Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

A Generalization of $su(2)$

Brian Curtin

Abstract. *We consider the following generalization of* $su(2)$*. Let* $P(q,x,y,z)$ *denote the associative algebra over any field* K *generated by* A_1*,* A_2*,* A_3 *with relations* $[A_1,A_2]_q = xA_3 + yI + z(A_1 + A_2,$ $[A_2,A_3]_q = xA_1 + yI + z(A_2 + A_3)$*,* $[A_3,A_1]_q = xA_2 + yI + z(A_3 + A_1)$ *for some* q*,* x*,* y*,* $z \in K$*. Assume that* $q \neq 0$ *is either 1 or not a root of unity and that* $x \neq 0$*. We describe the multiplicity-free finite-dimensional representations of this generalized algebra, and we describe an action of the modular group on this algebra.*

Résumé. *Nous considérons la généralisation suivante de* $su(2)$*. Soit* $P(q,x,y,z)$ *l'algèbre associative avec des générateurs* A_1*,* A_2*,* A_3 *et rélations* $[A_1,A_2]_q = xA_3 + yI + z(A_1 + A_2,$ $[A_2,A_3]_q = xA_1 + yI + z(A_2 + A_3)$*,* $[A_3,A_1]_q = xA_2 + yI + z(A_3 + A_1)$ *pour* q*,* x*,* y*,* $z \in K$*. Supposez que* $q \neq 0$ *est 1 ou pas une racine de l'unité, et supposez que Nous décrivons l* $x \neq 0$*.es représentations fini-dimensionnelles sans multiplicité de cette algèbre généralisé, et Nous décrivons une action du groupe modulaire sur cette algèbre.*

1. Introduction

Recall that the special unitary Lie algebra $su(2)$ is the Lie algebra with basis S_1, S_2, S_3 and relations

$$[S_1,S_2] = \mathrm{i}S_3, \qquad [S_2,S_3] = \mathrm{i}S_1, \qquad [S_3,S_1] = \mathrm{i}S_2. \tag{1.1}$$

We generalize $su(2)$ (or rather its enveloping algebra) as follows.

Definition 1.1. Let $\mathbb{K}$ denote any field. Pick q, x, y, $z \in \mathbb{K}$. Let $\mathcal{P} = \mathcal{P}(q,x,y,z)$ be the associative algebra over $\mathbb{K}$ generated by three symbols S_1, S_2, S_3 subject to the relations

$$[S_1,S_2]_q = xS_3 + yI + z(S_1 + S_2), \tag{1.2}$$
$$[S_2,S_3]_q = xS_1 + yI + z(S_2 + S_3), \tag{1.3}$$
$$[S_3,S_1]_q = xS_2 + yI + z(S_3 + S_1), \tag{1.4}$$

where $[x,y]_q = xy - qyx$.

Like the relations of (1.1), the relations (1.2) – (1.4) express (q-)commutators as linear expressions in the three generators (the two in the commutator having the same coefficient) and have a cyclic symmetry.

We describe the multiplicity-free irreducible finite-dimensional representations of $\mathcal{P}(q,x,y,z)$ when $x \neq 0$ and q is some nonzero element of $\mathbb{K}$ which is not a root of unity, other than perhaps 1 itself. We need some notation. Fix a field $\mathbb{K}$ and a vector space V over $\mathbb{K}$ of finite nonnegative dimension. Let $\mathrm{End}(V)$ denote the vector space of all $\mathbb{K}$-linear transformations from V to V. A square matrix over $\mathbb{K}$ is said to be *tridiagonal*

1991 *Mathematics Subject Classification.* Primary 17B37; Secondary 33C45, 33D45, 16G99.

Key words and phrases. special unitary Lie algebra, Leonard pair, Leonard triple, modular Leonard triple.

Project sponsored by the National Security Agency under Grant Number MDA904–03-1-0020.

whenever every nonzero entry appears on the diagonal, the superdiagonal, or the subdiagonal. A tridiagonal matrix is *irreducible* whenever the entries on the sub- and superdiagonals are all nonzero.

Definition 1.2. Let A_1, A_2, A_3 denote an ordered triple of elements taken from $\mathrm{End}(V)$. We call this triple a *Leonard triple on* V whenever for each $A \in \{A_1, A_2, A_3\}$ there exists a basis of V with respect to which the matrix representing A is diagonal and the matrices representing the other two operators in the triple are irreducible tridiagonal.

By an *antiautomorphism* of $\mathrm{End}(V)$, we mean a $\mathbb{K}$-linear bijection $\tau : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $\tau(XY) = \tau(Y)\tau(X)$ for all $X, Y \in \mathrm{End}(V)$.

Definition 1.3. Let A_1, A_2, A_3 denote a Leonard triple on V. Then this Leonard triple is said to be *modular* whenever for each $A \in \{A_1, A_2, A_3\}$ there exists an antiautomorphism of $\mathrm{End}(V)$ which fixes A and swaps the other two operators in the triple.

Our main result on the representations of $\mathcal{P}(q, x, y, z)$ is the following.

Theorem 1.4. *With reference to Definition 1.1, assume $x \neq 0$. Also assume that $q \neq 0$ is either 1 or not a root of unity. Let V denote an irreducible finite-dimensional module for $\mathcal{P}(q, x, y, z)$. Let $a_1 = S_1|_V$, $a_2 = S_2|_V$, $a_3 = S_3|_V$. Assume that a_1, a_2, a_3 are multiplicity-free. Then a_1, a_2, a_3 is a modular Leonard triple on V.*

The modular Leonard triples are completely characterized–we recall this characterization in Section 3. We conclude by showing that the modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{P}(q, x, y, z)$ when $x \neq 0$.

2. Multiplicity-free representations of $\mathcal{P}$

We show that the representations of $\mathcal{P}(q, x, y, z)$ of interest are closely related to Leonard pairs. We begin by recalling the notion of a Leonard pair.

Definition 2.1. Let A_1, A_2 denote an ordered pair of elements taken from $\mathrm{End}(V)$. We call this pair a *Leonard pair on* V whenever for each $A \in \{A_1, A_2\}$ there exists an ordered basis of V with respect to which the matrix representing A is diagonal and the matrix representing the other member of the pair is irreducible tridiagonal.

We need the following criterion.

Theorem 2.2 (Vidunas and Terwilliger [**VT**])**.** *Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let A, A_2 denote an ordered pair of elements of* $\mathrm{End}(V)$ *linear operators in* $\mathrm{End}(V)$. *Assume that*

(1) *A_1 and A_2 are multiplicity-free;*
(2) *V is irreducible as an (A_1, A_2)-module;*
(3) *there exist β, γ, γ^*, ρ, ρ^*, ω, η, $\eta^* \in \mathbb{K}$ such that*

$$A_1^2 A_2 - \beta A_1 A_2 A_1 + A_2 A_1^2 - \gamma(A_1 A_2 + A_2 A_1) - \rho A_2 = \gamma^* A_1^2 + \omega A_1 + \eta I, \tag{2.1}$$

$$A_2{}^2 A_1 - \beta A_2 A_1 A_2 + A_1 A_2{}^2 - \gamma^*(A_2 A_1 + A_1 A_2) - \rho^* A_1 = \gamma A_2{}^2 + \omega A_2 + \eta^* I; \tag{2.2}$$

(4) *no q satisfying $q + q^{-1} = \beta$ is a root of unity.*

Then A_1, A_2 is a Leonard pair on V.

Theorem 2.3. *With reference to Definition 1.1, assume $x \neq 0$. Then any two of S_1, S_2, S_3 satisfy (2.1) and (2.2) with*

$$\begin{aligned} \beta &= q + 1/q, \\ \gamma = \gamma^* &= z(q-1)/q, \\ \rho = \rho^* &= (z^2 - x^2)/q, \\ \omega = \omega^* &= (y(q-1) + z(z-x))/q, \\ \eta = \eta^* &= y(z-x)/q. \end{aligned}$$

PROOF. Each of S_1, S_2, S_3 appears linearly with coefficient x in one of equations one of (1.2)–(1.4). Solve for, say, S_3 in (1.2), and eliminate it in (1.3) and (1.4). □

Lemma 2.4. *With the notation and assumptions of Theorem 1.4, any two of* a_1, a_2, *and* a_3 *form a Leonard pair.*

PROOF. Observe that V is irreducible as, say, an (a_1, a_2)-module since V is irreducible as a $\mathcal{P}(q, x, y, z)$ and a_3 is expressed using a_1 and a_2. The result follows from Theorems 2.2 and 2.3. □

It turns out that the representations of $\mathcal{P}(q, x, y, z)$ of interest correspond to a special extension of a Leonard pair.

PROOF OF THEOREM 1.4. (sketch) By Lemma 2.4, any two of a_1, a_2, a_3 form a Leonard pair. Thus by Definition 2.1 there is a basis of V with respect to which the matrix representing, say, a_1 is irreducible tridiagonal and the matrix representing a_2 is diagonal. Substituting these forms into (1.2) gives that the matrix representing a_3 is also irreducible tridiagonal. Thus a_1, a_2, a_3 is a Leonard triple. It turns out that all Leonard pairs in Lemma 2.4 are isomorphic. (This follows from the fact that they all satisfy the same Askey-Wilson relations and some facts about canonical forms of a Leonard pair [**T4**]). Composing the antiautomorphism of $\mathrm{End}(V)$ which fixes a_1 and a_2 and the automorphism which swaps a_1 and a_2 gives an antiautomorphism which swaps a_1 and a_2. Applying this map to (1.2) gives that it fixes a_3. □

We conclude this section with some comments on Leonard pairs. Leonard pairs were introduced by P. Terwilliger [**T1, T3**] as an algebraic abstraction of work of D. Leonard concerning the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal polynomials. [**Len1, Len2**] (cf. [**BI**]). Leonard characterized these orthogonal polynomials in terms of hypergeometric series. This result is analogous to Askey and Wilson's characterization of similar orthogonal polynomials with continuous support [**AW1, AW2**] (cf. [**KS**]). The reference [**T5**] describes a bijective correspondence between the isomorphism classes of Leonard pairs and the appropriate orthogonal polynomials. In particular, results concerning Leonard pairs can be viewed as results concerning such orthogonal polynomials. This connection is further developed in [**T6**]. Relations (2.1) and (2.2) are called the *Askey-Wilson relations.* They were introduced by Zhedanov et. al. [**GLZ, Z**] in connection with the quadratic Askey-Wilson algebra.

3. The modular Leonard triples

We now recall a characterization of the modular Leonard triples [**C**]. We do so by first describing three examples of modular Leonard triples in Lemmas 3.1, 3.2, and 3.3, and then describing how, up to isomorphism, they are the only examples. We use the following conventions throughout. Given any square matrix X of order n with entries in $\mathbb{K}$, we view X as a linear operator on $\mathbb{K}^n$, acting by $v \mapsto Xv$. Let d denote a nonnegative integer. Write

$$\begin{aligned}
A_1 &= \mathrm{tridiag}\begin{pmatrix} b_0 & b_1 & \cdots & b_{d-1} & * \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ * & c_1 & \cdots & c_{d-1} & c_d \end{pmatrix}, \\
A_2 &= \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_d), \\
A_3 &= \mathrm{tridiag}\begin{pmatrix} b_0\nu_1 & b_1\nu_2 & \cdots & b_{d-1}\nu_d & * \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ * & c_1/\nu_1 & \cdots & c_{d-1}/\nu_{d-1} & c_d/\nu_d \end{pmatrix}.
\end{aligned}$$

Lemma 3.1. *([**C**]) Set*

$$
\begin{aligned}
\nu_i &= \nu q^{i-1} \qquad (1 \le i \le d),\\
\theta_i &= \theta_0 + h(1-q^i)(1-\nu^2 q^{i-1})q^{-i} \qquad (0 \le i \le d),\\
b_0 &= -\frac{h(1-q^d)(1+\nu^3 q^{d-1})}{q^d(1-\nu)},\\
b_i &= -\frac{h(1-q^{d-i})(1-\nu^2 q^{i-1})(1+\nu^3 q^{d+i-1})}{q^{d-i}(1-\nu q^i)(1-\nu^2 q^{2i-1})} \qquad (1 \le i \le d-1),\\
c_i &= \frac{h\nu(1-q^i)(1+\nu q^{d-i})(1-\nu^2 q^{d+i-1})}{q^{d-i+1}(1-\nu q^{i-1})(1-\nu^2 q^{2i-1})} \qquad (1 \le i \le d-1),\\
c_d &= \frac{h\nu(1-q^d)(1+\nu)}{q^{(}1-\nu q^{d-1})},\\
a_i &= \theta_0 - b_i - c_i \qquad (0 \le i \le d)\ (c_0 = 0,\ b_d = 0)
\end{aligned}
$$

for some scalars θ_0, h, ν, q in $\mathbb{K}$ such that $h\nu q \neq 0$, $q^i \neq 1$ $(1 \le i \le d)$, $\nu^3 q^{2d-1-i} \neq -1$ $(1 \le i \le d)$, and $\nu^2 q^i \neq 1$ $(0 \le i \le 2d-2)$. Then A_1, A_2, A_3 is a modular Leonard triple on $\mathbb{K}^{d+1}$.

Lemma 3.2. *([**C**]) Assume* $\operatorname{char}\mathbb{K}$ *is 0 or an odd prime greater than d. Set*

$$
\begin{aligned}
\nu_i &= -1 \qquad (1 \le i \le d),\\
\theta_i &= \theta_0 + hi(i+1+s) \qquad (0 \le i \le d),\\
b_0 &= \frac{-hd(3s+2d+4)}{4},\\
b_i &= \frac{h(i+1+s)(d-i)(2i+3s+2d+4)}{4(2i+1+s)} \qquad (1 \le i \le d-1),\\
c_i &= \frac{hi(i+s+d+1)(2i-s-2d-2)}{4(2i+1+s)} \qquad (1 \le i \le d-1),\\
c_d &= \frac{-hd(s+2)}{4},\\
a_i &= \theta_0 - b_i - c_i \qquad (0 \le i \le d)\ (c_0 = 0,\ b_d = 0)
\end{aligned}
$$

for some scalars θ_0, h, s in $\mathbb{K}$ such that $h \neq 0$, $s \neq -i$ $(2 \le i \le 2d)$, and $3s \neq -2i$ $(d+2 \le i \le 2d+1)$. Then A_1, A_2, A_3 is a modular Leonard triple on $\mathbb{K}^{d+1}$.

Lemma 3.3. *([**C**]) Assume* $\operatorname{char}\mathbb{K} = 0$ *or* $\operatorname{char}\mathbb{K} > d$. *Set*

$$
\begin{aligned}
\nu_i &= \nu \qquad (1 \le i \le d),\\
\theta_i &= \theta_0 + hi \qquad (0 \le i \le d),\\
b_i &= -\frac{h(d-i)(1-\nu+\nu^2)}{(1-\nu)^2} \qquad (0 \le i \le d-1),\\
c_i &= \frac{hi\nu}{(1-\nu)^2} \qquad (1 \le i \le d),\\
a_i &= \theta_0 - b_i - c_i \qquad (0 \le i \le d)\ (c_0 = 0,\ b_d = 0)
\end{aligned}
$$

for some scalars θ_0, h, ν in $\mathbb{K}$ such that $h\nu \neq 0$, $\nu \neq 1$, and $1-\nu+\nu^2 \neq 0$. Then A_1, A_2, A_3 is a modular Leonard triple on $\mathbb{K}^{d+1}$.

Definition 3.4. Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let A_1, A_2, A_3 denote a modular Leonard triple on V. We say that the triple A_1, A_2, A_3 is of *type I*, *type II*, or *type III*, respectively, whenever there exists a basis of V with respect to which the matrices representing A_1, A_2, A_3 are as in Lemma 3.1, Lemma 3.2, or Lemma 3.3, respectively.

Theorem 3.5 ([**C**]). *Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let A_1, A_2, A_3 denote a modular Leonard triple on V. Then A_1, A_2, A_3 is of type I, type II, or type III.*

Theorem 3.6. *Let A_1, A_2, A_3 denote a modular Leonard triple on V. Then there are scalars q, x, y, z in $\mathbb{K}$ with $x \neq 0$ such that (1.2)–(1.4) hold.*

Proof. Direct verification using the above classification of modular Leonard triples. □

4. A modular group action

We describe an action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{P}(q,x,y,z)$. This modular group action was first observed for the modular Leonard triples, hence their name. We begin with describing some antiautomorphisms for $\mathcal{P}(q,x,y,z)$.

Lemma 4.1. *With reference to Definition 1.1, assume $x \neq 0$. Then for any $T \in \{S_1, S_2, S_3\}$, there exists an antiautmorphism of $\mathcal{P}(q,x,y,z)$ which fixes T and swaps the other two generators.*

Proof. Let $\mu : \mathcal{P} \to \mathcal{P}$ denote a linear map which reverses the order of multiplication and swaps S_1 and S_2. Then μ fixes the q-commutator in (1.2). On the right-hand side of (1.2) the linear terms involving I and $S_1 + S_2$ are fixed, so S_3 is fixed by such a map. Observe that μ is indeed an antiautomorphism of $\mathcal{P}$. □

Lemma 4.2. *With reference to Definition 1.1, assume $x \neq 0$. Then for any $T \in \{S_1, S_2, S_3\}$, there exists an antiautmorphism of $\mathcal{P}(q,x,y,z)$ which fixes the elements of $\{S_1, S_2, S_3\}\backslash T$.*

Proof. Let $\alpha : \mathcal{P} \to \mathcal{P}$ denote a linear map which reverses the order of multiplication and swaps S_1 and S_2. Applying α to (1.2) gives an expression for $\alpha(S_3)$. Essentially the same computation as was performed in Theorem 2.3 shows that α is indeed an antiautomorphism of $\mathcal{P}$. □

Recall that $\mathrm{PSL}_2(\mathbb{Z})$ has presentation $\langle s,\, t \,|\, s^2 = 1,\, t^3 = 1\rangle$.

Lemma 4.3. *With reference to Definition 1.1, assume $x \neq 0$.*

(1) *Let σ denote the composition of the antiautomorphisms of $\mathcal{P}$ which respectively fix and swap S_1 and S_2. Then $\sigma^2 = I$.*
(2) *Let τ denote the composition of the antiautomorphisms of $\mathcal{P}$ which respectively swap S_1 and S_2 and swap S_2 and S_3. Then $\tau^3 = I$.*

In particular, $\mathrm{PSL}_2(\mathbb{Z})$ *acts on $\mathcal{P}$ as a group of automorphisms.*

Proof. It is easy to verify from their constructions that τ sends S_1 to S_3, S_2 to S_1, and S_3 to S_2, and that σ swaps S_1 and S_2. The result follows. □

References

[A] T.M. Apostol, *The Modular Group and Modular Functions.* Ch.Ê2 in Modular Functions and Dirichlet Series in Number Theory, 2nd ed, pp.Ê17 and pp. 26-46, Springer-Verlag, New York, 1997.

[AW1] R. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or* $6-j$ *symbols*, SIAM J. Math. Anal. **10** (1979), pp. 1008–1016.

[AW2] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985).

[BI] E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, 1984.

[BB] J.M. Borwein and P.B. Borwein, *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity*, p.Ê113, Wiley, New York, 1987.

[C] B. Curtin, *Modular Leonard triples*, preprint.

[GLZ] Ya.A. Granovskii, A.S. Lutzennko and A.S. Zhedanov, *Mutual integrability, quadratic algebras, and dynamical symmetry*, Ann. Phys **217** (1992), pp. 1–20.

[KS] Koekoek and Suwartouw, *The Askey scheme of hypergeometric orthogonal polynomials and its q-analog*, Reports of the Faculty of Technical Mathematics and Informatics, vol. 98-17, Delft, Netherlands, 1998.

[Lan] R.H. Landau *Quantum Mechanics II: A Second Course in Quantum Theory*, 2nd ed. Wiley, New York, 1996.

[Len1] D. Leonard, *Orthogonal polynomials, duality, and association schemes*, SIAM J. Math. Anal. **13** (1982), pp. 656–663.

[Len2] D. Leonard, *Parameters of association schemes that are both P- and Q-polynomial*, J. Combin. Theory Ser. A **36** (1984), pp. 355–363.

[T1] P. Terwilliger, *The subconstituent algebra of an association scheme*, J. Alg. Combin. Part I: **1** (1992), pp. 363–388; Part II: **2** (1993), pp. 73–103; Part III: **2** (1993), pp. 177–210.

[T2] P. Terwilliger, *Leonard pairs from 24 points of view*, Rocky Mountain Journal of Mathematics, **32** (2002), pp. 827–888.

[T3] P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra Appl. **330** (2001), pp. 149–203.

[T4] P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other: The TD-D and LB-UB canonical forms*, preprint.

[T5] P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other: Comments on the parameter array*, preprint.

[T6] P. Terwilliger, *Leonard pairs and the q-Racah polynomials*, preprint.

[VT] R. Vidunas and P. Terwilliger *Leonard pairs and Askey-Wilson relations*, preprint.

[Z] A.S. Zhedanov, *Hidden symmetry of Askey-Wilson polynomials*, Teoret. Mat. Fiz., **89** (1991), pp. 190–204.

Department of Mathematics, University of South Florida, Tampa, FL USA, 33620
E-mail address: `bcurtin@math.usf.edu`