text{The right most nonzero entry in each row is 1} \\ \text{and it is the first nonzero entry in its column.} \end{array}$$

Those "leading entries" form a permutation matrix corresponding to $\pi \in S_n$.

Now we can use π^{-1} to rearrange the columns of gb^{-1} in order to get $gb^{-1}\pi^{-1} = u \in \mathbb{U}_\pi$, i.e., $g = u\pi b$. This is called the *Bruhat decomposition* of the matrix g. One can prove that this decomposition is unique, and thus we have a partition of G into double cosets indexed by the elements of the Weyl group S_n.

If $\pi \in S_n$ then the double coset indexed by π decomposes into left $\mathbb{B}^+$-cosets in the following way: For every choice of $u \in \mathbb{U}_\pi$, $u\pi$ is a representative of the left coset $u\pi\mathbb{B}^+$. Thus a general representative of the double coset $\mathbb{U}_\pi$ can be taken as matrix of the form $(*)$, with every column filled with free parameters beyond the leading 1.

We summarize the information we gathered about the Bruhat decomposition for type A in the following:
Proposition 2.3. *The group $GL_n(\mathbb{F})$ is a disjoint union of double cosets of the form $\mathbb{U}_\pi\pi\mathbb{B}^+$, where π runs through S_n. Every double coset decomposes into cosets of the form $A\mathbb{B}^+$ where A is a general representative of the form $(*)$. The number of free parameters in A is equal to $\binom{n}{2} - \ell(\pi)$.*

Here is an example of the coset decomposition for $GL_3(\mathbb{Z}_2)$:

$$\mathbb{U}_1 1\mathbb{B}^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \mathbb{B}^+ \mid \alpha,\beta,\gamma \in \mathbb{Z}_2 \right\}$$

$$\mathbb{U}_{s_1} s_1\mathbb{B}^+ = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta & 1 \end{pmatrix} \mathbb{B}^+ \mid \alpha,\beta \in \mathbb{Z}_2 \right\}$$

$$\mathbb{U}_{s_2} s_2\mathbb{B}^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 0 & 1 \\ \beta & 1 & 0 \end{pmatrix} \mathbb{B}^+ \mid \alpha,\beta \in \mathbb{Z}_2 \right\}$$

$$\mathbb{U}_{s_2s_1}s_2s_1\mathbb{B}^+ = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 1 & 0 & 0 \end{pmatrix} \mathbb{B}^+ \mid \alpha \in \mathbb{Z}_2 \right\}$$

$$\mathbb{U}_{s_1s_2}s_1s_2\mathbb{B}^+ = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} \mathbb{B}^+ \mid \alpha \in \mathbb{Z}_2 \right\}$$

$$\mathbb{U}_{s_1s_2s_1}s_1s_2s_1\mathbb{B}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbb{B}^+$$

2.4. Bruhat Decomposition for Type C. In order to be able to present the Bruhat decomposition for type C, we must first define a Borel subgroup for $Sp_{2n}(\mathbb{F})$. We present this subject following [**11**]. Note that although the exposition of [**11**] deals with groups over algebraically closed fields, the results hold also over finite fields. Start with the Borel subgroup $\mathbb{B}^+$, chosen for type A, consisting of the upper triangular matrices.

If $X = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) \in \mathbb{B}^+$, then $\varphi(X) = \left(\begin{smallmatrix} J(C^T)^{-1}J & J(C^T)^{-1}B^T(A^T)^{-1}J \\ 0 & J(A^T)^{-1}J \end{smallmatrix}\right) \in \mathbb{B}^+$. (The automorphism φ was defined in Section 2.2). Moreover, the automorphism φ keeps the Borel subgroup $\mathbb{B}^+$, as well as the groups of diagonal and monomial matrices (denoted by H and $N(H)$ respectively in Section 2.1) invariant. Thus we can take $\mathbb{B}_C^+ = Sp_{2n}(\mathbb{F}) \cap \mathbb{B}^+$ and $\mathbb{B}_C^- = Sp_{2n}(\mathbb{F}) \cap \mathbb{B}^-$ as the Borel subgroup and the opposite Borel subgroup of $Sp_{2n}(\mathbb{F})$ respectively, and similarly for H and $N(H)$.

The Weyl group of type C can be realized as the group of those permutations $\pi \in S_{2n}$ such that $\pi(2n+1-i) = 2n+1-\pi(i)$. This group is isomorphic to the hyperoctahedral group B_n (See definition in Section **??**). The isomorphism can be seen by labelling the basis elements of the space on which $Sp_{2n}(\mathbb{F})$ acts by indices $n, n-1, ..., 1, ..., -n$.

We define also the groups $\mathbb{U}_C^+ = \mathbb{U}^+ \cap Sp_{2n}(\mathbb{F})$ and $\mathbb{U}_C^- = \mathbb{U}^- \cap Sp_{2n}(\mathbb{F})$ to be the upper and lower unipotent subgroups respectively. For every $\pi \in B_n$ we define $\mathbb{U}_\pi^C = \mathbb{U}_C^- \cap (\pi \mathbb{U}_C^- \pi^{-1})$. $\mathbb{U}_\pi^C$ is the intersection of $Sp_{2n}(\mathbb{F})$ with the set of matrices with 1's along the diagonal and zeros at entries in location (i,j) whenever $i < j$ or $\pi^{-1}(i) < \pi^{-1}(j)$. This is an affine space of dimension $n^2 - \ell(\pi)$. (Here, $\ell(\pi)$ is the length function of B_n).

Now, we can use the Bruhat decomposition of $GL_{2n}(\mathbb{F})$ to produce the Bruhat decomposition for $Sp_{2n}(\mathbb{F})$. Let $g \in Sp_{2n}(\mathbb{F})$. Consider g as an element of $GL_{2n}(\mathbb{F})$ and write $g = u\pi b$ where $\pi \in S_{2n}$, $u \in \mathbb{U}_\pi$ and $b \in \mathbb{B}^+$. We have:

$$g = \varphi(g) = \varphi(u)\varphi(\pi)\varphi(b),$$

but from the uniqueness of the decomposition in $GL_{2n}(\mathbb{F})$ we have:

$$\varphi(u) = u, \quad \varphi(\pi) = \pi h^{-1}, \quad \varphi(b) = hb$$

where h is diagonal and thus $\pi \in B_n$ and $b \in \mathbb{B}_C^+$. This gives us the Bruhat decomposition. The description of the double cosets and the coset representatives is similar to the one given for type A, with the exception that here we have to intersect with $Sp_{2n}(\mathbb{F})$.

We summarize the information we gathered about the Bruhat decomposition for type C in the following:

Proposition 2.4. *The group $Sp_{2n}(\mathbb{F})$ decomposes into double cosets of the form $\mathbb{U}_\pi^C \pi \mathbb{B}_C^+$, where π runs through B_n. Every double coset decomposes into cosets of the form $A\mathbb{B}_C^+$ where A is a general representative of the form (∗). The number of free parameters in A is equal to $n^2 - \ell(\pi)$.*

3. Sign Balance for Type A

3.1. Sign Balance over $\mathbb{Z}_2$. In this section we present the results concerning the imbalance of the groups of type A over the field $\mathbb{Z}_2$. The proofs are written in **??**.

Theorem 3.1.
$$\sum_{K\in GL_n(\mathbb{Z}_2)} (-1)^{o(K)} = -2^{\binom{n}{2}}[n-1]_2!$$
where $[k]_q = \frac{1-q^k}{1-q}$.

The following corollary is immediate:

Corollary 3.1. *The number of even matrices in* $GL_n(\mathbb{Z}_2)$ *is exactly*
$$[n-1]_2!2^{\binom{n}{2}}(2^{n-1}-1)$$
while the number of odd matrices in $GL_n(\mathbb{Z}_2)$ *is:*
$$[n-1]_2!2^{\binom{n}{2}+n-1}$$
□

3.2. Sign Balance for Prime Fields. In this section we present the results concerning the imbalance of the groups of type A over the field $\mathbb{Z}_p$. The proofs are written in **??**.

Let p be a prime number and denote by $\mathbb{Z}_p$ the field with p elements. The results of Section 3.1 can be extended to invertible matrices over the field $\mathbb{Z}_p$, provided we substitute a primitive complex p-th root of unity in the generating function of the sum of elements of a matrix mod p. Explicitly, we use the information we gathered in the previous section to get the following:

Theorem 3.2.
$$\sum_{K\in GL_n(\mathbb{Z}_p)} \omega_p^{s(K)} = -(p-1)^{n-1}p^{\binom{n}{2}}[n-1]_p!.$$

where $s(K)$ *is the sum (mod p) of the elements of the matrix* K*, and* ω_p *is a primitive complex* p*-th root of unity.*

The following corollary is immediate:

Corollary 3.2. *The number of matrices in* $GL_n(\mathbb{Z}_p)$ *whose sum of entries modulo* p *is* 0 *is exactly*
$$[n-1]_p!(p-1)^{n-1}p^{\binom{n}{2}}(p^{n-1}-1)$$
while for every $1\le i\le p-1$*, the number of matrices in* $GL_n(\mathbb{Z}_p)$ *whose entries add up to* i *modulo* p *is:*
$$[n-1]_p!(p-1)^{n-1}p^{\binom{n}{2}+n-1}.$$
□

4. Sign Balance for Type C

In this section we prove the following result:

Theorem 4.1.
$$\sum_{K\in Sp_{2n}(\mathbb{Z}_2)} (-1)^{o(K)} = -2^{n^2}\cdot[2]_2[4]_2\cdots[2n-2]_2$$

The following corollary is immediate:

Corollary 4.1. *The number of even matrices in* $Sp_{2n}(\mathbb{Z}_2)$ *is exactly*
$$2^{n^2-1}[2]_2\cdots[2n-2]_2([2n]_2-1)$$
while the number of odd matrices is
$$2^{n^2-1}[2]_2\cdots[2n-2]_2([2n]_2+1).\qquad\square$$

In order to prove the theorem, we take the following direction: Instead of summing over the whole group of matrices, we sum over every coset separately. It turns out that some of the cosets are sign-balanced, while the others have only odd matrices. We start with the following definition:

Definition 4.2. A coset consisting entirely of odd matrices is called an *odd coset.*

The following lemma identifies the sign-balanced cosets.

Lemma 4.2. *Let A be a general representative of the double coset $U_\pi^C \pi \mathbb{B}_C^+$ corresponding to $\pi \in B_n$. Make some substitution in the free parameters of A to get a coset representative, and call it $\tilde{A}$. If $\tilde{A}$ has an odd column which is not the last one, then the coset $[\tilde{A}] = \{\tilde{A}B | B \in \mathbb{B}_C^+\}$ is sign-balanced, i.e.,*

$$\sum_{K \in \tilde{A}\mathbb{B}_C^+} (-1)^{o(K)} = 0.$$

Proof. An element of $\mathbb{B}_C^+$ is an invertible upper triangular matrix which is also symplectic. The condition of being symplectic is expressed by imposing a set of equations on the columns of the matrix. If we take b to be an upper triangular matrix with a set of columns $\{v_1, ..., v_{2n}\}$ then, as was stated in Section 2.2, forcing it to be symplectic is equivalent to imposing the equations (note that we are working over $\mathbb{Z}_2$):

$$B(v_i, v_j) = \begin{cases} 1 & i + j = 2n + 1 \\ 0 & i + j \neq 2n + 1 \end{cases}$$

As is easy to check, the equations of the form $B(v_i, v_i) = 0$ are trivial over $\mathbb{Z}_2$. The equations of the form $B(v_i, v_{2n+1-i}) = 1$ are also trivial. (Indeed, $B(v_i, v_{2n+1-i}) = \sum_{k=1}^{2n} b_{k,i} \cdot b_{2n+1-k,2n+1-i}$ but since b is upper triangular, over $\mathbb{Z}_2$ we have $b_{ii} \cdot b_{2n+1-i,2n+1-i} = 1$ and the other summands vanish since for $k > i$ one has $b_{ki} = 0$ and for $k > 2n + 1 - i$ one has $b_{2n+1-k,2n+1-i} = 0$).

Now, the only non trivial equations involving the parameters appearing in the last column are the ones of the form:

$$B(v_i, v_{2n}) = 0, (2 \leq i \leq 2n - 1)$$

and each such equation can be written in such a way that the parameters of the last column are free while the parameters of the first row depend on them. Explicitly, we write the equation $B(v_i, v_{2n}) = 0$ as

$$b_{1i} = \sum_{k=2}^{2n} b_{ki} \cdot b_{2n+1-k,2n}.$$

Note that the elements of the last column of the matrix b have no appearance as a part of a linear combination in any place other than the first row. This is justified by the fact that every non trivial equation, involving the first row, which we have not treated yet must be of the form $B(v_i, v_j) = 0$ for $1 \leq i < j \leq 2n - 1$. Thanks to the upper triangularity of b, the elements laying in the first row vanish in these equations.

Let us look at the following example:

$$b = \begin{pmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The only non trivial equations involving the last column are: $B(v_2, v_4) = 0$ and $B(v_3, v_4) = 0$.

These equations can be written as:

$$b_{12} = b_{34}$$
$$b_{13} = b_{24} + b_{23} \cdot b_{34}$$

so after intersecting with $Sp_{2n}(\mathbb{Z}_2)$, the matrix b looks like:

$$b = \begin{pmatrix} 1 & b_{34} & b_{24} + b_{23} \cdot b_{34} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The elements of the last column appear only in the first row and in the equations of the form $B(v_i, v_j) = 0$ the elements located in the first row vanish.

Note that in this case all of the parameters outside the first row are free. This doesn't hold in general. Nevertheless, as we have proven, we can arrange the parameters such that the elements of the last column reappear only in the first row.

Returning now to the proof, we have two cases:

- The first column of $\tilde{A}$ is odd. In this case we can use the element located in place $(1, 2n)$ to construct a bijection between odd and even matrices inside the coset $\tilde{A}\mathbb{B}_C^+$. This is done in the same way described earlier for type A: Divide $\mathbb{B}_C^+$ into two disjoint subsets:

  $$\mathbb{B}_{C0}^+ = \{T = (t_{i,j}) \in \mathbb{B}_C^+ \mid t_{1,2n} = 0\}$$

  $$\mathbb{B}_{C1}^+ = \{T = (t_{i,j}) \in \mathbb{B}_C^+ \mid t_{1,2n} = 1\}.$$

  For every matrix $X \in \tilde{A}\mathbb{B}_C^+$, the k-th column of X is a linear combination of the first k columns of $\tilde{A}$. Now, due to the fact that the parameter appearing in the location $(1, 2n)$ has no other appearance, for every $B \in \mathbb{B}_{C0}^+$ there is some $B' \in \mathbb{B}_{C1}^+$ such that B and B' differ only in the entry numbered $(1, 2n)$.

  Note that $\tilde{A}B$ and $\tilde{A}B'$ are obtained from $\tilde{A}$ by the same sequence of column operations except for the first column which was used in producing AB but was not used in producing $\tilde{A}B'$. Hence $\tilde{A}B$ and $\tilde{A}B'$ have opposite parity. This gives us a bijection between the odd and the even matrices of the coset $A\mathbb{B}_C^+$.
- The first column of $\tilde{A}$ is even. Denoting by j the number of the first odd column of $\tilde{A}$, we use the element located in place $(j, 2n)$ to construct a bijection between the odd and even matrices inside $\tilde{A}\mathbb{B}_C^+$ in the same way as in the previous case. Note that since the element located in place $(j, 2n)$ in the matrices of the Borel subgroup can reappear only in the first row, it affects only the first column of $\tilde{A}$, which is even.

□

We turn now to treat the odd cosets.

Lemma 4.3. *Let $\pi \in B_n$. Let A be a general representative of the double coset $U_\pi \pi \mathbb{B}_C^+$ corresponding to $\pi \in B_n$. Make some substitution in the free parameters of A to get a coset representative, and call it $\tilde{A}$. If all of the first $2n-1$ columns of $\tilde{A}$ are even then all of the matrices belonging to the coset $\tilde{A}\mathbb{B}_C^+$ are odd. The imbalance calculated inside this coset is:*

$$\sum_{K \in \tilde{A}\mathbb{B}_C^+} (-1)^{o(K)} = -|\mathbb{B}_C^+| = -2^{n^2}.$$

Proof. The last column of $\tilde{A}$ is always odd and thus since all other columns of $\tilde{A}$ are even, $\tilde{A}$ itself is an odd matrix and the same holds for $\tilde{A}B$ for every $B \in \mathbb{B}_C^+$. The size of the coset $\tilde{A}\mathbb{B}^+$ is 2^{n^2}, and the result follows. □

Lemma 4.4. *Let $\pi \in B_n$. The double coset $U_\pi^C \pi \mathbb{B}_C^+$ contains odd cosets if and only if $\pi(2n) = 2n$.*

PROOF. Let A be a general representative of the double coset $U_{\pi}^{C}\pi\mathbb{B}_{C}^{+}$. Write $U = A\pi^{-1}$. Then $U \in \mathbb{U}_{\pi}^{C}$ is a lower triangular matrix and since $\pi(2n) = 2n$ (which implies also $\pi(1) = 1$), the first column as well as the last row of U contain $2n-1$ parameters. Note that $U^T \in \mathbb{B}_{C}^{+}$ and thus by the considerations described in Lemma 4.2, the parameters appearing in the last column of U^T can reappear only in the first row of U^T. We conclude that the parameters of the last row of U can reappear only in the first column of U. Now, for every column numbered $2 \le k \le 2n-1$ in U and for every choice of the first elements of the column numbered k, we are free to choose the parameter located in the bottom of this column, $(2n, k)$, in such a way that the column will be even. The parameter located in the place $(2n, 1)$ has no other appearance and thus we can choose all of the first $2n-1$ columns of U to be even. Getting back to the general representative A, since $\pi(2n) = 2n$, we have also $\pi(1) = 1$ and thus A and U differ only in the columns $1 < k < 2n$ so that the proof works also for A.

On the other hand, if $\pi(2n) \neq 2n$ then π contains a column numbered $k < 2n$ which has only one nonzero element, located in place $(2n, k)$. By the construction of the general representative A, there are only zeros above the 1 coming from the permutation and thus this odd column appears also in A. By the previous lemma, the coset $\{\tilde{A}B | B \in \mathbb{B}_{C}^{+}\}$ is sign-balanced. $\square$

Now, we have to count the imbalance on the odd cosets. By Lemma 4.4 we are interested only in the double cosets corresponding to the permutations $\pi \in B_{n-1}$. The following lemma shows how to count.

Lemma 4.5. *Let $\pi \in B_n$ such that $\pi(n) = n$. The double coset $\mathbb{U}_{\pi}^{C}\pi\mathbb{B}_{C}^{+}$ contains exactly $2^{(n-1)^2-\ell(\pi)}$ odd cosets.*

PROOF. Let A be representative of the double coset $\mathbb{U}_{\pi}^{C}\pi\mathbb{B}_{C}^{+}$. As was shown in the previous lemma, the parity of a each one of the first $2n-1$ columns of A is determined by the free parameter in its bottom. Since there are a total of $n^2 - \ell(\pi)$ free parameters and exactly $2n-1$ 'bottom parameters', the number of substitutions of parameters giving all of the $2n-1$ first columns even is $2^{n^2-\ell(\pi)-(2n-1)}$. This is also the number of odd cosets in the double coset $\mathbb{U}_{\pi}^{C}$. $\square$

$\square$

We turn now to the proof of Theorem 4.1. In order to calculate the imbalance we have to count only odd cosets. By Lemma 4.4, we are interested only in the double cosets corresponding to permutations $\pi \in B_{n-1}$. By Lemma 4.5, every such double coset contains $2^{(n-1)^2-\ell(\pi)}$ odd cosets. By Lemma 4.3, each odd coset contributes -2^{n^2} to the imbalance, and we have in total:

$$
\begin{aligned}
\sum_{K \in Sp_{2n}(\mathbb{Z}_2)} (-1)^{o(K)} &= \sum_{\substack{\pi \in B_n \\ \pi(n)=n}} -2^{n^2} \cdot 2^{(n-1)^2-\ell(\pi)} \\
&= -2^{n^2} \sum_{\pi \in B_{n-1}} 2^{(n-1)^2-\ell(\pi)} \\
&= -2^{n^2} \sum_{\pi \in B_{n-1}} 2^{\ell(\pi)} \\
&= -2^{n^2} [n-1]_2! \\
&= -2^{n^2} \cdot [2]_2 [4]_2 \cdots [2n-2]_2
\end{aligned}
$$

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