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Negative-descent representations for Weyl groups of type D

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Abstract. *We introduce a monomial basis for the coinvariant algebra of type D, that allows us to define a new family of representations of D_n. We decompose the homogeneous components of the coinvariant algebra into a direct sum of these representations and finally we give the decomposition of them into irreducible components. This algebraic setting is then applied to find new, and generalize various, combinatorial identities.*

Résumé. *On introduit une base monomiale de l'algèbre coinvariante de type D, ce qui nous permet de definir une nouvelle classe de representations de D_n. On decompose les composantes homogènes de l'algèbre coinvariante comme somme directe de ces representations et on decrit leur decomposition en modules irreductibles. Ce contexte algebrique est finalement utilisé pour decouvrir des nouvelles identités combinatoires.*

1. Introduction

Let W be one of the classical Weyl groups A_{n-1}, B_n or D_n and let I_n^W be the ideal of the polynomial ring $\mathbf{P}_n := \mathbf{C}[x_1, \ldots, x_n]$ generated by costant-term-free W-invariant polynomials. The quotient $R(W) := \mathbf{P}_n/I_n^W$ is called the coinvariant algebra of W and it's well know that it has dimension $|W|$ as a $\mathbf{C}$-vector space. The problem of finding a basis for the coinvariant algebra has been studied by a number of mathematicians (see, e.g., [**3, 4**], [**5**]). Garsia and Stanton presented a descent basis for a finite dimensional quotient of the Stanley-Reisner ring arising from a finite Weyl group (see [**10**]). For type A, unlike for other types, this quotient is isomorphic to $R(W)$ and in this case the basis elements are monomials of degree equal to the "major index" (maj) of the indexing permutation. On the other hand it is well known that $R(W)$ affords the left regular representation of W (see e.g., [**11**]), i.e. the multiplicity of each irreducible representation is its dimension. Moreover, the action of W preserves the natural grading induced from that of $\mathbf{P}_n$ by total degree, and so it is natural to ask about the multiplicity of each irreducible representation of W in the k-th homogeneous component R_k^W. In the case of the symmetric group S_n, the answer is given by a well known theorem, due independently to Kraskiewicz and Weymann [**13**] and Stanley [**18**], that expresses the multiplicity of the irreducible S_n-representations in $R_k^{S_n}$ in terms of the statistic maj defined on standard Young tableaux (SYT).

For type B these problems have been studied by Adin, Brenti and Roichman in [**1**]. They provide a descent basis of $R(B)$ and an extension of the construction of Solomon's descent representations (see [**17**]) for this type.

In this extended abstract we show how to extend these results to the Weyl groups of type D. We construct an analogue of the descent basis for the coinvariant algebra of type D via a Straightening Lemma. The basis elements are monomials of degree $Dmaj$, that is an analogous statistic of maj for D_n (see [**7**]). This basis leads to the definition of a new family of D_n-modules $R_{D,N}$, which have a basis indexed by the even-signed permutations having D and N as "descent set" and "negative set", respectively. For this reason we call them negative-descent representations. They are analogous but different from Solomon descent representations and Kazhdan-Lusztig representations (see [**12**]). We decompose $R_k^{D_n}$ into a direct sums of these $R_{D,N}$. Finally, we introduce the concept of D-standard Young bitableaux. By extending the definition of $Dmaj$ on them we give an explicit decomposition into irreducible modules of these negative-descent representations, refining a theorem of Stembridge [**20**]. This algebraic setting is then applied to obtain new multivariate combinatorial identities.

2. Notation and preliminaries

In this section we give some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} := \{1, 2, 3, \ldots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$. For $a \in \mathbf{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Given $n, m \in \mathbf{Z}$, $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$.

2.1. Statistics on Coxeter groups. We always consider the linear order on $\mathbf{Z}$

$$-1 \prec -2 \prec \cdots \prec -n \prec \cdots \prec 0 \prec 1 \prec 2 \prec \cdots \prec n \prec \cdots$$

instead of the usual ordering. Given a finite sequence $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{Z}^n$ we let

$$Inv(\sigma) := \{(i, j) \ : \ i < j \ , \ \sigma_i \succ \sigma_j\} \text{ and } inv(\sigma) := |Inv(\sigma)|.$$

The *set of descents* and the *descent number* of σ are respectively

$$Des(\sigma) := \{i \in [n-1] \ : \ \sigma_i \succ \sigma_{i+1}\} \text{ and } des(\sigma) := |Des(\sigma)|.$$

The number of descents in σ from position i on is denoted by

$$d_i(\sigma) := |\{j \in Des(\sigma) \ : \ j \geq i\}|. \tag{2.1}$$

The *major index* of σ (first defined by MacMahon in [**15**]) is

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i.$$

Note that $d_1(\sigma) = des(\sigma)$ and $\sum_{i=1}^n d_i(\sigma) = maj(\sigma)$. Moreover we let

$$Neg(\sigma) := \{i \in [n] : \sigma_i < 0\} \quad \text{and} \quad neg(\sigma) := |Neg(\sigma)|.$$

The generating function of the joint distribution of des and maj over S_n is given by the following Carlitz's Identity, (see, e.g., [**9**]). Let $n \in \mathbf{P}$. Then

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)}}{\prod_{i=0}^n (1 - tq^i)}$$

in $\mathbf{Z}[q][[t]]$, where $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$.

Let B_n be the group of all bijections β of the set $[-n, n] \setminus \{0\}$ onto itself such that $\beta(-i) = -\beta(i)$ for all $i \in [-n, n] \setminus \{0\}$, with composition as the group operation. We will usually identify $\beta \in B_n$ with the sequence $(\beta(1), \ldots, \beta(n))$ and we call this the *window* notation of β. Following [**2**] we define the *flag-major index* of $\beta \in B_n$ by $fmaj(\beta) := 2maj(\beta) + neg(\beta)$

It's known that *fmaj* is equidistributed with length on B_n and that it satisfies many other algebraic properties (see, for example, [**1**] and [**2**]).

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, i.e.

$$D_n := \{\gamma \in B_n \ : \ neg(\gamma) \equiv 0 \,(\text{mod } 2)\}.$$

Following [**7**] for $\gamma \in D_n$ we let

$$|\gamma|_n := (\gamma(1), \ldots, \gamma(n-1), |\gamma(n)|) \in B_n,$$

$$D_\gamma := Des(|\gamma|_n) \ \text{ and } \ N_\gamma := Neg(|\gamma|_n).$$

Then we define the *D-major index* of $\gamma \in D_n$ by

$$Dmaj(\gamma) := 2 \sum_{i \in D_\gamma} i + |N_\gamma|,$$

and the *D-descent number* of γ by

$$Ddes(\gamma) := 2|D_\gamma| + \eta_1(\gamma)$$

where

$$\eta_1(\gamma) := \begin{cases} 1, & \text{if } \gamma(1) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example if $\gamma = [2, -5, 3, 1, -4]$, then $D_\gamma = \{1, 3\}$ and $N_\gamma = \{2\}$ and hence $Dmaj(\gamma) = 9$ and $Ddes(\gamma) = 4$.

The statistic $Dmaj$ is Mahonian (i.e. equidistributed with length) on D_n and the generating function of the pair $(Ddes, Dmaj)$ is given by

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})} \tag{2.2}$$

in $\mathbf{Z}[q][[t]]$, (see [**7**, Theorem 4.3] for a proof).

2.2. Partitions and tableaux. A *partition* λ of a nonnegative integer n is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ and $|\lambda| := \sum_i \lambda_i = n$, denoted also $\lambda \vdash n$. We denote by λ' the conjugate partition of λ. The *dominance order* is a partial order defined on the set of partitions of a fixed nonnegative integer n as follows. Let μ and λ two partitions of n. We define $\mu \trianglelefteq \lambda$ if for all $i \geq 1$

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

A *standard Young tableau* of shape λ is obtained by inserting the integers $1, 2, \ldots, n$ (where $n = |\lambda|$) as *entries* in the cells of the Young diagram of shape λ in such a way that the entries increase along rows and columns. We denote by $SYT(\lambda)$ the set of all standard Young tableaux of shape λ. For example the tableau T in Figure 1 belongs to $SYT(5, 3, 2, 1)$.

T:=

1	3	5	8	10
2	6	7		
4	11			
9				

FIGURE 1

A *descent* in a standard Young tableau T is an entry i such that $i+1$ is strictly below i. We denote the set of descents in T by $Des(T)$. The *major index* of a tableau T is

$$maj(T) := \sum_{i \in Des(T)} i.$$

In the example in Figure 1 $Des(T) = \{1, 3, 5, 8, 10\}$ and so $maj(T) = 27$.

A *bipartition* of a nonnegative integer n is an ordered pair (λ, μ) of partitions such that $|\lambda| + |\mu| = n$ denoted by $(\lambda, \mu) \vdash n$. The *Young diagram* of shape (λ, μ) is obtained by the union of the Young diagrams of shape λ and μ by positioning the second to the south-west of the first. A *standard Young bitableau* $T = (T_1, T_2)$ of shape $(\lambda, \mu) \vdash n$ is obtained by inserting the integers $1, 2, \ldots, n$ in the corresponding Young diagram increasing along rows and columns.

Definition. Given two partitions λ, μ such that $|\lambda| + |\mu| = n$, we define a *D-standard bitableau* $T = (T_1, T_2)$ of type $\{\lambda, \mu\}$ as a standard Young bitableau of shape (λ, μ) or (μ, λ) such that n is an entry of T_1.

We let $Des(T)$ and $maj(T)$ be as above and we let $Neg(T)$ be the set of entries of T_2. The *D-major index* of a D-standard bitableau is defined by

$$Dmaj(T) := 2 \cdot maj(T) + |Neg(T)|.$$

For example T and S in Figure 2 are two D-standard bitableau of type $\{(3, 1), (2, 2, 1)\}$ and we have $Dmaj(T) = 2 \cdot 15 + 5 = 35$ and $Dmaj(S) = 2 \cdot 13 + 4 = 30$.

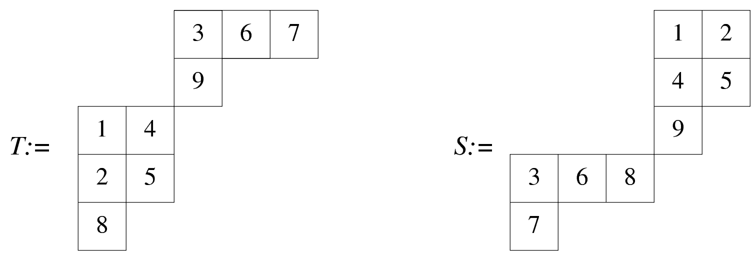

FIGURE 2

We denote by $DSYT\{\lambda, \mu\}$ the set of all D-standard bitableaux of type $\{\lambda, \mu\}$.

2.3. Irreducible representations of classical Weyl groups. Recall that the irreducible representations of the symmetric group S_n are indexed by partitions of n in a classical way (see, for example, [**19**, §7.18]) and denote S^λ the irreducible module corresponding to λ

In the case of B_n the irreducible representations are parametrized by ordered pairs of partitions such that the total sum of their parts is equal to n (see, for example, [**14**]), and we denote by $S^{\lambda,\mu}$ the irreducible module corresponding to (λ, μ).

Since D_n is a subgroup of index 2 of the Weyl group B_n, the restrictions of an irreducible representation of B_n to D_n is either irreducible, or splits up into two irreducible components. Let (λ, μ) be a pair of partitions with total size n. If $\lambda \neq \mu$ then the restrictions of the irreducible representations of B_n labeled by (λ, μ) and (μ, λ) are irreducible and equal. If $\lambda = \mu$ then the restriction of the character labeled by (λ, λ) splits into two irreducible components, which we denote by $(\lambda, \lambda)^+$ and $(\lambda, \lambda)^-$. Note that this can only happen if n is even. Hence we may denote all irreducible modules of D_n by $S^{\lambda,\mu,\epsilon}$ where λ and μ are two partitions such that $|\lambda| + |\mu| = n$, $\lambda \preceq \mu$ in some total order $\prec$ on the set of all integer partitions, and ϵ is equal to $\prec$ if $\lambda \neq \mu$ and ϵ is equal to $+$ or $-$ if $\lambda = \mu$.

3. Monomial bases of coinvariant algebras

Let $\mathbf{P}_n := \mathbf{C}[x_1, \ldots, x_n]$ and consider the natural action φ of a classical Weyl group W (with $W = A_{n-1}$, B_n, D_n) on $\mathbf{P}_n$ defined on the generators by

$$\varphi(w) : x_i \mapsto \frac{w(i)}{|w(i)|} x_{|w(i)|},$$

for all $w \in W$ and extended uniquely to an algebra homomorphism. Let I_n^W be the ideal of $\mathbf{P}_n$ generated by the elements in $\mathbf{P}_n^W$ without costant term. The quotient

$$R(W) := \mathbf{P}_n / I_n^W$$

is called the *coinvariant algebra* of W and it is well known that it has dimension $|W|$ as a $\mathbf{C}$-vector space. Moreover, W acts naturally as a group of linear operators on this space and it can be shown that this representation of W is isomorphic to the *regular representation* (see e.g., [**11**, § 3.6]). All these properties naturally lead to the problem of finding a "nice" basis for $R(W)$. A basis for the coinvariant algebra of type A has been found by Garsia and Stanton [**10**]. For $\sigma \in S_n$ they define

$$a_\sigma := \prod_{j \in Des(\sigma)} (x_{\sigma(1)} \cdots x_{\sigma(j)}).$$

It's immediate to see that $a_\sigma := \prod_{i=1}^n x_{\sigma(i)}^{d_i(\sigma)}$ where $d_i(\sigma)$ is defined in (2.1). They show that the set $\{a_\sigma + I_n^{S_n} \,:\, \sigma \in S_n\}$ is a basis of $R(S_n)$, called the *descent basis.* Note that the representatives a_σ of this basis are actually monomials with $deg(a_\sigma) = maj(\sigma)$.

Allen ([**4**]) constructed a non-monomial basis for $R(W)$ for all classical Weyl groups and Adin, Brenti and Roichman ([**1**]) defined for any $\beta \in B_n$ a monomial b_β of degree $fmaj(\beta)$ such that the set of the corresponding classes in the coinvariant algebra of type B is a linear basis of this vector space.

The first main goal of this work is to define a family of monomials, indexed by D_n, and to show that the corresponding classes form a basis of the coinvariant algebra of type D. To this end we present a straightening lemma for expanding an arbitrary monomial in $\mathbf{P}_n$ in terms of the descent basis with coefficients in $\mathbf{P}_n^{D_n}$. This algorithm is a generalization of the one presented in [**1**] for type A and B.

For $\gamma \in D_n$ and $i \in [n-1]$, we let

$$\delta_i(\gamma) :=| \{j \in D_\gamma \,:\, j \geq i\} |, \qquad \eta_i(\gamma) := \begin{cases} 1, & \text{if } \gamma(i) < 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_i(\gamma) := 2\delta_i(\gamma) + \eta_i(\gamma).$$

Note that

$$\sum_{i=1}^{n-1} h_i(\gamma) = Dmaj(\gamma) \;\text{ and }\; h_1(\gamma) = Ddes(\gamma). \tag{3.1}$$

Definition. For any $\gamma \in D_n$, we define

$$c_\gamma := \prod_{i=1}^{n-1} x_{|\gamma(i)|}^{h_i(\gamma)}.$$

For example, if $\gamma := (6, -4, -2, 3, -5, -1) \in D_6$, then $(h_1(\gamma), \ldots, h_5(\gamma)) = (6, 5, 3, 2, 1)$ and $c_\gamma = x_6^6 x_4^5 x_2^3 x_3^2 x_5^1$.

The goal of this section is to show how we can prove that the set $\{c_\gamma + I_n^D \,:\, \gamma \in D_n\}$ is a linear basis for the coinvariant algebra of type D. We call it the *negative-descent basis.* We denote by

$$f_i(x_1, \ldots, x_n) := \begin{cases} e_i(x_1^2, \ldots, x_n^2), & \text{for } i \in [n-1]; \\ x_1 \cdots x_n, & \text{for } i = n, \end{cases}$$

where e_i is the i-th elementary symmetric function. It is clear that the polynomials f_j are invariant under the action of D_n. Moreover, for any partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ with $\lambda_1 \leq n$, we define $f_\lambda := f_{\lambda_1} \cdots f_{\lambda_t}$.
Let's restrict our attention to the quotient $S := \mathbf{P}_n/(f_n)$ and we denote by $\pi : \mathbf{P}_n \to S$ the natural projection. We start by associating to any monomial $M \in S$ an even-signed permutation $\gamma(M)$ and a partition $\mu(M)$. Let M be a monomial such that $\pi(M) \neq 0$, $M = \prod_{i=1}^n x_i^{p_i}$ (note that $p_i = 0$ for some $i \geq 1$). We define $\gamma = \gamma(M) \in D_n$ as the unique even-signed permutation such that, for $i \in [n-1]$,

i) $p_{|\gamma(i)|} \geq p_{|\gamma(i+1)|}$;
ii) $p_{|\gamma(i)|} = p_{|\gamma(i+1)|} \Longrightarrow |\gamma(i)| < |\gamma(i+1)|$;
iii) $p_{|\gamma(i)|} \equiv 0 \,(\mathrm{mod}\ 2) \Longleftrightarrow \gamma(i) > 0$.

Note that the last condition determines also the sign of $\gamma(n)$.

We show how to determine $\gamma(M)$ with an example. For $n = 6$, let $M = x_1^7 x_2 x_3^6 x_5 x_6^4$. Reorder the variables in such a way that the exponents are weakly decreasing without inverting the variables having the same exponent. We obtain $M = x_1^7 x_3^6 x_6^4 x_2^1 x_5^1 x_4^0$. Then $\gamma(M)$ is given by the indices of M reordered in this way and we put a minus sign in the first six entries according to the parity of the corresponding exponent in M. Hence we obtain $\gamma(M) = (-1, 3, 6, -2, -5, -4)$. To define the partition $\mu(M)$ we first need the following observation.

Lemma 3.1. *Let $M = \prod_{i=1}^n x_i^{p_i}$ such that $\pi(M) \neq 0$. Then the sequence $(p_{|\gamma(i)|} - h_i(\gamma(M)))$, $i = 1, \ldots, n-1$, consists of nonnegative even integers and is weakly decreasing.*

We denote by $\mu(M)$ the partition conjugate to $\left(\frac{p_{|\gamma(i)|}-h_i(\gamma)}{2}\right)_{i=1}^{n-1}$, where $\gamma = \gamma(M)$ (note that $\mu(M)_1 < n$). In our running example we have $(h_1(\gamma), \ldots, h_5(\gamma)) = (3, 2, 2, 1, 1)$ and hence $\mu(M) = (3, 2)$.

Now we introduce a technical partial order on the monomials of the same total degree that we will use later on.

Definition. Let M and M' be monomials such that $\pi(M) \neq 0$ and $\pi(M') \neq 0$ with the same total degree and such that the exponents of x_i in M and M' have the same parity for every $i \in [n]$. Then we write $M' < M$ if one of the following holds

1. $\lambda(M') \lhd \lambda(M)$, or
2. $\lambda(M') = \lambda(M)$ and $inv(|\gamma(M')|_n) > inv(|\gamma(M)|_n)$.

Lemma 3.2 (Straightening Lemma). *Let M be a monomial in S. Then M admits the following expression*

$$M = f_{\mu(M)} \cdot c_{\gamma(M)} + \sum_{M' < M} n_{M',M} f_{\mu(M')} \cdot c_{\gamma(M')},$$

where $n_{M,M'}$ are integers.

For example, let $n = 4$ and $M = x_1^4 x_2 x_4^4$. We have $\gamma(M) = [1, 4, -2, -3]$, $(h_1, h_2, h_3) = (2, 2, 1)$, $c_{\gamma(M)} = x_1^2 x_2 x_4^2$ and $\mu(M) = (2)$. Then, if we set $M_1 = x_1^4 x_2^3 x_4^2$ and $M_2 = x_1^2 x_2^3 x_4^4$, we have that

$$M = c_{\gamma(M)} f_2 - M_1 - M_2$$

in S, with $M_i < M$ for $i = 1, 2$. One can easily verifies that $\gamma(M_1) = [1, -2, 4, -3]$, $\mu(M_1) = \emptyset$, $\gamma(M_2) = [4, -2, 1, -3]$ and $\mu(M_2) = (3)$ and concludes that

$$M = c_{\gamma(M)}\, f_2 - c_{\gamma(M_1)} - c_{\gamma(M_2)}\, f_3.$$

Now the main result of this section is a mere consequence of Lemma 3.2.

Theorem 3.1. *The set*

$$\{c_\gamma + I_n^D \, : \, \gamma \in D_n\}$$

is a basis for $R(D_n)$.

4. Negative-descent representations of D_n

The coinvariant algebra has a natural grading induced from the grading of $\mathbf{P}_n$ by total degree and we denote by R_k its k-th homogeneous component, so that

$$R(W) = \bigoplus_{k \geq 0} R_k.$$

In the case of the symmetric group the major index on standard Young tableaux plays a crucial role in the decomposition of R_k into irreducible representations. The following theorem due independently to Kraskiewicz and Weymann [**13**] and Stanley [**18**, Proposition 4.11] (see also, [**16**, Theorem 8.8]) holds.

Theorem 4.1. *In type A, for $0 \leq k \leq \binom{n}{2}$, the representation R_k is isomorphic to the direct sum $\oplus m_{k,\lambda} S^{\lambda}$, where λ runs through all partitions of n, S^{λ} is the corresponding irreducible S_n-representation, and*

$$m_{k,\lambda} = \mid \{T \in SYT(\lambda) \, : \, maj(T) = k\} \mid .$$

The following is the analogous result for D_n and was proved by Stembridge [**20**] (see also [**4**]). Here we state it in our terminology.

Theorem 4.2. *In type D, for $0 \leq k \leq n^2 - n$, the representation R_k^D is isomorphic to the direct sum $\oplus m_{k,(\lambda,\mu,\epsilon)} S^{\lambda,\mu,\epsilon}$, where $S^{\lambda,\mu,\epsilon}$ is the irreducible representation of D_n labelled as in §2.3, and*

$$m_{k,(\lambda,\mu,\epsilon)} := \mid \{T \in DSYT\{\lambda, \mu\} \, : \, Dmaj(T) = k\} \mid .$$

Now we introduce a new family of D_n-modules $R_{D,N}$. We decompose $R_k^{D_n}$ into a direct sum of these modules and finally we compute the multiplicity of each irreducible representation of D_n in $R_{D,N}$. This result is a refinement of Theorem 4.2.
For any $D \subseteq [n-1]$ we define the partition $\lambda_D := (\lambda_1, \ldots, \lambda_{n-1})$, where $\lambda_i := |D \cap [i, n-1]|$. For $D, N \subseteq [n-1]$, we define the vector

$$\lambda_{D,N} := 2 \cdot \lambda_D + \mathbf{1}_N,$$

where $\mathbf{1}_N \in \{0,1\}^{n-1}$ is the characteristic vector of N. If $\lambda_{D,N}$ is a partition we say that (D, N) is an admissible couple. It is easy to see that (D_γ, N_γ) is admissible for all $\gamma \in D_n$. If (D, N) and (D', N') are two admissible couples then we write $(D, N) \leq (D', N')$ if $\lambda_{D,N} \trianglelefteq \lambda_{D',N'}$. A direct consequence of Lemma 3.2 is that, for all $\gamma, \xi \in D_n$, we have

$$\xi \cdot c_\gamma = \sum_{\{u \in D_n \, : \, (D_u, N_u) \leq (D_\gamma, N_\gamma)\}} n_u c_u + p,$$

where $n_u \in \mathbf{Z}$ and $p \in I_n^D$. It clearly follows that

$$J_{D,N}^{\leq} := \mathrm{span}_{\mathbf{C}}\{c_\gamma + I_n^D \mid \gamma \in D_n, \, (D_\gamma, N_\gamma) \leq (D, N)\}$$

and

$$J_{D,N}^{<} := \mathrm{span}_{\mathbf{C}}\{c_\gamma + I_n^D \mid \gamma \in D_n, \, (D_\gamma, N_\gamma) < (D, N)\}$$

are two submodules of R_k^D, where $k = |\lambda_{D,N}|$, for all admissible couples (D, N). Their quotient is still a D_n-module denoted by

$$R_{D,N} := \frac{J_{D,N}^{\trianglelefteq}}{J_{D,N}^{<}}.$$

If (D, N) is not admissible we let $R_{D,N} := 0$.

Proposition 4.1. *For any D, $N \subseteq [n-1]$, the set*

$$\{\bar{c}_\gamma \, : \, \gamma \in D_n, \, D_\gamma = D \text{ and } N_\gamma = N\},$$

where $\bar{c}_\gamma$ is the image of c_γ in the quotient $R_{D,N}$, is a linear basis of $R_{D,N}$.

By the previous proposition it is natural to call the D_n-module $R_{D,N}$ a *negative-descent representation.* Now we are ready to state the following decomposition of the homogeneous components of the coinvariant algebra.

Theorem 4.3. *For every $0 \leq k \leq n^2 - n$,*

$$R_k^D \cong \bigoplus_{D,N} R_{D,N}$$

as D_n-modules, where the sum is over all $D, N \in [n-1]$ such that $2 \cdot \sum_{i \in D} i + |N| = k$.

Our next goal is to show a simple combinatorial way to compute the multiplicities of the irreducible representations of D_n in $R_{D,N}$.

For any standard Young bitableau $T = (T_1, T_2)$ of shape (λ, μ), following [**1**], we define for $i \in [n]$,

$$h_i(T) := 2 \cdot d_i(T) + \epsilon_i(T), \tag{4.1}$$

where $d_i(T) := |\{j \geq i \ : \ j \in Des(T)\}$, and $\epsilon_i(T) := 1$, if $i \in Neg(T)$ and $\epsilon_i(T) := 0$ otherwise.

The following technical lemma is the key ingredient in the proof of the next theorem.

Lemma 4.2. *Let $T = (T_1, T_2)$ be a Young standard bitableau of total size n such that $n \in T_1$. Then*

$$h_i(T_1, T_2) = h_i(T_2, T_1) + 1$$

for all $i = 1, \ldots, n$.

Theorem 4.4. *For any pair of subset $D, N \subseteq [n-1]$, and a bipartition of n $(\lambda, \mu) \vdash n$, the multiplicity of the irreducible D_n-representation corresponding to $(\lambda, \mu)^\epsilon$ in $R_{D,N}$ is*

$$m_{D,N,(\lambda,\mu)^\epsilon} := |\{T \in DSYT\{\lambda, \mu\} \ : \ Des(T) = D \text{ and } Neg(T) = N\}|.$$

Theorem 4.2 easily follows from this and Theorem 4.3, by observing that $\sum_{i=1}^{n-1} h_i(T) = Dmaj(T)$, for any $T \in DSYT\{\lambda, \mu\}$.

5. Combinatorial Identities

In this last section we compute the Hilbert series of the polynomial ring $\mathbf{P}_n$ with respect to multi-degree rearranged into a weakly decreasing sequence in two different ways and we deduce from this some new combinatorial identities. In particular we obtain one of the main results of [**7**, Corollary 4.4] as a special case of Corollary 5.1.

Following [**6**] we recall the negative statistics on D_n. For $\gamma \in D_n$ we define the *D-negative descent multiset*

$$DDes(\gamma) = Des(\gamma) \biguplus \{Neg(\gamma^{-1})\} \setminus \{n\}. \tag{5.1}$$

and we let

$$ddes(\gamma) := |DDes(\gamma)| \ \text{ and } \ dmaj(\gamma) := \sum_{i \in DDes(\gamma)} i.$$

The Hilbert series of $\mathbf{P}_n$ can be computed by considering the even-signed descent basis for the coinvariant algebra of type D and applying the Straightening Lemma. It's easy to see that the map $\mathbf{P}_n \to D_n \times \mathcal{P}(n)$ given by

$$M \mapsto (\gamma(M), \bar{\mu}(M)'), \tag{5.2}$$

is a bijection, where, if $M = f_n^t M'$, with $M' \in S$, then $\bar{\mu}(M) = ((n)^t, \mu(M'))$. For a partition λ we let $m_j(\lambda) := |\{i \in [n] \ : \ \lambda_i = j\}|$, and

$$\binom{n}{\bar{m}(\lambda)} := \binom{n}{m_0(\lambda), m_1(\lambda), \ldots},$$

be the multinomial coefficient.

Theorem 5.1. *Let $n \in \boldsymbol{P}$. Then*

$$\sum_{\ell(\lambda) \leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\gamma \in D_n} \prod_{i=1}^{n-1} q_i^{2\delta_i(\gamma) + \eta_i(\gamma)}}{(1 - q_1 \cdots q_n) \prod_{i=1}^{n-1} (1 - q_1^2 \cdots q_i^2)},$$

in $\boldsymbol{Z}[[q_1, \ldots, q_n]]$.

Now we compute the Hilbert series in a different way using the following observation. Let $T := \{\sigma \in D_n : des(\sigma) = 0\}$. It is well known, and easy to see, that

$$D_n = \biguplus_{u \in S_n} \{\sigma u : \sigma \in T\}, \tag{5.3}$$

where $\biguplus$ denotes disjoint union. Now define $\bar{n}_i(\gamma) := |\{j \geq i \; : \; j \in Neg(|\gamma|_n)\}|$. It follows that

$$ddes(\gamma) = d_1(\gamma) + \bar{n}_1(\gamma). \tag{5.4}$$

Theorem 5.2. *Let $n \in \boldsymbol{P}$. Then*

$$\sum_{\ell(\lambda) \leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\gamma \in D_n} \prod_{i=1}^{n-1} q_i^{d_i(\gamma)+\bar{n}_i(\gamma^{-1})}}{\prod_{i=1}^{n-1}(1 - q_1^2 \cdots q_i^2)(1 - q_1 \cdots q_n)},$$

in $\boldsymbol{Z}[[q_1, \ldots, q_n]]$.

The following beautiful identity easily follows by Theorems 5.1 and 5.2.

Corollary 5.1. *Let $n \in \boldsymbol{P}$. Then*

$$\sum_{\gamma \in D_n} \prod_{i=1}^{n-1} q_i^{d_i(\gamma)+\bar{n}_i(\gamma^{-1})} = \sum_{\gamma \in D_n} \prod_{i=1}^{n-1} q_i^{2\delta_i(\gamma)+\eta_i(\gamma)}.$$

□

The two pair of statistics $(ddes, dmaj)$ and $(Ddes, Dmaj)$ have the same distribution on D_n, (see [**7**, Corollary 4.4]) given by (2.2). Now it is clear that this result follows directly by Corollary 5.1 by setting $q_1 = qt$ and $q_i = q$ for $i \geq 2$.

Corollary 5.2. *Let $n \in \boldsymbol{P}$. Then*

$$\sum_{\gamma \in D_n} t^{ddes(\gamma)} q^{dmaj(\gamma)} = \sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}.$$

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