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Generalised Schur P–Functions and Weyl's Denominator Formula

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Abstract. *We derive a general identity that relates generalised P–functions to the product of a Schur function and*

$$\prod_{1\le i<j\le n}(x_i+y_j).$$

This result generalises a number of well–known results in Robbins and Rumsey, Chapman, Tokuyama, and Macdonald. We also interpret our result in terms of μ–alternating sign matrices.

Résumé. *Nous dérivons une identité générale reliant les P–fonctions généralisées et, le produit d'une fonction de Schur et $\prod_{1\le i<j\le n}(x_i+y_j)$. Ce résultat est une généralisation des travaux de Robbins et Rumsey, Chapman, Tokuyama, et Macdonald. Nous en donnons aussi une variante avec des μ–matrices à signes alternants.*

1. Introduction

The fundamental expression

$$\prod_{1\le i<j\le n}(x_i+y_j) \tag{1.1}$$

appears in a number of contexts in symmetric function theory. Given $\mathbf{y}=y_1,y_2,\ldots,y_n$ and $\mathbf{x}=x_1,x_2,\ldots,x_n$, when $\mathbf{y}=-\mathbf{x}$, equation (1.1) is the Weyl denominator formula (also called the Vandermonde determinant):

$$\det(x_i^{n-j})=\prod_{1\le i<j\le n}(x_i-x_j). \tag{1.2}$$

For $\mathbf{y}=\lambda\mathbf{x}$, expression (1.1) becomes the λ–determinant formula of Robbins and Rumsey [**RR86**]:

$$\prod_{1\le i<j\le n}(x_i+\lambda x_j)=\sum_{A\in\mathcal{A}_n}\lambda^{SE(A)}(1+\lambda)^{NS(A)}\prod_{i=1}^{n}x_i^{NE_i(A)+SE_i(A)+NS_i(A)}, \tag{1.3}$$

where the exponents are various parameters associated with alternating sign matrices and defined in Section 3. Bressoud [**B01**] asked for a combinatorial proof of (1.3) which was provided by Chapman [**C01**] who generalised it to:

$$\prod_{1\le i<j\le n}(x_i+y_j)=\sum_{A\in\mathcal{A}_n}\prod_{i=1}^{n}x_i^{NE_i(A)}y_i^{SE_i(A)}(x_i+y_i)^{NS_i(A)}. \tag{1.4}$$

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For $\mathbf{y} = t\mathbf{x}$, there is also the t deformation of the Weyl denominator formula due to Tokuyama [**T88**]:

$$\prod_{1\le i<j\le n}(x_i+tx_j)s_\lambda(\mathbf{x})s_{1^n}(\mathbf{x}) = \sum_{ST\in\mathcal{ST}^\mu} t^{hgt(ST)}(1+t)^{str(ST)-n}x^{wgt(ST)}, \tag{1.5}$$

where the sum is over semistandard shifted tableaux ST and where hgt, str, and wgt are parameters associated with semistandard shifted tableaux and defined in Section 2. Note also that $s_\lambda(\mathbf{x})$ is the Schur function, and $s_{1^n}(\mathbf{x}) = x_1x_2\dots x_n$ is the Schur function of shape 1^n.

Here we present a general identity that unifies results (1.2)-(1.5) and we also demonstrate a connection to a generalisation of Schur P–functions. Our identity can also easily be re–interpreted in terms of Schur Q–functions—see Section 2.

The Main Result:

Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with λ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \dots, 1)$. In addition, let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$. Then

$$P_\mu(\mathbf{x}/\mathbf{y}) = s_{1^n}(\mathbf{x})\ s_\lambda(\mathbf{x}) \prod_{1\le i<j\le n}(x_i+y_j), \tag{1.6}$$

where $P_\mu(\mathbf{x}/\mathbf{y})$ is the generalised P–function defined in Section 2. Our paper is arranged as follows. Section 2 introduces the necessary background. Section 3 gives a formal statement of the result and provides a proof and detailed example. Section 4 demonstrates the connection to alternating sign matrices. Section 5 explores future directions involving other root systems.

2. Background

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0$ be a partition of weight $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_p$ and length $\ell(\lambda) = p$, where each λ_i is a positive integer for all $i = 1, 2, \dots, p$. Then λ defines a Young diagram F^λ consisting of p rows of boxes of lengths $\lambda_1, \lambda_2 \dots, \lambda_p$ left-adjusted to a vertical line.

A partition $\mu = (\mu_1, \mu_2, \dots, \mu_q)$ of length $\ell(\mu) = q$ is said to be a strict partition if all the parts of μ are distinct, that is $\mu_1 > \mu_2 > \cdots > \mu_q > 0$. A strict partition μ defines a shifted Young diagram SF^μ consisting of q rows of boxes of lengths $\mu_1, \mu_2, \dots, \mu_q$ left-adjusted this time to a diagonal line.

For any partition λ of length $\ell(\lambda) \le n$ let $\mathcal{T}^\lambda(n)$ be the set of all semistandard tableaux T obtained by numbering all the boxes of F^λ with entries taken from the set $\{1, 2, \dots, n\}$, subject to the usual total ordering $1 < 2 < \cdots < n$. The numbering must be such that the entries are:

T1 weakly increasing across each row from left to right;
T2 strictly increasing down each column from top to bottom.

The weight of the tableau T is given by $\mathrm{wgt}(T) = (w_1, w_2, \dots, w_n)$, where w_k is the number of times k appears in T for $k = 1, 2, \dots, n$.

By the same token, for any strict partition μ of length $\ell(\mu) \le n$ let $\mathcal{ST}^\mu(n)$ be the set of all semistandard shifted tableaux ST obtained by numbering all the boxes of SF^μ with entries taken from the set $\{1, 2, \dots, n\}$, subject to the total ordering $1 < 2 < \cdots < n$. The numbering must be such that the entries are:

ST1 weakly increasing across each row from left to right;
ST2 weakly increasing down each column from top to bottom;
ST3 strictly increasing down each diagonal from top-left to bottom-right.

The weight of the tableau ST is again given by

$$\mathrm{wgt}(ST) = (w_1, w_2, \dots, w_n),$$

where w_k is the number of times k appears in ST for $k = 1, 2, \dots, n$. The rules ST1-ST3 serve to exclude any 2×2 blocks of boxes all containing the same entry, and as a result each $ST \in \mathcal{ST}^\mu(n)$ consists of a sequence of ribbon strips of boxes containing identical entries. Any given ribbon strip may consist of a number of disjoint connected components. Let $str(ST)$ denote the total number of disjoint connected components of all the ribbon strips. Let $hgt(ST)$ be the height of the tableaux, defined $hgt(ST) = \sum_{k=1}^{n}(row_k(ST) - con_k(ST))$, where $row_k(ST)$ is the number of rows of S containing an entry k, and $con_k(ST)$ is the number of connected components of the ribbon strip of ST consisting of all the entries k.

Refining this construct, for any strict partition μ with $\ell(\mu) \leq n$ let $\mathcal{PST}^\mu(n)$ be the set of all primed, or marked, semistandard shifted tableaux PST obtained by numbering all the boxes of SF^μ with entries taken from the set $\{1', 1, 2', 2, \ldots, n', n\}$, subject to the total ordering $1' < 1 < 2' < 2 < \cdots < n' < n$. The numbering must be such that the entries are:

- PST1 weakly increasing across each row from left to right;
- PST2 weakly increasing down each column from top to bottom;
- PST3 with no two identical unmarked entries in any column;
- PST4 with no two identical marked entries in any row;
- PST5 with no marked entries on the main diagonal.

The passage from $\mathcal{ST}^\mu(n)$ to $\mathcal{PST}^\mu(n)$ is effected merely by adding marks to the entries of each $ST \in \mathcal{ST}^\mu(n)$ in all possible ways that are consistent with PST1-5 to give some $PST \in \mathcal{PST}^\mu(n)$. The only entries for which any choice is possible are those in the lower left hand box at the beginning of each connected component of a ribbon strip. Thereafter in that connected component of the ribbon strip entries in the boxes of its horizontal portions are unmarked and those in the boxes of its vertical portions are marked. It should be noted that all the boxes on the main diagonal are necessarily at the lower left hand end of a connected component of a ribbon strip, but their entries remain unmarked by virtue of PST5. The marked weight of the tableau PST is then defined to be the vector $\text{wgt}(PST) = (u_1, u_2, \ldots, u_n/v_1, v_2, \ldots, v_n)$, where u_k and v_k are the number of times k and k', respectively, appear in PST for $k = 1, 2, \ldots, n$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector of n indeterminates and let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a vector of n non-negative integers. Then

$$\mathbf{x}^\mathbf{w} = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}.$$

With this notation it is well known that each partition λ of length $\ell(\lambda) \leq n$ specifies a Schur function $s_\lambda(\mathbf{x})$ with combinatorial definition:

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n)} \mathbf{x}^{\text{wgt}(T)} \tag{2.1}$$

Similarly, each strict partition μ of length $\ell(\mu) \leq n$ specifies a Schur Q-function whose combinatorial defintion takes the form:

$$Q_\mu(\mathbf{x}) = \sum_{ST \in \mathcal{ST}^\mu(n)} 2^{\text{str}(ST)} \mathbf{x}^{\text{wgt}(ST)}. \tag{2.2}$$

The corresponding Schur P-function takes the form:

$$P_\mu(\mathbf{x}) = \sum_{ST \in \mathcal{ST}^\mu(n)} 2^{\text{str}(ST) - \ell(\mu)} \mathbf{x}^{\text{wgt}(ST)}. \tag{2.3}$$

Let $\mathbf{z} = (\mathbf{x}/\mathbf{y}) = (x_1, x_2, \ldots, x_n/y_1, y_2, \ldots, y_n)$, where $\mathbf{x}$ and $\mathbf{y}$ are two vectors of n indeterminates, and let $\mathbf{w} = (\mathbf{u}/\mathbf{v}) = (u_1, u_2, \ldots, u_n/v_1, v_2, \ldots, v_n)$ where $\mathbf{u}$ and $\mathbf{v}$ are two vectors of n non-negative integers. Then let $\mathbf{z}^\mathbf{w} = (\mathbf{x}/\mathbf{y})^{(\mathbf{u}/\mathbf{v})} = \mathbf{x}^\mathbf{u}\,\mathbf{y}^\mathbf{v} = x_1^{u_1} \cdots x_n^{u_n}\, y_1^{v_1} \cdots y_n^{v_n}$. With this notation each strict partition μ of length $\ell(\mu) \leq n$ serves to specify a generalised Schur P-function that may be denoted by $P_\mu(\mathbf{x}/\mathbf{y})$ and defined by

$$P_\mu(\mathbf{x}/\mathbf{y}) = \sum_{PST\in\mathcal{P}ST^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PST)} \tag{2.4}$$

Since the map back from $PST \in \mathcal{P}ST^\mu(n)$ to some $|ST| \in \mathcal{S}T^\mu(n)$ is effected merely by deleting marks, and there are no marks on the main diagonal, it follows that

$$Q_\mu(\mathbf{x}) = 2^{\ell(\mu)} P_\mu(\mathbf{x}) \quad \text{with} \quad P_\mu(\mathbf{x}) = P_\mu(\mathbf{x}/\mathbf{x}). \tag{2.5}$$

It might be noted that $s_\lambda(\mathbf{x})$, $P_\mu(\mathbf{x})$ and $Q_\mu(\mathbf{x})$ are nothing other than the specialisations $P_\lambda(\mathbf{x};0)$, $P_\mu(\mathbf{x};-1)$ and $Q_\mu(\mathbf{x};-1)$, respectively, of the Hall-Littlewood functions $P_\mu(\mathbf{x};t)$ and $Q_\mu(\mathbf{x};t)$. In fact $s_\lambda(\mathbf{x}) = P_\lambda(\mathbf{x};0) = Q_\lambda(\mathbf{x};0)$, see Macdonald [**M95**] pp 208 and p225, and this is true for all partitions λ.

Rather than generalise $P_\mu(\mathbf{x})$ we could equally well have generalised $Q_\mu(\mathbf{x})$. If we replace PST1-4 by identical conditions QST1-4, but drop the condition PST5, the corresponding marked shifted tableaux $QST \in \mathcal{Q}ST^\mu(n)$, with marks now allowed on the diagonal entries, serve to define

$$Q_\mu(\mathbf{x}/\mathbf{y}) = \sum_{QST\in\mathcal{Q}ST^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)}. \tag{2.6}$$

With this definition, the result analagous to (1.6) takes the form:

$$Q_\mu(\mathbf{x}/\mathbf{y}) = s_\lambda(\mathbf{x}) \prod_{1\le i\le j\le n} (x_i + y_j). \tag{2.7}$$

3. The Bijection

3.1. Main Result. The generalisation from $P_\mu(\mathbf{x})$ to $P_\mu(\mathbf{x}/\mathbf{y})$ allows us to formulate the following

Theorem 3.1. *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with λ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \dots, 1)$. In addition, let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$. Then*

$$P_\mu(\mathbf{x}/\mathbf{y}) = s_{1^n}(\mathbf{x})\ s_\lambda(\mathbf{x}) \prod_{1\le i<j\le n} (x_i + y_j). \tag{3.1}$$

Here $s_{1^n}(\mathbf{x}) = x_1 x_2 \cdots x_n$ is associated with the unmarked entries $1, 2, \dots, n$ that must appear on the main diagonal of each $PST \in \mathcal{P}ST^\mu(n)$ in the case $\ell(\mu) = n$.

Our main result, is to provide a bijective proof of the above identity from which follow a number of corollaries. The case $\lambda = 0$ is equivalent to an alternating sign matrix identity attributed to Robbins and Rumsey [**RR86**], proved combinatorially by Chapman[**C01**]. The case $\mathbf{x} = \mathbf{y}$ is an example of Macdonald (Sec. III.8, Ex. 2) [**M95**]. The case $\mathbf{y} = t\mathbf{x} = (tx_1, tx_2, \dots, tx_n)$ is equivalent to a Weyl denominator deformation Theorem due to Tokuyama [**T88**] for the Lie algebra $gl(n)$ and proved combinatorially by Okada [**O90**].

It should be stressed that the above Theorem is restricted to the case of a strict partition μ of length $\ell(\mu) = n$, although a similar result applies in the case $\ell(\mu) = n - 1$ which may be obtained from the above by dividing both sides by $s_{1^n}(\mathbf{x}) = x_1 x_2 \cdots x_n$.

Proof of Theorem 3.1: Given a primed semistandard shifted tableau, PST, of shape $\mu = \lambda + \delta$, we will show how to decompose it into a semistandard tableau of shape λ and a primed (not necessarily semistandard) shifted tableau of shape δ satisfying: 1) k' appears only in column k; 2) k appears only in row k, and; 3) there are no primed entries on the main diagonal.

Apply jeu de taquin for generalised marked shifted tableaux ([**S87**], [**W84**], [**SS89**], [**M95**],[**HH92**]) to the primed entries k' in turn (starting with the $1'$) by moving them to the left as far as but no farther than the kth column. For this purpose we assume k' is less than i for $i = 1, 2, \dots, k-1$. If there is more than one k' we start with the highest one. In doing this we must always be careful not to violate PST4—thus identical primed entries must be in different rows at all times even if they are separated by unprimed entries. Note that at each stage we can blank out all entries greater than k in the right hand portion and remove all columns to the left of the kth.

When the k''s have all been moved to their own column, the tableau that results will have unprimed elements on the main diagonal. Now permute the other entries so as to leave all the unprimed entries in their own rows. We can divide the resulting tableau at column n to give a primed semistandard shifted tableau of shape δ and a semistandard tableau of shape λ.

To undo the above transformation, reverse the steps taken. First move all the primed entries to the top of their own columns. Then play jeu de taquin in reverse with primed entries k' taken in turn from bottom to top. These entries move in a south easterly direction with k' now assumed to be larger than i for $i = 1, \dots, k-1$ but less than j for $j = k, k+1, \dots,$ with the semistandardness conditions applying to all unprimed entries at all times. $\diamondsuit$

We can derive a number of corollaries of Theorem 3.1. We will derive a further corollary in Section 4.

Setting $\lambda = 0$ in Theorem 3.1 we obtain the following corollary:

Corollary 3.2.

$$s_{1^n}(\mathbf{x}) \prod_{1 \le i < j \le n} (x_i + y_j) = P_\delta(\mathbf{x}/\mathbf{y}). \tag{3.2}$$

The case $\mathbf{y} = t\mathbf{x} = (tx_1, tx_2, \dots, tx_n)$ is equivalent to a Weyl denominator deformation Theorem due to Tokuyama [**T88**] for the Lie algebra $gl(n)$. There is also a combinatorial proof due to Okada [**O90**].

Corollary 3.3.

$$\prod_{1 \le i < j \le n} (x_i + tx_j) s_\lambda(\mathbf{x}) = \sum_{ST \in \mathcal{ST}^\mu} t^{hgt(ST)} (1+t)^{str(ST)-n} x^{wgt(ST)}, \tag{3.3}$$

Finally, when $\mathbf{x} = \mathbf{y}$ we derive a formula appearing in Macdonald (Sec. III.8, Ex. 2, p.259):

Corollary 3.4.

$$P_\mu(\mathbf{x}) = s_\lambda(\mathbf{x}) \prod_{1 \le i < j \le n} (x_i + x_j).$$

where $\mu = \lambda + \delta$ with $\ell(\mu) = n$.

3.2. Example. Consider the case $\mu = (9, 8, 6, 4, 3, 1)$ and the shifted standard tableau:

$$S = \begin{array}{ccccccccc} 1&1&1&2&3&3&4&4&4 \\ &2&2&2&3&4&5&5&5 \\ &&3&4&4&4&5&6& \\ &&&4&5&5&6&& \\ &&&&5&6&6&& \\ &&&&&6&&& \end{array} \in \mathcal{ST}^{9,8,6,4,3,1} \tag{3.4}$$

Now let us assign $'$'s to those entries for which it is essential; that is, for every entry lying immediately above the same entry and some of those for which it is optional (those entries off the main diagonal that are at the start of any continuous strip of equal entries).

This gives, for example,

$$PST = \begin{array}{ccccccccc} 1 & 1 & 1 & 2' & 3' & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \in \mathcal{P}ST^{9,8,6,4,3,1} \tag{3.5}$$

Now we move all the primed entries k' to the left by means of jeu du taquin as far as but no further than their own column, that is with $1'$'s at the top of column 1, $2'$'s at the top of column 2 etc. In doing this it is assumed that k' is less that i for all $i = 1, 2, \ldots, k-1$.

First moving the single $2'$ as far as possible in a north-westerly direction, but no further than column 2.

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 2' & 3' & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 1 & 2' & 1 & 3' & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 2' & 1 & 1 & 3' & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \tag{3.6}$$

Then do the same for the two $3'$'s, moving the upper one first,

$$\longrightarrow \begin{array}{ccccccccc} 1 & 2' & 1 & 3' & 1 & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 1 & 1 & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \tag{3.7}$$

and then the second $3'$

$$\longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 1 & 1 & 3 & 4 & 4 & 4 \\ & 2 & 2 & 3' & 2 & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 1 & 1 & 3 & 4 & 4 & 4 \\ & 2 & 3' & 2 & 2 & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \tag{3.8}$$

Now for the two $4'$'s, again moving the upper $4'$ first

$$\longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 1 & 1 & 4' & 4 & 4 & 4 \\ & 2 & 3' & 2 & 2 & 3 & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 1 & 4' & 1 & 4 & 4 & 4 \\ & 2 & 3' & 2 & 2 & 3 & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1 & 2' & 3' & 4' & 1 & 1 & 4 & 4 & 4 \\ & 2 & 3' & 2 & 2 & 3 & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \tag{3.9}$$

and then the other $4'$

$$\longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&1&1&4&4&4\\ &2&3'&4'&2&3&5'&5&5\\ &&3&2&4&4&5&6&\\ &&&4&5'&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \tag{3.10}$$

Now for the two $5'$'s, the upper one first

$$\longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&1&1&5'&4&4\\ &2&3'&4'&2&3&4&5&5\\ &&3&2&4&4&5&6&\\ &&&4&5'&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&1&5'&1&4&4\\ &2&3'&4'&2&3&4&5&5\\ &&3&2&4&4&5&6&\\ &&&4&5'&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&2&3&4&5&5\\ &&3&2&4&4&5&6&\\ &&&4&5'&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \tag{3.11}$$

and then the other $5'$

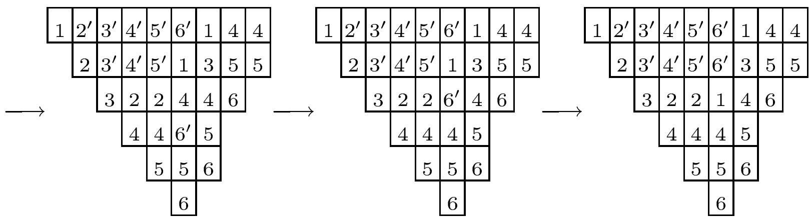
$$\longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&2&3&4&5&5\\ &&3&2&5'&4&5&6&\\ &&&4&4&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&5'&3&4&5&5\\ &&3&2&2&4&5&6&\\ &&&4&4&5&6'&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \tag{3.12}$$

Finally, for the two $6'$'s, first the upper one

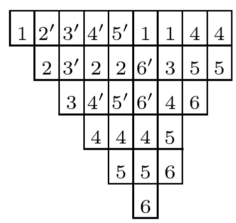
$$\longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&5'&3&4&5&5\\ &&3&2&2&4&6'&6&\\ &&&4&4&5&5&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&5'&3&6'&5&5\\ &&3&2&2&4&4&6&\\ &&&4&4&5&5&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&1&1&4&4\\ &2&3'&4'&5'&6'&3&5&5\\ &&3&2&2&4&4&6&\\ &&&4&4&5&5&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \tag{3.13}$$

$$\longrightarrow \begin{array}{ccccccccc} 1&2'&3'&4'&5'&6'&1&4&4\\ &2&3'&4'&5'&1&3&5&5\\ &&3&2&2&4&4&6&\\ &&&4&4&5&5&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \tag{3.14}$$

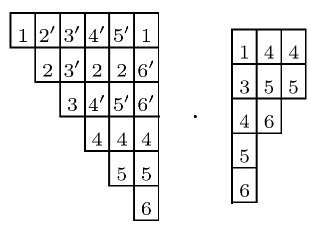
and then the final $6'$

(3.15) 

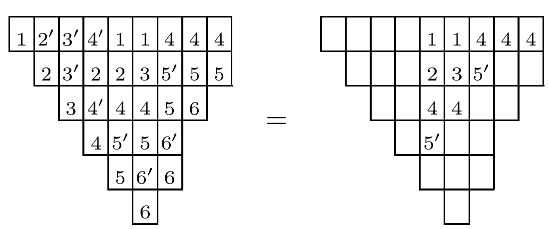
Finally notice that in each of the first 6 columns the entry on the main diagonal is always unprimed and we permute the other entries so as to leave all the unprimed entries in their own rows. This operation still leaves all the primed entries in their own column.

(3.16) 

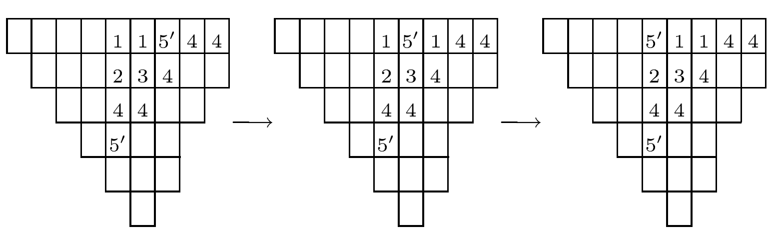
This results in a primed semistandard shifted tableau juxtaposed with a semistandard Young tableau:

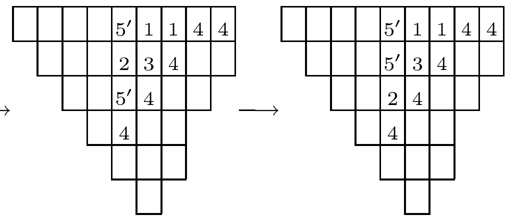
(3.17) 

Note that at an individual stage, say the shifting of the $5'$'s, we can blank out the entries greater than $5'$ in the right hand portion and also strip off the columns to the left of the first column that contains a $5'$. This reduces the problem to a classical jeu de taquin problem. We start with

(3.18) 

Now play the jeu du taquin

(3.19) 

(3.20) $\longrightarrow$ 

4. Connection to Alternating Sign Matrices

In this section we show how to move from PST to alternating sign matrices. Using this relationship, a result of Chapman [**C01**] is a straightforward consequence of Theorem 3.1.

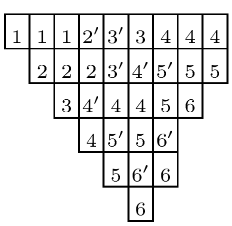
An alternating sign matrix (ASM) is an $n \times n$ matrix filled with 0's, 1's, and -1's such that the first and last nonzero entries of each row and column are 1's and the nonzero entries within a row or column alternate in sign. There is a famous formula, conjectured by Mills, Robbins, and Rumsey [**MRR83**] and proved by Zeilberger [**Z96**], that counts the number of ASM of size n as $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. See also Bressoud [**B99**].

We work with a generalisation of ASM called μ–ASM [**O93**] that can be associated with shifted tableaux. Given a partition μ with distinct parts and such that $\ell(\mu) = n$ and $\mu_1 \leq n$, the set of μ–alternating sign matrices, μ–ASM, is the set of $n \times m$ matrices that satisfy the following conditions:

ASM1 $a_{iq} \in \{-1, 0, 1\}$ for $1 \leq i \leq n, 1 \leq q \leq m$;
ASM2 $\sum_{q=p}^{m} a_{iq} \in \{0, 1\}$ for $1 \leq i \leq n, 1 \leq p \leq m$;
ASM3 $\sum_{i=j}^{n} a_{iq} \in \{0, 1\}$ for $1 \leq j \leq n, 1 \leq q \leq m$
ASM4 $\sum_{q=1}^{m} a_{iq} = 1$ for $1 \leq i \leq n$;
ASM5 $\sum_{i=1}^{n} a_{iq} = 1$ if $q = \mu_j$ for some j; or $\sum_{i=1}^{n} a_{iq} = 0$ otherwise; for $1 \leq q \leq m$.

The bijection to μ–ASM is a special case of our bijection between μ–UASM and symplectic shifted tableaux [**HK03**]. Briefly, associate to each primed shifted tableaux PST of shape μ with $\ell(\mu) = n$ and $\mu_1 = m$ an $n \times m$ matrix filled with the entries from the primed shifted tableaux and with zeros such that if there is an i (resp. i') on diagonal j of the PST (where the main diagonal is diagonal 1 and the last box in the first row is diagonal $\mu_1 = m$), then there is an i (resp. i') in row i (resp. i), column j of the matrix. All other positions are zero.

For example, given a primed shifted tableau of shape $\mu = 9, 8, 6, 4, 3, 1$:

(4.1) $PST =$ 

$$\Longrightarrow M(PST) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2' & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3' & 3' & 3 & 0 & 0 & 0 \\ 4 & 4' & 4 & 4 & 4' & 0 & 4 & 4 & 4 \\ 5 & 5' & 5 & 0 & 5 & 5' & 5 & 5 & 0 \\ 6 & 6' & 6 & 6' & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$$

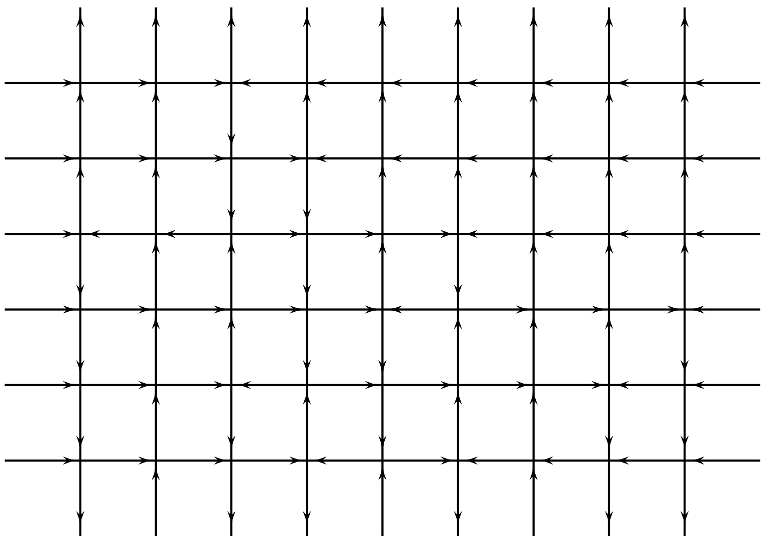
This can be converted into a μ–alternating sign matrix by replacing the rightmost entry of each continuous sequence of nonzero entries by a 1 and each zero immediately to the left of a nonzero entry by -1, leaving all other entries 0.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \end{bmatrix} \in \mathcal{A}^{986431} \tag{4.2}$$

Square ice provides a further refinement of the above bijection. Square ice is a directed graph that models the orientation of oxygen and hydrogen molecules in frozen water. The vertices are laid out in an $n \times n$ grid and each vertex has two incoming and two outgoing edges in a north, south, east, west orientation. At each vertex there are six possible orientations of the four directed edges. The horizontal orientation (with both horizontal edges directed in) corresponds to $+1$ and the vertical orientation (with both vertical edges directed in) corresponds to -1; the other four orientations correspond to 0. Accordingly there are northwest zeros (with edges pointing in the north and west directions), southwest zeros, northeast zeros, and southeast zeros. Northwest zeros are those whose nearest nonzero neighbour to the right, if it has one, is -1, and whose nearest nonzero neighbour below is 1. Southwest zeros are those whose nearest nonzero neighbour to the right, if it has one, is -1, and whose nearest nonzero neighbour below, if it has one, is -1. Northeast zeros are those whose nearest nonzero neighbour to the right is 1, and whose nearest nonzero neighbour below is 1. Southeast zeros are those whose nearest nonzero neighbour to the right is 1, and whose nearest nonzero neighbour below, if it has one, is -1.

$$\begin{array}{cccccc} WE & NS & NE & SW & NW & SE \\ \\ \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\ \longrightarrow \cdot \leftarrow & \leftarrow \cdot \longrightarrow & \longrightarrow \cdot \longrightarrow & \leftarrow \cdot \leftarrow & \leftarrow \cdot \leftarrow & \longrightarrow \cdot \longrightarrow \\ \downarrow & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow \\ \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array} \tag{4.3}$$

The equivalent expression in square ice is



We can also derive a "compass points" matrix:

$$CM = \begin{bmatrix} NE & NE & WE & NW & NW & NW & NW & NW & NW \\ NE & NE & SE & WE & NW & NW & NW & NW & NW \\ WE & NW & NS & SE & NE & WE & NW & NW & NW \\ SE & NE & NE & SE & WE & NS & NE & NE & WE \\ SE & NE & WE & NS & SE & NE & NE & WE & SW \\ SE & NE & SE & WE & NS & WE & NW & SW & SW \end{bmatrix} \tag{4.4}$$

The entries NE in the kth row may be associated with an entry k in PST and correspondingly to a weight factor x_k. The entries SE in the kth row may be associated with an entry k' in PST and correspondingly to a weight factor y_k. The entries NS in the kth row are to be associated with the two possible labels k and k' of the first box of each connected component of $\mathrm{str}_k(PST)$ other than the one starting on the main diagonal. Correspondingly each NS in row k is associated with a weight factor $(x_k + y_k)$. It should be pointed out that the above weighting excludes the weight $x_1 x_2 \cdots x_n$ arising from the entries $1, 2, \ldots, n$ on the main diagonal of each PST.

Combining the weight factors we have a total weight associated with each $A \in \mathcal{A}^\mu$ given by

$$\sum_{A \in \mathcal{A}^\mu} \prod_{k=1}^{n} x_k^{NE_k(A)} \, y_k^{SE_k(A)} \, (x_k + y_k)^{NS_k(A)} \tag{4.5}$$

Corollary 4.1.

$$\prod_{1 \le i < j \le n} (x_i + y_j) s_{1^n}(\mathbf{x}) s_\lambda(\mathbf{x}) = \sum_{A \in \mathcal{A}^\mu} \prod_{k=1}^{n} x_k^{NE_k(A)} y_k^{SE_k(A)} (x_k + y_k)^{NS_k(A)}.$$

where $\mu = \lambda + \delta$.

This generalises a result of Chapman [**C01**]. In his original paper he weights by column instead of row so the parameters in his paper correspond to the transpose matrix.

Corollary 4.2 (Chapman [**C01**])**.**

$$\prod_{1\le i<j\le n}(x_i+y_j)=\sum_{A\in\mathcal{A}}\prod_{k=1}^{n}x_k^{NE_k(A)}y_k^{SE_k(A)}(x_k+y_k)^{NS_k(A)}.$$

5. Other Directions

Okada [**O93**] contains a number of t–deformations of Weyl's denominator formula for root systems B_n, C_n, and D_n. These are similar in form to the Robbins and Rumsey [**RR86**] formula, (1.3), which can be seen as a deformation for A_n. Deformations for B_n and C_n also appear in Simpson [**S97a**][**S97b**] and Hamel and King [**HK02**]. We anticipate that the methods presented here would also apply to these root systems and would enable combinatorial proofs of y generalisations of these t–deformations similar in spirit to (1.4), Chapman's generalisation [**C01**] of Robbins and Rumsey.

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