Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

The Octahedron Recurrence and $\mathfrak{gl}_n$ Crystals

André Henriques and Joel Kamnitzer

Abstract. *We study the hives of Knutson, Tao, and Woodward by means of a modified octahedron recurrence. We define a tensor category where tensor product is given by hives and where the associator and commutor are defined using our recurrence. We then prove that this category is equivalent to the category of crystals for the Lie algebra $\mathfrak{gl}_n$. The proof of this equivalence uses a new connection between the octahedron recurrence and the Jeu de Taquin and Schützenberger involution procedures on Young Tableaux.*

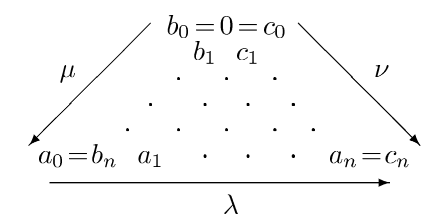
Résumé. *Nous étudions les hives de Knutson, Tao, et Woodward avec une récurrence octaèdre modifiée. Nous définissons une catégorie tensorielle où le produit tensoriel est donné par les hives et où l'associateur et le commuteur sont définies en termes de notre récurrence. Nous montrons que cette catégorie est équivalente à la catégorie des cristaux pour l'algèbre de Lie $\mathfrak{gl}_n$. La preuve de cette équivalence emploie une connection nouvelle entre la récurrence d'octaèdre et, les procédures de Jeu de Taquin et de l'involution de Schützenberger sur les tableaux de Young.*

1. Hives

In [**KTW**], Knutson, Tao, and Woodward introduced hives for studying tensor product multiplicities of $\mathfrak{gl}_n$ representations. Consider the triangle $\{(x, y, z) : x + y + z = n, x, y, z \geq 0\}$. This has $\binom{n+2}{2}$ integer points; call this finite set $\triangle_n$. We will draw it in the plane and put $(n, 0, 0)$ at the lower left, $(0, 0, n)$ at the top, and $(0, n, 0)$ in the lower right.

Let P be a function $P : \triangle_n \to \mathbb{Z}$. We say that P satisfies the *hive condition* if for any unit rhombus in a hive, the sum across the short diagonal is greater than the sum across the long diagonal.

A *hive* is an equivalence class of functions satisfying the hive condition, where two functions are considered to be equivalent if their difference is a constant function. We will usually picture a hive in terms of its representative that takes the value 0 at $(0, 0, n)$.



$\in \mathtt{HIVE}^{\nu}_{\lambda,\mu}$

1991 *Mathematics Subject Classification.* Primary 05E10.
Key words and phrases. hives, crystals, tableaux.
The second author was supported by an NSERC postgraduate scholarship.

By adding together rhombus inequalities along the bottom of the hive, we see that $(a_1 - a_0, a_2 - a_1, \ldots, a_n - a_{n-1})$ is a weakly decreasing sequence of integers. Similarly, the sides labelled by b and c give weakly decreasing sequences of integers.

Let Λ_+ denote the set of weakly decreasing sequences of integers of length n. We can identify Λ_+ with the set of dominant weights of $\mathfrak{gl}_n$.

For $\lambda, \mu, \nu \in \Lambda_+$, let $\mathtt{HIVE}^{\nu}_{\lambda\mu}$ denote the set of hives of size n such that

- the difference on the bottom $(a_1 - a_0, a_2 - a_1, \ldots, a_n - a_{n-1}) = \lambda$
- the differences on the upper left side $(b_1 - b_0, b_2 - b_1, \ldots, b_n - b_{n-1}) = \mu$
- the differences on the upper right side $(c_1 - c_0, c_2 - c_1, \ldots c_n - c_{n-1}) = \nu$

Example 1.1. We will use the following two examples of hives throughout the paper:

$$T = \begin{matrix} & & & 0 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 5 & & 7 & & 8 & & 8 \end{matrix} \in \mathtt{HIVE}^{(3,3,2)}_{(2,1,0),(2,2,1)} \qquad U = \begin{matrix} & & & 0 & & & \\ & & 1 & & 2 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 3 & & 4 & & 5 \end{matrix} \in \mathtt{HIVE}^{(2,2,1)}_{(2,1,1),(1,0,0)}$$

In [**KTW**], Knutson, Tao, and Woodward define a ring with basis b_λ for $\lambda \in \Lambda_+(\mathfrak{gl}_n)$ and multiplication:

$$b_\lambda b_\mu = \sum_\nu c^{\nu}_{\lambda\mu} b_\nu$$

where $c^{\nu}_{\lambda\mu}$ is the size of the set $\mathtt{HIVE}^{\nu}_{\lambda\mu}$. They then prove that their ring is isomorphic to the representation ring of $\mathfrak{gl}_n$. The most difficult step in their proof is to show that their ring is associative.

To prove this associativity they use the octahedron recurrence of [**RR**] to construct a bijection:

$$\bigcup_\delta \mathtt{HIVE}^{\rho}_{\lambda\delta} \times \mathtt{HIVE}^{\delta}_{\mu\nu} \Longrightarrow \bigcup_\gamma \mathtt{HIVE}^{\gamma}_{\lambda\mu} \times \mathtt{HIVE}^{\rho}_{\gamma\nu} \tag{1.1}$$

The purpose of this paper is to modify the octahedron recurrence in order to construct a bijection:

$$\mathtt{HIVE}^{\nu}_{\lambda\mu} \Longrightarrow \mathtt{HIVE}^{\nu}_{\mu\lambda} \tag{1.2}$$

and to understand the structure of these bijections. This structure is most easily seen as giving us an associator and a commutor for a certain tensor category Hives whose simple objects are indexed by Λ_+ and whose tensor product is defined using hives. For the purposes of the present paper, a tensor category is a category with a tensor product along with a natural isomorphism called the *associator* making the tensor product associative and a natural isomorphism called the *commutor* making the tensor product commutative.

1.1. The category Hives.

We now define the category Hives. An object in Hives is not a hive; rather an object A is a choice of finite set A_λ for each $\lambda \in \Lambda_+$ such that only finitely many A_λ are non-empty. A morphism from A, B is just a set map from A_λ to B_λ for each λ.

We think of A as being a representation of $\mathfrak{gl}_n$ along with a direct sum decomposition into irreducible subrepresentations with the elements of A_λ labelling those summands isomorphic to V_λ.

Now we use our hives to define the tensor product on the category. We define:

$$(A \otimes B)_\nu = \bigcup_{\lambda,\mu} A_\lambda \times B_\mu \times \mathtt{HIVE}^{\nu}_{\lambda\mu}$$

Note that:

$$(A \otimes (B \otimes C))_\rho = \bigcup_{\delta,\lambda,\mu,\nu} A_\lambda \times B_\mu \times C_\nu \times \mathtt{HIVE}^{\rho}_{\lambda\delta} \times \mathtt{HIVE}^{\delta}_{\mu\nu}$$

$$((A \otimes B) \otimes C)_\rho = \bigcup_{\gamma,\lambda,\mu,\nu} A_\lambda \times B_\mu \times C_\nu \times \mathtt{HIVE}^{\gamma}_{\lambda\mu} \times \mathtt{HIVE}^{\rho}_{\gamma\nu}$$

So in order to define a natural isomorphism $A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ (an associator) we need a bijection:

$$\bigcup_{\delta} \mathtt{HIVE}^{\rho}_{\lambda\delta} \times \mathtt{HIVE}^{\delta}_{\mu\nu} \Longrightarrow \bigcup_{\gamma} \mathtt{HIVE}^{\gamma}_{\lambda\mu} \times \mathtt{HIVE}^{\rho}_{\gamma\nu}$$

Similarly, to make a natural isomorphism $A \otimes B \to B \otimes A$ (a commutor) we need a bijection:

$$\mathtt{HIVE}^{\nu}_{\lambda\mu} \Longrightarrow \mathtt{HIVE}^{\nu}_{\mu\lambda}$$

To construct these bijections we now introduce the octahedron reccurence.

2. The Octahedron Recurrence

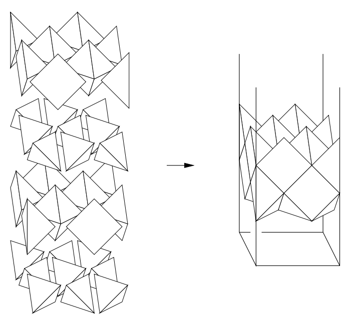


FIGURE 1. The tiling of space-time.

Fix $m, n \in \mathbb{Z}_{>0}$. Let us call *space-time* the space $Y = [0,m] \times [0,n] \times \mathbb{R}$. It contains the lattice $\mathcal{L} = \{(x,y,t) \in \mathbb{Z}^3 \cap Y : \ x+y+z \text{ is even}\}$ on which the recurrence will take place. Y has two compact spatial dimensions and one time dimension. The lattice $\mathcal{L}$ is the set of vertices of a tiling of Y by tetrahedra, octahedra, 1/2-octahedra, and 1/4-octahedra as shown in Figure 1. The tetrahedra are given by

$$\begin{array}{ll} \mathit{conv}\{(x,y,t),(x+1,y+1,t),(x+1,y,t+1),(x,y+1,t+1)\}, & x+y+t \text{ even}, \\ \mathit{conv}\{(x+1,y,t),(x,y+1,t),(x,y,t+1),(x+1,y+1,t+1)\}, & x+y+t \text{ odd}, \end{array}$$

while the octahedra, 1/2-octahedra and 1/4-octahedra are given by

$$Y \cap \mathit{conv}\{(x+1,y,t),(x,y+1,t),(x,y,t+1),(x-1,y,t),(x,y-1,t),(x,y,t-1)\},$$

for $x+y+t$ odd.

A *section* is a connected subcomplex S of the 2-skeleton of the above tiling which contains exactly one point over each (x,y). In particular, S is the graph $S = \{(x,y,h(x,y))\}$ of a continuous map $h : [0,m] \times [0,n] \to \mathbb{R}$. A point $(x,y,t) \in \mathcal{L}$ is said to be in the *future* of a section S if there exists $(x,y,t') \in S$ with $t' \le t$.

A *state* of a subset $A \subset Y$ is an integer valued function $f : A \cap \mathcal{L} \to \mathbb{Z}$. In particular we may speak of the state of a section. The state f of a section S determines the state (again denoted by f) of the set of all points in its future, according to the following modified octahedron recurrence:

(2.1) $$f(x,y,t+1) =$$

$$\begin{array}{ll}
\max\Big(f(x+1,y,t)+f(x-1,y,t), f(x,y+1,t)+f(x,y-1,t)\Big)-f(x,y,t-1) & \\
 & \text{if } 0<x<m, 0<y<n,\\
f(x+1,y,t)+f(x-1,y,t)-f(x,y,t-1) & \text{if } 0<x<m, y=0 \text{ or } n,\\
f(x,y+1,t)+f(x,y-1,t)-f(x,y,t-1) & \text{if } 0<y<n, x=0 \text{ or } m,\\
f(x+1,y,t)+f(x,y+1,t)-f(x,y,t-1) & \text{if } (x,y)=(0,0),\\
f(x+1,y,t)+f(x,y-1,t)-f(x,y,t-1) & \text{if } (x,y)=(0,n),\\
f(x-1,y,t)+f(x,y+1,t)-f(x,y,t-1) & \text{if } (x,y)=(m,0),\\
f(x-1,y,t)+f(x,y-1,t)-f(x,y,t-1) & \text{if } (x,y)=(m,n).
\end{array}$$

So we have one rule if our new point is in the interior (this is the recurrence in [**KTW**] which is the tropicalization of the original octahedron recurrence in [**RR**]), another rule if it lies on a wall, and a third if it lies on a vertical edge. These rules can be seen in Figure 2.

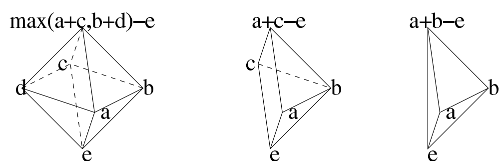


FIGURE 2. The modified octahedron recurrence.

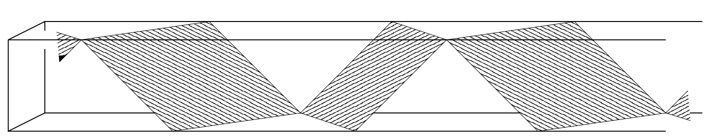
2.1. The hive Condition. We want to use the octahedron recurrence to define operantions on hives. We therefore need to understand how the hive condition propagates through the octahedron recurrence. A *rhombus* in Y is a subcomplex consisting of two coplanar unit triangles touching each other by one edge. A rhombus R has two obtuse vertices and two acute vertices. Given a state f, we say that f satisfies the *hive condition* at R if $f(\text{obtuse vertex})+f(\text{other obtuse vertex})\geq f(\text{acute vertex})+f(\text{other acute vertex})$. We say that f satisfies the hive condition on a section S if it satisfies the above inequality for all rhombi $R\subset S$.

Let S, S' be two sections with S' in the future of S. Let f be a state on S which is extended to a state of S' by the octahedron recurrence. Suppose that f satisfies the hive condition on S. We will now investigate the problem of which hive conditions will be satisfied by f on S'.

A *wavefront* is a subcomplex $W\subset Y$ of the form

$$\begin{aligned}
&W=\big\{(x,y,t)\in Y\big|\exists k\in\mathbb{Z}:\ |t+2k(m+n)+c|=x+y\big\}\\
\text{or}\quad &W=\big\{(x,y,t)\in Y\big|\exists k\in\mathbb{Z}:\ |t+2k(m+n)+c|=x+(n-y)\big\},
\end{aligned}$$

for some constant c. We gave wavefronts their name because one can think of them as world-surfaces of a linear waves propagating at speed 1, and reflecting on the corners of space. A wave front W is composed of big rhombi, touching each other at their acute vertices. Call these acute vertices the *cutpoints* of W.



A wavefront.

We say that a section S is *transverse* to a wavefront W if $W\cap S$ is one dimensional and if no cutpoint of W is contained in S. Given an edge $\alpha\subset W\setminus\partial Y$, let R_α be the rhombus that has α as its small diagonal and that is not contained in W.

Given a state f of S and a wavefront W which is tranverse to it, we say that f satisfies the hive condition at $W \pitchfork S$ if it satisfies the hive condition at each rhombus R_α for $\alpha \subset W \cap S$.

We see that the hive condition propogates along wavefronts in the following way:

Lemma 2.1. *Let S, S', f be as above. Let W be a wavefront transverse to both S and S'. Then f satisfies the hive condition at $W \pitchfork S$ if and only if it satisfies the hive condition at $W \pitchfork S'$.*

3. Operations on Hives

We can define an associator and a commutor for category Hives using the octahedron recurrence. Throughout this section we fix our spacetime to have size $n = m$.

3.1. Associator. Consider the section S which is the graph of the function $|x-y|$. This section is composed of two equilateral triangles which meet along a common edge. Now suppose we have two hives $T \in \mathtt{HIVE}^{\rho}_{\lambda\delta}$ and $U \in \mathtt{HIVE}^{\delta}_{\mu\nu}$. Then the northwest edge of T is the same as the northeast edge of U. Now we have two natural maps $\triangle_n \to S$ by $(x,y,z) \mapsto (x, n-z, y)$ and $(x,y,z) \mapsto (n-z, y, x)$. The images of these two maps are the two equilateral triangles discussed above. Use these maps transport T and U onto S. Since T and U agree on an edge and the points of $\triangle_n$ are all mapped into $\mathcal{L}$, we get a state f of S.

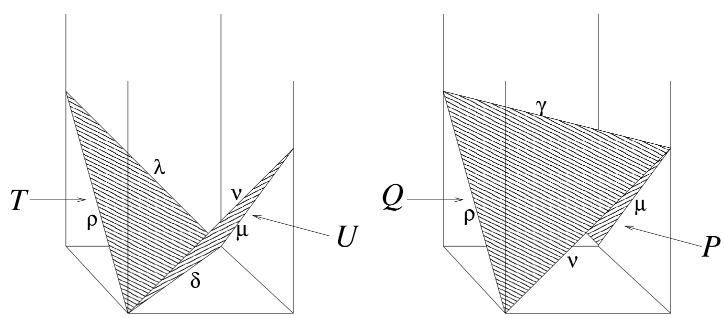


FIGURE 3. The spacelike section S with old hives T, U and the section S' with new hives P, Q.

Once we have f on S, we can use the octahedron recurrence to get the state of any future point. In particular consider the section S' defined as the graph of $n - |n-x-y|$. Note that S' is in the future of S and that four of the edges of S' match four of the edges of S. We again have two natural maps taking $\triangle_n \to S'$, namely $(x,y,z) \mapsto (x,y,n-z)$ and $(x,y,z) \mapsto (n-y, n-z, n-x)$. The state f on S' induces two integer labellings P and Q of $\triangle_n$.

To show that P and Q are hives, consider the set $\mathcal{W}$ of wavefronts W which are transverse to S. It consists of all the wavefronts except the ones that contain a facet of the big tetrahedron $A = \{(x,y,t) : |x-y| \leq t \leq n - |n-x-y|\}$. The wavefronts in $\mathcal{W}$ are also the ones which are transverse to S'. Now, saying that T and U are hives is equivalent to say that f satisfies the hive condition at $S \pitchfork W$ for all $W \in \mathcal{W}$. By Lemma 2.1, this implies the hive condition at $S' \pitchfork W$ for all $W \in \mathcal{W}$. Hence, P and Q are hives.

Example 3.1. Consider the hives T, U from Example 1.1. We apply the octahedron ecurrence and get a state on the region A. Here is its state, shown by a sequence of horizonal slices through A:

0 2 4 5   3 5 7 4 1 3   6 8 6 4 1 3   8 7 4 1

Hence the resulting P, Q are:

$$P = \begin{array}{c} \mu \swarrow \quad 1 \quad \searrow \gamma \\ 3 \;\; 4 \\ 4 \;\; 6 \;\; 7 \\ 5 \;\; 7 \;\; 8 \;\; 8 \\ \xrightarrow{} \\ \lambda \end{array} \qquad Q = \begin{array}{c} \nu \swarrow \quad 0 \quad \searrow \rho \\ 1 \;\; 3 \\ 1 \;\; 4 \;\; 6 \\ 1 \;\; 4 \;\; 7 \;\; 8 \\ \xrightarrow{} \\ \gamma \end{array}$$

Proposition 3.2 ([**KTW**]). *The map:*

$$\bigcup_{\delta} \mathtt{HIVE}^{\rho}_{\lambda\delta} \times \mathtt{HIVE}^{\delta}_{\mu\nu} \to \bigcup_{\gamma} \mathtt{HIVE}^{\gamma}_{\lambda\mu} \times \mathtt{HIVE}^{\rho}_{\gamma\nu}$$
$$(T, U) \mapsto (P(T, U), Q(T, U))$$

is a bijection.

Now for $A, B, C \in \mathsf{Hives}$ we can define the associator:

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$
$$(a, (b, c, U), T) \mapsto ((a, b, P), c, Q)$$

This map is an isomorphism by Proposition 3.2.

3.2. Commutor. We also have a commutor in Hives. Let $P \in \mathtt{HIVE}^{\nu}_{\lambda\mu}$. Let $S = \{(x, y, t) : x + y = t \leq n\}$ (half of a section). Embed P into S by the map $(x, y, z) \mapsto (y, z, n - x)$ and use the octahedron recurrence to evolve this state to the region $A = \{(x, y, t) : x + y \leq t \leq 2n - x - y\}$ (a big 1/4-octahedron). Consider an embedding of $\triangle_n$ into the spacetime by $(x, y, z) \mapsto (y, z, n + x)$. This gives us $P^\star : \triangle_n \to \mathbb{Z}$. It's again possible to check that a wavefront W is transverse to the bottom face S if and only if it is transverse to the top face. We apply Lemma 2.1 and deduce that P is a hive if and only if P^* is.

By examining the octahedron recurrence on the boundary of the spacetime, we see that $P^\star$ has boundary μ, λ, ν and hence $P^\star \in \mathtt{HIVE}^{\nu}_{\mu\lambda}$.

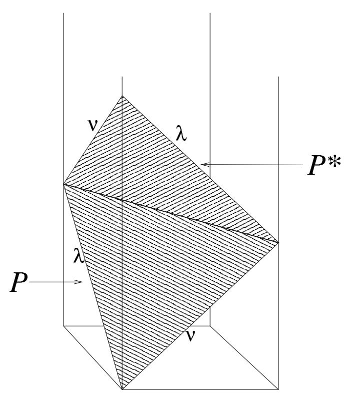


FIGURE 4. The old hive P and the new hive $P^\star$.

Example 3.3. Consider the hive:

$$P = \begin{array}{c} \mu \swarrow \quad 0 \quad \searrow \nu \\ 4 \;\; 4 \\ 6 \;\; 7 \;\; 7 \\ 6 \;\; 8 \;\; 8 \;\; 8 \\ \xrightarrow{} \\ \lambda \end{array}$$

We follow the above procedure and give a state to A. Here is the state as shown by a sequence of horizonal slices through A:

$$8 \qquad \begin{smallmatrix}8 & \\ & 7\end{smallmatrix} \qquad \begin{smallmatrix}8 & & \\ & 7 & \\ 7 & & 4\end{smallmatrix} \qquad \begin{smallmatrix}6 & & & \\ & 6 & & \\ 7 & & 4 & \\ & 4 & & 0\end{smallmatrix} \qquad \begin{smallmatrix}5 & & \\ & 4 & \\ 4 & & 0\end{smallmatrix} \qquad \begin{smallmatrix}2 & \\ & 0\end{smallmatrix} \qquad -2$$

Hence the resulting $P^\star$ is:

$$P^\star = \begin{array}{ccccccc} & & & -2 & & & \\ & \lambda & 0 & & 2 & \nu & \\ & 0 & & 4 & & 5 & \\ 0 & & 4 & & 6 & & 6 \\ & & & \mu & & & \end{array}$$

Proposition 3.4. *The map:*

$$\begin{aligned} \mathtt{HIVE}^{\nu}_{\lambda\mu} &\to \mathtt{HIVE}^{\nu}_{\mu\lambda} \\ P &\mapsto P^\star \end{aligned}$$

is a bijection.

We define the commutor $\sigma_{A,B}$ in Hives by:

$$\begin{aligned} \sigma_{A,B} : A \otimes B &\to B \otimes A \\ (a, b, P) &\mapsto (b, a, P^\star) \end{aligned} \tag{3.1}$$

4. $\mathfrak{gl}_n$ Crystals

We would like to relate the category Hives to $\mathfrak{gl}_n$ crystals. We will study $\mathfrak{gl}_n$ crystals using tableaux. We begin by recalling this connection. These results have generally appeared elsewhere, see for example [**Sh**].

Proposition 4.1. *There exists a crystal structure on the set B_λ of semistandard Young tableaux of shape λ. Moreover this family $\{B_\lambda\}$ is the unique closed family of irreducible highest weight $\mathfrak{gl}_n$ crystals.*

If T, U are two tableaux of shape λ and μ respectively, we can form their skew product denoted $T \star U$ which is the skew tableau made by putting U up and to the right of T. Denote the resulting skew shape by $\lambda \star \mu$.

Example 4.2.

$$\text{If:} \quad \widehat{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \widehat{U} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} 3 & \\ \cline{1-1}\end{array} \qquad \text{then:} \quad \widehat{T} \star \widehat{U} = \begin{array}{cccc} & & 1 & 2 \\ & & 2 & \\ & & 3 & \\ 1 & 3 & & \\ 2 & & & \end{array}$$

Given a skew tableau X, let $J(X)$ be the tableau that results by applying Jeu de Taquin to X. Suppose that X and Y are skew tableaux of the same shape. Choose a particular order for performing Jeu de Taquin. Then X and Y are said to be *dual equivalent* if the shapes of X and Y are the same throughout the Jeu de Taquin process.

We have the following connection between Jeu de Taquin and tensor product:

Theorem 4.1. *The map $B_\lambda \otimes B_\mu \to \cup B_\nu$ by $(T, U) \mapsto J(T \star U)$ is a map of crystals. Moreover, (T, U) and (T', U') are in the same component of $B_\lambda \otimes B_\mu$ iff $T \star U$ and $T' \star U'$ are dual equivalent.*

4.1. Category of crystals. The category $\mathfrak{gl}_n$-$\mathsf{Crystals}$ is the category whose objects are crystals B such that each connected component of B is isomorphic to some B_λ. For the rest of this paper, crystal means an object in this category. A morphism of crystals is a map of the underlying sets that commutes with all the crystal operators. We have the following version of Schur's Lemma:

Lemma 4.3. $\mathrm{Hom}(B_\lambda, B_\mu)$ *contains just the identity if* $\lambda = \mu$ *and is empty otherwise. Hence if* B *is a crystal there is exactly one way to identify each of its components with a* B_λ. □

Let B be a crystal and let $b \in B$ be a high weight element of weight λ. By lemma 4.3, the component of B generated by b is isomorphic to B_λ via a unique isomorphism. So if $T \in B_\lambda$, we let $T(b)$ denote the image of b under this isomorphism. We refer to $T(b)$ as the T-element of the subcrystal generated by b.

The category $\mathfrak{gl}_n$-$\mathsf{Crystals}$ acquires a tensor product by the usual tensor product of crystals. The associator is this category is very simple because if A, B, C are crystals then $A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ by $(a, (b, c)) \mapsto ((a, b), c)$ is an isomorphism of crystals.

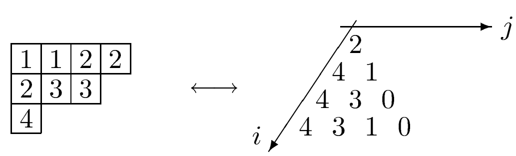
4.2. Commuter for crystals. The basic idea to construct the commuter is to first produce an involution $\xi_B : B \to B$ for each crystal B that reverses the crystal structure. Then the commuter is defined by $(a, b) \mapsto \xi(\xi(b), \xi(a))$. This idea was originally suggested by Arkady Berenstein and is carried out for general $\mathfrak{g}$ in [**HK**]. For our case $\mathfrak{g} = \mathfrak{gl}_n$, the map ξ will be the Schützenberger involution on tableaux. We will now define this involution.

First, recall the defintion of Gelfand-Tsetlin patterns. A *Gelfand-Tsetlin* pattern of size n is a map $T : (i, j) : 1 \le j \le i \le n \to \mathbb{Z}$ such that $T(i, j) \ge T(i-1, j) \ge T(i, j+1)$. We will usually draw a GT pattern in a triangle like a hive of size $n-1$, but we use a different indexing convention than for hives to emphasize that GT pattern are less symmetric. We will index them by pairs (i, j) with $(0, 0)$ on the top $(n, 0)$ on the bottom left and (n, n) on the bottom right.

The *base* of a Gelfand-Tsetlin pattern is the sequence of integers that appear on the bottom row, and the *weight* of a GT pattern is the sequence of difference of row sums from top to bottom.

Recall that there is a bijection between GT patterns of base λ and weight μ and tableaux of shape λ and weight μ. This bijection sends a tableau T to the GT pattern whose value at (i, j) is the number of $1 \ldots i$ on the jth row of T.

Example 4.4. Here is a tableau and the corresponding GT pattern:



This bijection is so natural that we will use the same letter to denote both the tableau and the corresponding GT pattern, so that if T is a tableau, $T(i, j)$ denotes the number of $1 \ldots i$ on row j of T.

For each $1 \le i < n$, we have the *Bender-Knuth move* s_i. This map takes GT patterns of weight λ to themselves by:

$$s_i(T)(k, j) = \begin{cases} \min(T(i+1, j), T(i-1, j-1) + \\ \max(T(i+1, j+1), T(i-1, j)) - T(i, j) & \text{if } k = i \\ T(k, j) \quad \text{otherwise} \end{cases}$$

We can now define the *Schüzenberger involution* by:

$$\begin{aligned} \xi_\lambda : B_\lambda &\to B_\lambda \\ T &\mapsto s_1(s_2 s_1)\cdots(s_{n-1}\cdots s_1)(T) \end{aligned}$$

Example 4.5. Consider:

$$\widehat{P} = \begin{array}{ccccc} & & 1 & & \\ & 3 & & 1 & \\ 4 & & 2 & & 0 \end{array} \overset{s_1}{\mapsto} \begin{array}{ccccc} & & 3 & & \\ & 3 & & 1 & \\ 4 & & 2 & & 0 \end{array} \overset{s_2}{\mapsto} \begin{array}{ccccc} & & 3 & & \\ & 4 & & 1 & \\ 4 & & 2 & & 0 \end{array} \overset{s_1}{\mapsto} \begin{array}{ccccc} & & 2 & & \\ & 4 & & 1 & \\ 4 & & 2 & & 0 \end{array} \quad \text{so} \quad \xi(\widehat{P}) = \begin{array}{ccccc} & & 2 & & \\ & 4 & & 1 & \\ 4 & & 2 & & 0. \end{array}$$

Proposition 4.6. *The Schützenberger involution has the following properties with respect to the crystal structure on* B_λ*:*

$$\begin{aligned} e_i \cdot \xi(T) &= \xi(f_{n-i} \cdot T) \\ f_i \cdot \xi(T) &= \xi(e_{n-i} \cdot T) \\ \mathrm{wt}(\xi(T)) &= w_0 \cdot \mathrm{wt}(T) \end{aligned}$$

where w_0 *denotes the long element in the symmetric group.*

Extend ξ to a map $\xi_B : B \to B$ for all crystals B by applying the appropriate ξ_λ to each connected component of B.

Let A, B by crystals. We define:

$$\begin{aligned} \sigma_{A,B} : A \otimes B &\to B \otimes A \\ (a, b) &\mapsto \xi_{B\otimes A}(\xi_B(b), \xi_A(a)) \end{aligned} \tag{4.1}$$

Theorem 4.2. $\sigma_{A,B}$ *is a natural isomorphism of crystals.*

5. Equivalence of Categories

Recall that a tensor functor Φ : Crystals $\to$ Hives is a functor Φ along with natural isomorphisms

$$\phi_{A,B} : \Phi(A) \otimes \Phi(B) \to \Phi(A \otimes B)$$

such that the following diagrams commute:

$$\begin{array}{ccc} \Phi(A) \otimes (\Phi(B) \otimes \Phi(C)) & \xrightarrow{\ \alpha\ } & (\Phi(A) \otimes \Phi(B)) \otimes \Phi(C) \\ \downarrow{\scriptstyle \phi\circ 1\otimes\phi} & & \downarrow{\scriptstyle \phi\circ\phi\otimes 1} \\ \Phi(A \otimes (B \otimes C)) & \xrightarrow[\Phi(\alpha)]{} & \Phi((A \otimes B) \otimes C). \end{array} \tag{5.1}$$

$$\begin{array}{ccc} \Phi(A) \otimes \Phi(B) & \xrightarrow{\ \sigma\ } & \Phi(B) \otimes \Phi(A) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \Phi(A \otimes B) & \xrightarrow{\Phi(\sigma)} & \Phi(B \otimes A) \end{array} \tag{5.2}$$

An equivalence of tensor categories is a pair of tensor functors which give rise to an equivalence of categories.

The rest of the paper will be devoted to establishing the following result:

Theorem 5.1. *There exists an equivalence of tensor categories between* Crystals *and* Hives*.*

We define start by defining functors $\Phi : \mathsf{Crystals} \to \mathsf{Hives}$ and $\Psi : \mathsf{Hives} \to \mathsf{Crystals}$ by

$$\Phi(B)_\lambda = \{\text{set of highest weight elements of B of weight } \lambda\}$$
$$\Psi(A) = \bigcup_\lambda A_\lambda \times B_\lambda$$

Clearly these functors provide an equivalence of categories, so it remains to enrich one of them to a tensor functor.

5.1. From Tableaux to Hives. Because of the way we have defined Φ, it will be very important for us to think about high weight elements of crystals. In particular we must consider the high weight elements of tensor products. Let B be a crystal. Recall that we have a map $\varepsilon_i : B \to \mathbb{Z}$ such that $\varepsilon_i(b)$ is the number of times we can apply e_i to b. We say that $b \in B$ is μ-dominant if $\varepsilon_i(b) \leq \langle \mu, \alpha_i^\vee \rangle$ for all $i \in I$. Examining the definition of tensor product formula we have the following observation which we first found in [**St**]:

Lemma 5.1. *Let $b \in B$ and $c \in C$. Then (b, c) is high weight iff b is μ-dominant and c is high weight of weight μ.*

A *quasi-hive* is an equivalence class of maps $P : \triangle_n \to \mathbb{Z}$ which satisfies the two horizontal rhombus axioms, but not neccessarily the vertical rhombus axiom.

Given a quasi-hive, we can produce a GT pattern $\widehat{P}$ by defining $\widehat{P}(i, j) = P(i - j, j, n - i) - P(i - j + 1, j - 1, n - i)$.

Example 5.2. For hives T, U from Example 1.1 we get the GT patterns:

$$\widehat{T} = \begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \end{matrix} \qquad \widehat{U} = \begin{matrix} & & 1 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 1 \end{matrix}$$

which correspond to the tableaux of Example 4.2.

The following bijection was established by Pak and Vallejo [**PV2**], following similar bijections due to Berenstein and Zelevinsky, and others.

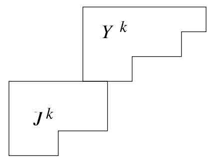
Theorem 5.2. *If P is a quasi-hive, then $\widehat{P}$ is a GT pattern. Moreover, the map $P \mapsto \widehat{P}$ provides a bijection between $\mathtt{HIVE}^\nu_{\lambda\mu}$ and the set of μ-dominant tableaux of shape λ and weight $\nu - \mu$.*

Now we can define the natural isomorphisms $\phi_{A,B}$ for $A, B \in \mathsf{Crystals}$ by:

$$\phi_{A,B} : \Phi(A) \otimes \Phi(B) \to \Phi(A \otimes B)$$
$$(a, b, P) \mapsto (\widehat{P}(a), b)$$

To see that this makes sense, note that a is a high weight element of A of weight λ, b is a high weight element of B of weight μ and $P \in \mathtt{HIVE}^\nu_{\lambda\mu}$. Then by Lemma 5.1 and Theorem 5.2 $(\widehat{P}(a), b)$ is a high weight element of $A \otimes B$. It is of weight ν since $\widehat{P}(a)$ has weight $\nu - \mu$ and b has weight μ.

5.2. Associator. In order to prove that (5.1) commutes we need to better understand what happens to tableaux in tensor products. Let X, Y be tableaux. One way to perform the Jeu de Taquin process on $X \star Y$ is to first excavate all the empty boxes to the left of the last row of Y, then those to the left of the second last row, etc. After excavating the boxes to the left of rows $n, \ldots, k+1$ of Y, the resulting skew tableau will be of the form:



where J^k is some tableau and Y^k denotes the first k rows of Y. Note that $J^n = X$ and $J^0 = J(X \star Y)$.

Example 5.3. If $\widehat{T}$ and $\widehat{U}$ are as in Example 4.2, the Jeu de Taquin process produces:

$$\widehat{T} = J_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad J_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad J_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad J_0 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Now let λ^k denote the shape of J^k. We define a recording tableau $R(X,Y)$ for the Jeu de Taquin process. We define it in terms of the associated GT pattern by:

$$R(X,Y)(i,j) = \sum_{r \geq j} \lambda_r^{i-j+1} - \sum_{r \geq j+1} \lambda_r^{i-j}$$

Example 5.4. For the above example,

$$\lambda^3 = (2,1,0),\ \lambda^2 = (3,1,0),\ \lambda^1 = (3,2,0),\ \lambda^0 = (3,3,1)$$

So as a GT pattern:

$$R(\widehat{T}, \widehat{U}) = \begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \end{array}$$

We establish the following refinement of Theorem 4.1:

Theorem 5.3. *If $X \in B_\lambda$, $Y \in B_\mu$, then (X,Y) sits in the component of $B_\lambda \otimes B_\mu$ with high weight element $(R(X,Y), b_\mu)$ and represents the $J(X \star Y)$ element of that crystal.*

Returning to our proof that (Φ, ϕ) is a tensor functor, we want to prove that the following diagram commutes:

$$\begin{array}{ccc} \Phi(A) \otimes (\Phi(B) \otimes \Phi(C)) & \xrightarrow{\alpha} & (\Phi(A) \otimes \Phi(B)) \otimes \Phi(C) \\ \downarrow {\scriptstyle \phi \circ \phi \otimes 1} & & \downarrow {\scriptstyle \phi \circ 1 \otimes \phi} \\ \Phi(A \otimes (B \otimes C)) & \xrightarrow[\Phi(\alpha)]{} & \Phi((A \otimes B) \otimes C)) \end{array}$$

Let us carefully examine what we need to prove. Let $(a, (b, c, U), T) \in (\Phi(A) \otimes (\Phi(B) \otimes \Phi(C)))_\rho$. Then for some δ:

$$a \in \Phi(A)_\lambda,\ b \in \Phi(B)_\mu,\ c \in \Phi(C)_\nu,\ T \in \mathtt{HIVE}^\rho_{\lambda\delta},\ U \in \mathtt{HIVE}^\delta_{\mu\nu}$$

Let $P = P(T,U), Q = Q(T,U)$. Following the diagram along the top and then down gives $(\widehat{P}(\widehat{Q}(a), b), c)$. Following the diagram down and then along the bottom gives $((\widehat{T}(a), \widehat{U}(b)), c)$.

Hence we must show that in the tensor product of $B_\lambda \otimes B_\mu$, $(\widehat{T}, \widehat{U})$ lies in the same component as $(\widehat{Q}, b_\mu)$ and that it represents the $\widehat{P}$ element of that crystal. By Theorem 5.3, we see that it suffices to prove the following result which was conjectured by Pak and Vallejo in [**PV1**]:

Theorem 5.4. *We have the following relation between the octahedron recurrence and Jeu de Taquin:*

$$J(\widehat{T} \star \widehat{U}) = \widehat{P} \quad R(\widehat{T}, \widehat{U}) = \widehat{Q}$$

In fact more is true. Each stage of J^k of the Jeu de Taquin procedure can be read off from the octahedron recurrence:

Proposition 5.5. *Use T, U to give a state f to S as in section 3.1. Use the octahedron recurrence to extend this state to the region $A = \{(x, y, t) : |x - y| \leq t \leq n - |n - x - y|\}$.*

Then for each k define the map

$$
\begin{aligned}
&r^k : \triangle_n \to A \\
&(x, y, z) \mapsto \begin{cases} (n - z, x, y) \text{ for } x \leq k \\ (y + k, x, n - k - z) \text{ for } x \geq k \end{cases}
\end{aligned}
\tag{5.3}
$$

Use r^k to define a quasi-hive $P^k = f \circ r^k$.
Then $\widehat{P^k} = J^k(\widehat{T}, \widehat{U})$.

Example 5.6. Choosing hives T, U from Example 1.1, produces the state in Example 3.1. Reading off the P^k from this state gives:

$$
P^3 = T = \begin{array}{ccccccc} &&&0&&& \\ &&2&&3&& \\ &4&&5&&6& \\ 5&&7&&8&&8 \end{array}
\quad P^2 = \begin{array}{ccccccc} &&&0&&& \\ &&2&&3&& \\ &4&&5&&6& \\ 4&&7&&8&&8 \end{array}
\quad P^1 = \begin{array}{ccccccc} &&&0&&& \\ &&2&&3&& \\ &3&&5&&6& \\ 3&&6&&8&&8 \end{array}
\quad P^0 = P = \begin{array}{ccccccc} &&&0&&& \\ &&1&&3&& \\ &1&&4&&6& \\ 1&&4&&7&&8 \end{array}
$$

These hives T, U correspond to the tableaux $\widehat{T}, \widehat{U}$ from Example 4.2. Applying Jeu de Taquin to this pair of tableaux produced the intermediate tableaux J^k in Example 5.3. Note that the hives P^k correspond to these tableaux J^k and that the hive Q from Example 3.1 corresponds to the recording tableau $R(\widehat{T}, \widehat{U})$ from Example 5.4.

5.3. Commuter. To prove that the commuter diagram commutes we begin with the following consideration. Let $P \in \texttt{HIVE}^{\nu}_{\lambda\mu}$. By Lemma 5.1 and Theorem 5.2, $(\widehat{P}, b_\mu)$ is a high weight element of $B_\lambda \otimes B_\mu$. P can also be turned into a tableau $\widetilde{P}$ of shape μ by the formula:

$$
\widetilde{P}(i, j) = P(j, n - i, i - j) - P(j - 1, n - i, i - j + 1) \tag{5.4}
$$

Example 5.7. If P is as in Example 3.3, then as GT pattern:

$$
\widetilde{P} = \begin{array}{ccccc} &&1&& \\ &3&&1& \\ 4&&2&&0 \end{array}
$$

Note that each crystal B_λ posseses a *lowest weight element*, that is a $c_\lambda \in B_\lambda$ that is killed by all f_i and such that B_λ is generated by e_i acting on c_λ. In terms of tableaux, c_λ is the tableau with n at the end of every row, $n - 1$ second from the end of every row, etc. Also note that $\xi(b_\lambda) = c_\lambda$.

Recall that for $P \in \texttt{HIVE}^{\nu}_{\lambda\mu}$, $(\widehat{P}, b_\mu)$ was a highest weight element of the crystal $B_\lambda \otimes B_\mu$. We have the following related result for $\widetilde{P}$:

Lemma 5.8. *$(c_\lambda, \widetilde{P})$ is the lowest weight element of connected component of $B_\lambda \otimes B_\mu$ with highest weight $(\widehat{P}, b_\mu)$.*

Returning to the commutor diagram, we need to prove that the following commutes:

$$
\begin{array}{ccc}
\Phi(A) \otimes \Phi(B) & \xrightarrow{\sigma} & \Phi(B) \otimes \Phi(A) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
\Phi(A \otimes B) & \xrightarrow{\Phi(\sigma)} & \Phi(B \otimes A)
\end{array}
$$

Let $(a, b, P) \in (\Phi(A) \otimes \Phi(B))_\nu$, where:

$$
a \in \Phi(A)_\lambda, \quad b \in \Phi(B)_\mu, \quad P \in \texttt{HIVE}^{\nu}_{\lambda\mu}
$$

Following the diagram along the top and then down gives us

$$\phi(b, a, P^\star) = (\widehat{P^\star}(b), a) \tag{5.5}$$

Following the diagram down and then along the bottom gives:

$$\Phi(\sigma)(P(a), b) = \xi \otimes \xi \circ \text{flip} \circ \xi(P(a), b) = (\xi \otimes \xi)(\widetilde{P}(b), \xi(a)) = (\xi(\widetilde{P}(b), a)) \tag{5.6}$$

by Lemma 5.8 and the fact that $\xi(T(a)) = \xi(T)(a)$ which follows from the way we extended ξ componentwise.

Comparing (5.5) and (5.6), we see that it suffices prove the following relation between the Schützenberger involution and the octahedron recurrence:

Theorem 5.5. *Let P be a hive. Then:*

$$\widehat{P^\star} = \xi(\widetilde{P})$$

As for the Jeu de Taquin, each stage of the Schützenberger involution can be seen.

Let $A = \{(x, y, t) : x + y \leq t \leq 2n - x - y\}$, the region used to compute the commuter map $P \mapsto P^\star$. Let $r : \triangle_n \to A$ be an inclusion. We say that r is *standard* if it is of the form:

$$(x, y, z) \mapsto (x, y, h(z))$$

for some continuous function $h : [0 \dots n] \to [0 \dots 2n]$ with $h(0) = n$ and $h(z-1) \in \{h(z)+1, h(z)-1\}$ for $z \in \{1, \dots, n\}$.

If $0 < i < n$, we say that such an r is *i-flippable* if $h(n-i+1) = h(n-i-1) = h(n-i)+1$. We say that r is *0-flippabble* if $h(n-1) = h(n)+1$. If r is i-flippable, we define $t_i(r)$ by the formula:

$$t_i(r)(x, y, z) = \begin{cases} r(x, y, z) + (0, 0, 2) & \text{if } z = n - i \\ r(x, y, z) & \text{otherwise} \end{cases}$$

Now, let M be a quasi-hive and let r be a standard i-flippable embedding. Use M to given a state f to $\operatorname{im}(r)$. This determines a state on the image of $s_i(r)$ by the octahedron recurrence. Note that $t_i(M) := f \circ (r_i(m))$ is a quasi-hive by consideration of appropriate wavefronts.

Proposition 5.9. *With the above setup, if $i \neq 0$ we have:*

$$\widehat{t_i(M)} = s_i(\widehat{M})$$

where on the RHS we are using the Bender-Knuth move.

Also, $\widehat{t_0(M)} = \widehat{M}$.

Example 5.10. Let P be as in Example 3.3. Let r be the embedding $(x, y, z) \mapsto (x, y, n - z)$. Using r and P we get a state to the region A as shown in Example 3.3. From there we can read off:

$$P \overset{t_0}{\mapsto} \begin{array}{c} 7 \\ 7 \;\; 8 \\ 4 \;\; 7 \;\; 8 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array} \overset{t_1}{\mapsto} \begin{array}{c} 7 \\ 4 \;\; 7 \\ 4 \;\; 7 \;\; 8 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array} \overset{t_2}{\mapsto} \begin{array}{c} 7 \\ 4 \;\; 7 \\ 0 \;\; 4 \;\; 5 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array}$$

$$\overset{t_0}{\mapsto} \begin{array}{c} 4 \\ 4 \;\; 7 \\ 0 \;\; 4 \;\; 5 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array} \overset{t_1}{\mapsto} \begin{array}{c} 4 \\ 0 \;\; 2 \\ 0 \;\; 4 \;\; 5 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array} \overset{t_0}{\mapsto} \begin{array}{c} -2 \\ 0 \;\; 2 \\ 0 \;\; 4 \;\; 5 \\ 0 \;\; 4 \;\; 6 \;\; 6 \end{array} = P^\star$$

$\widetilde{P}$ is shown in Example 5.7 and the computation of $\xi(\widetilde{P})$ is shown using Bender-Knuth moves in Example 4.5. Note that the intermediate stages of that computation match the intermediate stages shown above.

Acknoledgements. We would like to thank Arkady Berenstein, Cilanne Boulet, Mark Haiman, Allen Knutson, Igor Pak, David Speyer, and Dylan Thurston for helpful conversations.

References

[HK] A. Henriques and J. Kamnitzer, *Crystals and bubbled categories*, In preparation.

[KTW] A. Knutson, T. Tao and C. Woodward, *A positive proof of the Littlewood-Richardson rule using the octahedron recurrence*, arXiv:math.CO/0306274.

[PV1] I. Pak and E. Vallejo, *Reductions of Young tableau bijections*, In preparation.

[PV2] I. Pak and E. Vallejo, *Combinatorics and geometry of Littlewood-Richardson cones*, Preprint available at: `http://www-math.mit.edu/~pak/research.html`.

[RR] D.P. Robbins and H. Rumsey, *Determinants and alternating-sign matrices*, Adv. Math. **62** (1986), 169–184.

[Sh] M. Shimozono, *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties*, J. Algebraic Combin. **15** (2002), no. 2, 151–187.

[St] J. Stembridge, *Combinatorial models for Weyl characters*, Adv. Math. **168** (2002), no. 1, 96–131.

Department of Mathematics, MIT, Cambridge, MA, USA
E-mail address: `andrhenr@math.mit.edu`

Department of Mathematics, UC Berkeley, Berkeley, CA, USA
E-mail address: `jkamnitz@math.berkeley.edu`