Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Intersecting Schubert Varieties

Sara Billey

In the late 1800's, H. Schubert was studying classical intersections of linear subspace arrangements. A typical *Schubert problem* asks how many lines in $\mathbb{C}^3$ generically meet 4 given lines? The generic answer, 2, can be obtained by doing a computation in the cohomology ring of the Grassmannian variety of 2-dimensional planes in $\mathbb{C}^4$. During the past century, the study of the Grassmannian has been generalized to the flag manifold where one can ask similar questions in enumerative geometry.

The flag manifold $\mathcal{F}_n(\mathbb{C}^n)$ consists of all complete flags $F_i = F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$ where F_i is a vector space of dimension i. A modern Schubert problem asks how many flags have relative position u, v, w to three fixed flags X_i, Y_i and Z_i. The solution to this problem used over the past twenty years, due to Lascoux and Schützenberger, is to compute a product of Schubert polynomials and expand in the basis of Schubert polynomials. The coefficient indexed by u, v, w is the solution. This represents a computation in the cohomology ring of the flag variety. It has been a long standing open problem to give a combinatorial rule for expanding these products proving the coefficients $c_{u,v,w}$ are nonnegative integers. It is known from the geometry that these coefficients are nonnegative because they count the number of points in a triple intersection of Schubert varieties with respect to three generic flags.

The main goal of this talk is to describe a method for directly identifying all flags in $X_u(F_i) \cap X_v(G_i) \cap X_w(H_i)$ when $\ell(u) + \ell(v) + \ell(w) = \binom{n}{2}$ and F_i, G_i, H_i are generic, thereby computing $c_{u,v,w}$ explicitly. In 2000, Eriksson and Linusson have shown that the rank tables of intersecting flags are determined by a combinatorial structure they call *permutation arrays*. We prove there is a unique permutation array for each nonempty 0-dimensional intersection of Schubert varieties with respect to flags in generic position. Then we use the structure of this permutation array to solve a small subset of the rank equations previously needed to identify flags in the given intersection. These equations are also useful for determining monodromy and Galois groups on specified collections of flags. This is joint work with Ravi Vakil at Stanford University.

Department of Mathematics, University of Washington, Seattle, WA 98195-4350