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A Signed Analog of the Birkhoff Transform

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Abstract. *We construct a family of posets, called signed Birkhoff posets, that may be viewed as signed analogs of distributive lattices. Our posets are generally not lattices, but they are shown to posses many combinatorial properties corresponding to well known properties of distributive lattices. They have the additional virtue of being face posets of regular cell decompositions of spheres. We give a combinatorial description the* **cd**-*index of a signed Birkhoff poset in terms of peak sets of linear extensions of an associated labeled poset. Our description is closely related to a result of Billera, Ehrenborg, and Readdy's expressing the* **cd**-*index of an oriented matroid in terms of the flag f-vector of the underlying geometric lattice. As an analog of the Distributive Lattice Conjecture, we conjecture that the chain polynomial of a signed Birkhoff poset has only real zeros.*

1. Introduction

This paper introduces a signed analog of the standard construction of a distributive lattice $J(P)$ from a finite poset P. Beginning with the work of Birkhoff [**Bi**], distributive lattices have been well studied fom a combinatorial viewpoint. Nowadays they are often analyzed in conjunction with notions such as P-partitions, linear extensions, and R-labelings; see, e.g., [**Sta4**, Chapter 3]. Our construction will give rise to a family of Eulerian posets that are amenable to similar types of analyses. Stembridge's enriched P-partitions [**Ste**] turn out to play a role in the enumeration theory of these posets that is analogous to the role of Stanley's P-partitions [**Sta1**] for distributive lattices. Our enumerative analysis is motivated by the work of Billera, Ehrenborg, and Readdy on the **cd**-index of oriented matroids [**BER**]. Although the posets that we construct are not directly related to face lattices of oriented matroids, the flag vectors of these two classes of posets are seen to have some remarkably similar properties.

Given a positive integer n and a poset P on the set $[n] := \{1, 2, \ldots, n\}$ partially ordered by $\leq_P$, let $\pm P$ be the poset on $\{\pm 1, \ldots, \pm n\}$ ordered so that $p <_{\pm P} q$ if and only if $|p| <_P |q|$. A *filter* of a poset Q is a subset X of Q such that whenever $q \in X$ and $q <_Q q'$ then $q' \in X$. The *Birkhoff transform of P* is the poset (distributive lattice) $J(P)$ consisting of the filters of P ordered by reverse inclusion.[1] Define a *signed P-filter* to be a filter X of $\pm P$ such that if p is a minimal element of X then $-p \notin X$. We now define the main object of study.

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[1] Usually $J(P)$ is defined as the poset of order ideals of P under inclusion, rather than as the filters under reverse inclusion; these two definitions yield isomorphic posets. Filters turn out to be more convenient for us notationally.

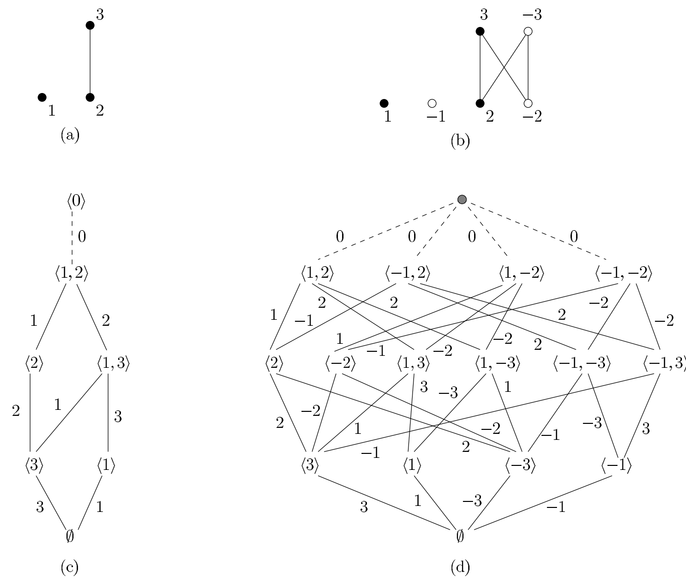

FIGURE 1. (a) A naturally labeled poset P; (b) the induced labeled poset $\pm P$; (c) the dual Birkhoff transforms $J(P)^*$ (without top element) and $J(P_0)^*$ (with top element), with edge-labeling induced by P; (d) the signed Birkhoff transform $B(P)$ (without top element) and $\widehat{B}(P)$ (with top element), with edge-labeling induced by P.

Definition 1.1. The *signed Birkhoff transform of* P is the poset $B(P)$ consisting of all signed P-filters ordered by inclusion.

Note that one could define the signed Birkhoff transform more abstractly without identifying P with $[n]$. This identification is made here for notational convenience and without loss of generality. Let $\widehat{B}(P)$ denote the poset $B(P)$ with a unique maximal element $\hat{1}$ added. Any poset of the form $B(P)$ or $\widehat{B}(P)$ is called a *signed Birkhoff poset.*[2] For clarity we sometimes call $\widehat{B}(P)$ a *graded signed Birkhoff poset* (cf. Proposition 2.2).

Figure 1 illustrates both the ordinary and signed Birkhoff transforms of a three element poset. Filters in the figure are denoted by $\langle p_1, \ldots, p_m \rangle$, where $p_1, \ldots, p_m$ are the minimal generators of the filter. Let us also point out two interesting families of examples. First recall that the *face poset* $P(\Gamma)$ of a finite regular cell complex Γ is the poset of cells of Γ, along with the empty cell, ordered by inclusion of their closures.

Example 1.2. If P is an n-element chain, then $B(P)$ is isomorphic to the face poset of a regular cell decomposition of the $(n-1)$-sphere with exactly two cells in each dimension. Such a poset is sometimes called a ladder.

Example 1.3. If P is an n-element antichain, then $B(P)$ is isomorphic to the face poset of the boundary of an n-dimensional hyperoctahedron.

Our main results are summarized below.

In Section 2 we discuss basic structural properties of signed Birkhoff posets, the highlight being a "pairing procedure" (Theorem 2.5) that allows one to recover P uniquely (up to isomorphism) from $B(P)$. This is analogous in part to Birkhoff's fundamental theorem for finite distributive lattices, which asserts that every

[2]To our knowledge, there is no direct connection between signed Birkhoff posets and the hyperoctahedral analogs of posets, called signed posets, introduced by Reiner [**R**].

finite distributive lattice L is isomorphic to the poset of order ideals of the subposet of join irreducibles of L. Presently lacking in this analogy is an intrinsic characterization of signed Birkhoff posets that avoids reference to an underlying poset P. Interestingly, $\widehat{B}(P)$ is *not* a lattice unless P is an antichain (Proposition 2.1), so the pairing procedure does not involve lattice notions such as join irreducibility.

Section 3 deals with shellability properties of signed Birkhoff posets. We show that the edge-labeling of $\widehat{B}(P)$ induced by a natural labeling of P is an EL-labeling and a dual R-labeling (Theorem 3.1). This implies that $\widehat{B}(P)$ is Gorenstein* for every P. The Gorenstein* property is also a consequence of the fact that $B(P)$ is the face poset of a regular shellable decomposition of a sphere (Theorem 3.4). This result, first established by Billera and the author, is proved here by showing that $\widehat{B}(P)$ admits a recursive coatom ordering (Theorem 3.3) then invoking a theorem of Björner's on cellular interpretations of posets [**Bj2**].

Section 4 covers enumerative aspects of signed Birkhoff posets. Let P_0 denote the poset P with a unique minimal element added. We establish the identity (Theorem 4.1)

$$2F_{\widehat{B}(P)^*} = \widetilde{K}_{P_0} \tag{1.1}$$

relating Ehrenborg's F-quasisymmetric function (which encodes the flag f-vector) of the dual poset $\widehat{B}(P)^*$ to the weight enumerator for enriched P_0-partitions. This fundamental identity follows easily from Stembridge's original work on enriched P-partitions [**Ste**] as well as from Bergeron, Mykytiuk, Sottile, and van Willigenburg's theory of Eulerian Pieri operators [**BMSW**, Section 7]. The latter work is relevant because of the close connection between the signed Birkhoff transform and the doubled réseau of a distributive lattice. A corollary of (1.1) is a description of the zeta polynomial of $\widehat{B}(P)$ in terms of the enriched order polynomial of P_0. Using recent work of Billera, Hsiao, and van Willigenburg [**BHW**] connecting the **cd**-index to Stembridge's peak algebra, we derive from (1.1) a combinatorial interpretation of the **cd**-index of $\widehat{B}(P)$ in terms of peak sets of linear extensions of P_0 (Theorem 4.4). Our description implies that the **cd**-index of $\widehat{B}(P)$ is coefficient-wise maximized when P is an antichain and minimized when P is a chain. There is an elegant reformulation of (1.1) that directly relates the **cd**-index of a signed Birkhoff poset to the flag f-vector of its underlying distributive lattice (Theorem 4.12). Our formula is essentially identical to the expression provided by Billera, Ehrenborg, and Readdy [**BER**] relating the **cd**-index of an oriented matroid to the flag f-vector of its geometric lattice of flats (Theorem 4.11).

In Section 5 we conjecture that the chain polynomial of $\widehat{B}(P)$ has only real zeros. This is a signed analog of the Distributive Lattice Conjecture, which is equivalent to the Neggers-Stanley Poset Conjecture for naturally labeled posets [**Br1**]. We show that ours is equivalent to Stembridge's Enriched Poset Conjecture for naturally labeled posets having a unique minimal element.

All posets in this paper are assumed to be finite unless otherwise indicated. A graded poset is always assumed to have a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Unexplained terminology and further background related to posets can be found in [**Sta4**, Chapter 3].

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2. Signed Birkhoff posets

Assume throughout this section that $n > 0$ is fixed and P is a poset on $[n]$ ordered by $\leq_P$. Sometimes P is called a *labeled poset*. Say that P is *naturally labeled* if $p <_P q$ implies $p < q$ as integers. Let P_0 denote the labeled poset obtained from P by adding a unique minimal element labeled 0.

2.1. Basic properties. Some familiar properties of ordinary Birkhoff transforms carry over to signed transforms without much difficulty. For instance, as with the identity $J(P \sqcup Q) \cong J(P) \times J(Q)$, it is straightforward to show that

$$B(P \sqcup Q) \cong B(P) \times B(Q), \tag{2.1}$$

where $\sqcup$ and $\times$ denote, respectively, the disjoint union and cartesian product operations for posets.

Unlike the class of distributive lattices, the class of signed Birkhoff posets is not closed under taking intervals. For instance, the poset in Figure 1(d) has several intervals that are isomorphic to the Boolean lattice of rank 3, which itself is not a signed Birkhoff poset. The following result points to another significant difference between these two classes of posets.

Proposition 2.1. *$\widehat{B}(P)$ is a lattice if and only if P is an antichain.*

In the sequel it will be useful to relate ordinary and signed transforms via the order-reversing surjective map $\varphi : \widehat{B}(P) \to J(P_0)$ defined by

$$\varphi(X) = \begin{cases} \{|p| : p \in X\} & \text{if } X \in B(P), \\ P_0 & \text{if } X = \hat{1}. \end{cases}$$

Note that φ restricts to a map from $B(P)$ onto $J(P)$.

The cover relations in $J(P)$ are precisely those relations of the form $A \cup \{p\} < A$ for some maximal element p of $P \backslash A$. Thus $J(P)$ is graded of rank n with rank function given by $rk(A) = n - \#A$. The corresponding assertions for signed Birkhoff posets follow easily:

Proposition 2.2. *The cover relations in $B(P)$ are precisely those relations of the form $X < X \cup \langle p \rangle$ such that p and $-p$ are maximal elements of $\pm P \backslash X$ or, equivalently, $|p|$ is a maximal element of $P \backslash \varphi(X)$. Thus $\widehat{B}(P)$ is a graded poset of rank $n+1$ with rank function given by $rk(X) = \#\varphi(X)$.*

It is a basic property of the Birkhoff transform that a sequence $(p_1, \ldots, p_n) \in P^{\times n}$ is in $\mathcal{L}(P)$, the set of linear extensions of P, if and only if $\{p_1, \ldots, p_n\} < \{p_2, \ldots, p_n\} < \cdots < \{p_n\} < \emptyset$ is a maximal chain of $J(P)$. By Proposition 2.2, if $c = \{\emptyset = X_0 \lessdot X_1 \lessdot \cdots \lessdot X_n\}$ is a maximal chain of $B(P)$ then there exists a sequence $\lambda(c) = (p_1, \ldots, p_n) \in (\pm P)^{\times n}$ such that $X_i = X_{i-1} \cup \langle p_i \rangle$ for all i. Such sequences can be characterized as "signed linear extensions" of P:

Proposition 2.3. *Let $\pi \in (\pm P)^{\times n}$. Then $\pi = \lambda(c)$ for some (unique) maximal chain c of $B(P)$ if and only if $\pi = \varepsilon\sigma$ for some $(\varepsilon, \sigma) \in \{\pm 1\}^{\times n} \times \mathcal{L}(P)$.*

Remark 2.4. The *doubled réseau* $\delta J(P)$ studied by Bergeron, et al. in [**BMSW**] is the directed graph obtained by replacing each labeled edge $A \cup \{p\} \xrightarrow{p} A$ in the Hasse diagram of $J(P)$ with the two labeled edges $A \cup \{p\} \overset{p}{\underset{-p}{\rightrightarrows}} A$. In light of Proposition 2.3, we may view signed Birkhoff posets as "poset realizations" of doubled réseaux of distributive lattices. It is then possible to infer a direct connection between flag enumeration in $\widehat{B}(P)$ and weight enumeration of enriched P-partitions via the theory of Eulerian Pieri operators developed in [**BMSW**, Section 7]; see Theorem 4.1 and Remark 4.2.

2.2. The pairing procedure. Let $B = B(P)$. We describe a procedure for recovering P from B. Define an equivalence relation on B by putting $X \equiv X'$ if and only if X and X' cover exactly the same set of elements, so in particular X and X' are of the same rank. Let $T_1, \ldots, T_m$ be the non-singleton equivalence classes in $B/\equiv$, indexed so that $i < j$ whenever the elements of T_i have rank greater than those of T_j. Our goal is to inductively construct posets $B_1, \ldots, B_m$ whose isomorphism types depend only on the isomorphism type of B; the result is that $B_m \cong P^*$.

It is easy to see that $\langle p \rangle \equiv \langle -p \rangle$ for all $p \in P$ and that every T_i is the union of sets of the form $\{\langle p \rangle, \langle -p \rangle\}$. Fix a partition of T_1 into blocks of size two and let B_1 be the antichain consisting of these blocks. Assume by induction that the poset B_{i-1} has been constructed for some $i > 1$. Given $X, X' \in T_i$, write $X \equiv_i X'$ provided that for every $j < i$ and $Y \in T_j$ we have $X < Y$ if and only if $X' < Y$. Each equivalence class in $T_i/\equiv_i$ has even size because $\langle p \rangle \equiv_i \langle -p \rangle$ for any p. Now partition each equivalence class in $T_i/\equiv_i$ arbitrarily into blocks of size two. Define the poset B_i by adjoining these two-element blocks to B_{i-1} and, for any such block $\{X, X'\}$ and any $\{Y, Y'\} \in B_{i-1}$, putting $\{X, X'\} <_{B_i} \{Y, Y'\}$ if and only if X and X' are both less than Y and Y'.

Theorem 2.5. *The pairing procedure, when applied to $B(P)$, always produces a poset that is isomorphic to P^*. Thus P is uniquely determined by $B(P)$ (up to isomorphism).*
Example 2.6. Let $B = B(P)$ be the poset from Figure 1(d). Then the pairing procedure yields the following:

1. $T_1 = \{\langle 2\rangle, \langle -2\rangle\}$;
2. $T_2 = \{\langle 1\rangle, \langle -1\rangle, \langle 3\rangle, \langle -3\rangle\}$;
3. B_1 is the one-element antichain $\{\{\langle 2\rangle, \langle -2\rangle\}\}$;
4. $T_2/\equiv_2 = \{\{\langle 1\rangle, \langle -1\rangle\}, \{\langle 3\rangle, \langle -3\rangle\}\}$;
5. B_2 is the poset on the set $\{\langle 2\rangle, \langle -2\rangle\}, \{\langle 1\rangle, \langle -1\rangle\}, \{\langle 3\rangle, \langle -3\rangle\}$ with exactly one relation, $\{\langle 3\rangle, \langle -3\rangle\} <_{B_2} \{\langle 2\rangle, \langle -2\rangle\}$.

Note that B_2 is isomorphic to P^* via the map $\{\langle p\rangle, \langle -p\rangle\} \mapsto |p|$.

3. Shellability and sphericity

Assume throughout this section that $n > 0$ is fixed and P is a poset on $[n]$.

3.1. ***EL*-shellability.** An *edge-labeling* of a poset is a map from its cover relations to the integers. The edge-labeling of $J(P)$ *induced by* P is defined by mapping each cover relation $A \cup \{p\} \lessdot A$ to p. Similarly, the edge-labeling of $B(P)$ induced by P is defined by mapping the cover relation $X \lessdot X \cup \langle p\rangle$ to p. We extend this to an edge-labeling of $\widehat{B}(P)$ by mapping each cover relation of the form $X \lessdot \hat{1}$ to 0. Figure 1 illustrates induced edge-labelings.

Let λ be an edge-labeling of a graded poset Q. Given a maximal chain $c = \{q_0 \lessdot q_1 \lessdot \cdots \lessdot q_m\}$ of some interval $[q_0, q_m]$ of Q, say that c is *increasing* if its label-sequence $\lambda(c) := (\lambda(q_0, q_1), \ldots, \lambda(q_{m-1}, q_m))$ is a weakly increasing sequence, and say that c is *decreasing* if $\lambda(c)$ is a strictly decreasing sequence. Call λ an *R-labeling* if every interval I has a unique increasing chain, which we denote by a_I. Call λ an *EL-labeling* if it is an R-labeling and for every interval I, $\lambda(a_I)$ is lexicographically smaller than $\lambda(c)$ for any other maximal chain c of I. Call λ a *dual R-labeling* of it is an R-labeling of the dual poset Q^*. If Q has an EL-labeling, then the lexicographic ordering of its maximal chains determines a *shelling* of the order complex of Q [**Bj1**]. For this reason we call such a poset *EL-shellable*. If P is naturally labeled, then the induced edge-labeling of $J(P)$ is well-known (and easily shown) to be an EL-labeling.
Theorem 3.1. *If P is naturally labeled then the induced edge-labeling of $\widehat{B}(P)$ is both an EL-labeling and a dual R-labeling.*

A graded poset Q with rank function rk is called *Eulerian* if its Möbius function satisfies $\mu_Q(p, q) = (-1)^{rk(q)-rk(p)}$ for every $p \leq_Q q$. It is called *Cohen-Macaulay* (over the rationals) if the homology of the order complex (i.e. simplicial complex of chains) of every open interval in Q vanishes below the top dimension. Say that Q is *Gorenstein** if it is Eulerian and Cohen-Macaulauy.
Corollary 3.2. *$\widehat{B}(P)$ is Gorenstein*.*

3.2. Recursive coatom ordering. Sphericity. Let Q be a graded poset. A *coatom* of Q is an element covered by $\hat{1}$. Let $coat(Q)$ denote the set of coatoms of Q. Following [**BW**], we say that Q *admits a recursive coatom ordering* if its rank is 1, or if its rank is greater than 1 and there is an ordering $x_1, x_2, \ldots, x_m$ of its coatoms such that the following conditions hold:

(i) For all $j = 1, \ldots, m$, $[\hat{0}, x_j]$ admits a recursive coatom ordering in which the elements in $coat([\hat{0}, x_j]) \cap (\cup_{i<j} coat([\hat{0}, x_i]))$ come first.

(ii) For all $i < j$, if $y < x_i, x_j$ then there exist $k < j$ and $z \in \widehat{B}(P)$ such that $y \leq z \lessdot x_k, x_j$.
Theorem 3.3. *$\widehat{B}(P)$ admits a recursive coatom ordering.*

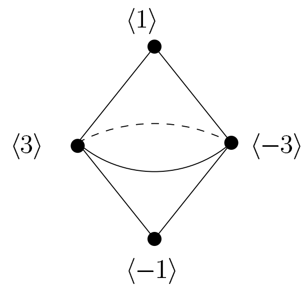

FIGURE 2. A cell decomposition of the 2-sphere into four 0-cells, six 1-cells, and four 2-cells. The face poset of this sphere is the signed Birkhoff poset in Figure 1(d).

The recursive coatom ordering property is a purely combinatorial formulation of the concept of shellability for a regular cell complex. It also generalizes the notion of *EL*-shellability. For a graded poset Q,

$$Q \text{ is } EL\text{-shellable} \Longrightarrow Q^* \text{ admits a recursive coatom ordering.}$$

These shelling properties make it possible to interpret intervals in signed Birkhoff posets (and their duals) as regular decompositions of spheres. Given a finite regular cell complex Γ, let $\widehat{P}(\Gamma)$ denote the face poset $P(\Gamma)$ with a unique maximal element added. Call a graded poset *thin* if every interval of rank 2 has size 4. Björner [**Bj2**] showed that a graded poset Q of rank n is isomorphic to $\widehat{P}(\Gamma)$ for Γ a shellable regular cell decomposition of the $(n-2)$-sphere if and only if Q is thin and admits a recursive coatom ordering. It is easy to prove directly that graded signed Birkhoff posets are thin. (This also follows from the fact that they are Eulerian.) Thus Björner's theorem together with Theorem 3.1 and Theorem 3.3 yield the following:

Theorem 3.4 (Billera and Hsiao). *Let $[X, Y]$ be an interval in $\widehat{B}(P)$ or $\widehat{B}(P)^*$. Then $[X, Y]$ is isomorphic to the face poset of a shellable regular decomposition of the $(rk(Y) - rk(X) - 2)$-sphere.*

Figure 2 illustrates a cell complex whose face poset is the signed Birkhoff poset from Figure 1(d).

Remark 3.5. A different proof that $B(P)$ is the face poset of a regular sphere was originally found by Billera and the author via an explicit geometric description of the cell decomposition. The geometric aspects of signed Birkhoff posets will be studied in greater detail elsewhere. We thank Sergey Fomin for pointing us to Björner's result.

4. Enumerative properties

4.1. Quasisymmetric generating functions. Let $\mathcal{Q} = \bigoplus_{n\geq 0} \mathcal{Q}^n$ denote the *graded algebra of quasisymmetric functions* over $\mathbb{Q}$ in the variables $x_1, x_2, \ldots$ The vector space $\mathcal{Q}^n$ consists of those homogeneous power series in $\mathbb{Q}[[x_1, x_2, \ldots]]$ of degree n for which the coefficients of $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ and $x_{i_1}^{a_1} x_{i_1}^{a_2} \cdots x_{i_k}^{a_k}$ are equal whenever $i_1 < \cdots < i_k$ and $a_1, \ldots, a_k$ is a sequence of positive integers summing to n. Set $\mathcal{Q}^0 = \mathbb{Q}$. For each $n \geq 1$, the *fundamental basis* for $\mathcal{Q}^n$ is the linear basis consisting of the 2^{n-1} elements

$$L_S := \sum_{\substack{i_1 \leq \cdots \leq i_n: \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (S \subseteq [n-1]).$$

This notation suppresses the dependence of L_S on n. See [**Sta5**] for general background and references on quasisymmetric functions.

Let Q be a graded poset (with $\hat{0}$ and $\hat{1}$) of rank n with rank function rk. If $s \leq t \in Q$ then write $rk(s,t) = rk(t) - rk(s)$. To study the flag enumerative invariants of Q, it will be useful to work with the following quasisymmetric generating function introduced by Ehrenborg [**E1**]:

$$F_Q := \sum_{\substack{k\geq 1,\\ \hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}}} x_1^{rk(t_0,t_1)} x_2^{rk(t_1,t_2)} \cdots x_k^{rk(t_{k-1},t_k)},$$

where the sum is over all multichains of Q from $\hat{0}$ to $\hat{1}$ in which $\hat{1}$ occurs exactly once. We review some essential facts about this generating function.

Recall that the *descent set* of a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of integers is defined by $Des(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$. If Q has an R-labeling λ, then

$$F_Q = \sum_c L_{Des(\lambda(c))}, \tag{4.1}$$

where the sum is over all maximal chains c of Q. In general, when Q does not necessarily have an R-labeling, the vector of coefficients of F_Q in the fundamental basis is the *flag h-vector* of Q.

Given a poset P, let $\mathcal{A}(P)$ denote the set of P*-partitions*; i.e. order-preserving maps from P to the positive integers.[3] The *weight enumerator* for P-partitions is the quasisymmetric function

$$K_P := \sum_{\sigma\in\mathcal{A}(P)} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Gessel [**G**] first studied quasisymmetric weight enumerators for more general objects called (P,ω)-partitions [**Sta1**], the motivation being that these weight enumerators generalize Schur functions in a combinatorially useful way. It is easy to verify using (4.1) (see [**Sta5**, page 359]) that

$$F_{J(P)} = K_P. \tag{4.2}$$

Theorem 4.1 below expresses a similar relationship between $F_{\widehat{B}(P)^*}$ and Stembridge's enriched weight enumerator.

4.2. Enumeration in the peak algebra. The *peak set* of a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ of integers is defined to be

$$Peak(\sigma) := \{i \in \{2, 3, \ldots, n-1\} : \sigma_{i-1} < \sigma_i > \sigma_{i+1}\}.$$

Let $Peak_n$ denote the set of all possible peak sets of sequences of length n. Thus, $S \in Peak_n$ if and only if (i) $1, n \notin S$ and (ii) $i \in S$ implies $i-1 \notin S$. For each $S \in Peak_n$, the *peak function* $\theta_S \in \mathcal{Q}^n$ is defined by

$$\theta_S := 2^{\#S+1} \sum_{T\subseteq[n-1]:S\subseteq T\triangle(T+1)} L_T,$$

where $T \triangle U := (T\backslash U) \cup (U\backslash T)$ and $T+1 := \{i+1 : i \in T\}$. The peak functions are linearly independent and span a proper subalgebra Π of $\mathcal{Q}$, called the *peak algebra* [**Ste**].

Let $\pm\mathcal{P}$ be the linear order $-1 \prec +1 \prec -2 \prec +2 \prec -3 \prec +3 \prec \cdots$ on the set of non-zero integers. An *enriched P-partition* of a poset P is an order-preserving map $\sigma : P \to \pm\mathcal{P}$ such that if $\sigma(p) = \sigma(q)$ then $\sigma(p) > 0$. Let $\mathcal{E}(P)$ denote the set of enriched P-partitions. The *enriched weight enumerator* for P-partitions is the quasisymmetric function

$$\widetilde{K}_P := \sum_{\sigma\in\mathcal{E}(P)} x_{|\sigma(1)|} x_{|\sigma(2)|} \cdots x_{|\sigma(n)|}.$$

[3] What we call a P-partition here is what Stanley [**Sta1**] originally calls a reverse P-partition.

Stembridge [**Ste**] originally defined enriched weight enumerators in the more general context of enriched (P,ω)-partitions.[4] His theory of enriched (P,ω)-partitions was motivated by the study of Schur's Q-functions. A basic property of enriched weight enumerators is that

$$\widetilde{K}_P = \sum_{\sigma\in\mathcal{L}(P)} \theta_{Peak(\sigma)} \tag{4.3}$$

when P is naturally labeled.

Theorem 4.1. *For any poset P,*

$$2F_{\widehat{B}(P)^*} = \widetilde{K}_{P_0}.$$

PROOF. We may assume without loss of generality that P is a naturally labeled poset on $[n]$. It follows from [**Ste**, Theorem 3.6 and (1.4)] that

$$\begin{aligned}\widetilde{K}_{P_0} &= \sum_{(\varepsilon,\sigma)\in\{\pm1\}^{\times(n+1)}\times\mathcal{L}(P_0)} L_{Des(\varepsilon\sigma)} \\ &= 2\sum_{(\varepsilon,\sigma)\in\{\pm1\}^{\times n}\times\mathcal{L}(P)} L_{Des(0.\varepsilon\sigma)}.\end{aligned} \tag{4.4}$$

The last expression equals $2F_{\widehat{B}(P)^*}$ by Proposition 2.3 and the fact that, by Theorem 3.1, the induced edge-labeling of $\widehat{B}(P)^*$ is an R-labeling. $\square$

Remark 4.2. In [**BMSW**, Example 7.5] it is observed that $\widetilde{K}_{P_0} = \sum_c L_{Des(c)}$, the sum being over all maximal chains in the doubled reséau $\delta J(P_0)$. This formula is essentially (4.4) and thus provides an alternate approach to proving Theorem 4.1. Yet another proof can be adapted from that of [**BER**, Theorem 3.1]; see Remark 4.13.

The *enriched order polynomial* $\Omega'(P,m)$ is the number of enriched P-partitions $\sigma : P \to \pm\mathcal{P}$ such that $\sigma(p) \preccurlyeq m$ for all $p \in P$. As an enriched analog of the familiar equation $Z(J(P),m) = \Omega(P,m)$ relating the zeta polynomial of $J(P)$ to the order polynomial of P [**Sta4**], we obtain the following:

Corollary 4.3. *For any poset P,*

$$2\,Z(\widehat{B}(P),m) = \Omega'(P_0,m).$$

4.3. The cd-index. Theorem 4.1 may be used to give a combinatorial interpretation of the **cd**-index of $\widehat{B}(P)$, as we now explain. For a graded poset Q of rank n, let $(f_S(Q) : S \subseteq [n-1])$ denote the flag f-vector of Q; i.e., $f_S(Q)$ is the number of chains of size $\#S$ in Q whose elements have ranks precisely in S. Define a polynomial of degree $n-1$ in the non-commuting variables $\mathbf{a}$ and $\mathbf{b}$ of degree 1 by

$$\Psi_Q := \sum_{S\subseteq[n-1]} f_S(Q)u_S,$$

where $u_S = u_1\cdots u_{n-1}$, $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}-\mathbf{b}$ if $i \notin S$. Fine [**BK**] observed that when Q is Eulerian, Ψ_Q can be written as a polynomial in the variables $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$, called the **cd**-index of Q; for a sampling of work on the **cd**-index, see [**Sta3**], [**BER**], [**BE**], [**ER**], and [**E2**]. If Γ is a cell complex such that $\widehat{P}(\Gamma)$ is Eulerian, we may refer to $\Psi_{\widehat{P}(\Gamma)}$ as the **cd**-index of Γ or $P(\Gamma)$.

To connect the **cd**-index to our work, we set up a one-to-one correspondence $w \mapsto S_w$ between the set of **cd**-words of degree $n-1$ and $Peak_n$ given by

$$\mathbf{c}^{a_1}\mathbf{d}\mathbf{c}^{a_2}\mathbf{d}\cdots\mathbf{c}^{a_k}\mathbf{d}\mathbf{c}^{a_{k+1}} \mapsto \{\deg(\mathbf{c}^{a_1}\mathbf{d}), \deg(\mathbf{c}^{a_1}\mathbf{d}\mathbf{c}^{a_2}\mathbf{d}), \dots, \deg(\mathbf{c}^{a_1}\mathbf{d}\cdots\mathbf{c}^{a_k}\mathbf{d})\}.$$

[4]Our definition of an enriched P-partition agrees with Stembridge's original definition of an enriched (P,ω)-partition when ω is a natural labeling of P.

For fixed n, let w_S denote the **cd**-word of degree $n-1$ associated to the peak set $S \in Peak_n$. For instance, $S_{\mathbf{cddccdc}} = \{3,5,9\} \in Peak_{11}$ and $w_{\{3,5,9\}} = \mathbf{cddccdc}$. Given an Eulerian poset Q of rank n and a **cd**-word w of degree $n-1$, let $[w]$ denote the coefficient of the word w in Ψ_Q. A link between the **cd**-index and the peak algebra is provided by the identity [**BHW**, Corollary 2.2]

$$F_Q = \sum_{S \in Peak_n} \frac{[w_S]}{2^{1+\#S}} \cdot \theta_S. \tag{4.5}$$

This formula together with Theorem 4.1 and (4.3) yield the following:

Theorem 4.4. *For a naturally labeled poset P,*

$$\Psi_{\widehat{B}(P)^*} = \sum_{\sigma \in \mathcal{L}(P_0)} 2^{\#Peak(\sigma)} w_{Peak(\sigma)}.$$

In particular, the **cd***-indices of $\widehat{B}(P)^*$ and $\widehat{B}(P)$ have non-negative coefficients.*

Note that Ψ_{Q^*} is obtained from Ψ_Q by changing every **cd**-word w to w^*, the word consisting of the letters of w in reverse order [**BER**].

Example 4.5. If P is the poset from Figure 1(a) then

$$\begin{aligned}\Psi_{\widehat{B}(P)^*} &= w_{Peak(0123)} + w_{Peak(0213)} + w_{Peak(0231)}\\ &= w_\emptyset + 2w_{\{2\}} + 2w_{\{3\}}\\ &= \mathbf{ccc} + 2\mathbf{dc} + 2\mathbf{cd}\end{aligned}$$

and

$$\Psi_{\widehat{B}(P)} = \mathbf{ccc}^* + 2\mathbf{dc}^* + 2\mathbf{cd}^* = \mathbf{ccc} + 2\mathbf{cd} + 2\mathbf{dc}.$$

Theorem 4.4 provides further evidence for Stanley's Gorensetin* conjecture [**Sta3**, Conjecture 2.1], which is known to hold for face lattices of convex polytopes and oriented matroids:

Conjecture 4.6 (Stanley)**.** *The coefficients of the* **cd***-index of a Gorenstein* poset are non-negative.*

Remark 4.7. Conjecture 4.6 has received special attention in connection with a conjecture of Charney and Davis [**CD**] on the sign of the quantity

$$\kappa(\Gamma) := 1 - \frac{1}{2}f_0 + \frac{1}{4}f_1 - \cdots + \left(-\frac{1}{2}\right)^{d+1} f_d,$$

where f_i is the number of i-cells of the d-dimensional cell complex Γ. The Charney-Davis Conjecture predicts that $(-1)^m\kappa(\Gamma) \geq 0$ whenever Γ is a flag complex triangulating a $(2m-1)$-sphere. If Γ is the order complex of $P\backslash\{\hat{0},\hat{1}\}$, where P is an Eulerian poset of rank $2m+1$, then $(-1)^m 2^{2m}\kappa(\Gamma)$ is the coefficient of $\mathbf{d}^m$ of the **cd**-index of P; see [**Sta2**] for additional details. For the face poset Q of a cell complex Γ, the order complex of $Q\backslash\{\hat{0}\}$ is a flag complex and is the barycentric subdivision of Γ. Thus Theorem 4.4 proves a special case of the Charney-Davis Conjecture by supplying a combinatorial interpretation of the quantity $(-1)^m\kappa(\Gamma)$ when Γ is the barycentric subdivision of a cellular sphere whose face poset is a signed Birkhoff poset.

Let $\mathfrak{S}_n^0$ be the set of permutations of $0, 1, \ldots, n$ that start with 0. Taking P to be the antichain on $[n]$ in Theorem 4.4 yields [**BER**, Proposition 8.1]:

Corollary 4.8 (Billera, Ehrenborg, and Readdy)**.** *Let $\mathcal{C}_n$ be the face lattice of the n-dimensional cube. Then*

$$\Psi_{\mathcal{C}_n} = \sum_{\pi \in \mathfrak{S}_n^0} 2^{\#Peak(\pi)} w_{Peak(\pi)}.$$

On the other hand, if P is an n-element chain then a direct computation shows that $\Psi_{\widehat{B}(P)} = \mathbf{c}^n$. For an arbitrary, naturally labeled poset P on $[n]$, $\mathcal{L}(P_0)$ is a subset of $\mathfrak{S}_n^0$. Thus Theorem 4.4 and Corollary 4.8 imply the following:

Corollary 4.9. *The* **cd***-index of a signed Birkhoff poset of rank $n+1$ is coefficient-wise maximized by the* **cd***-index of the n-dimensional hyperoctahedron and minimized by $\mathbf{c}^n$. In other words, $\Psi_{\widehat{B}(P)}$ is coefficient-wise maximized when P is an antichain minimized when P is a chain.*

4.4. Comparisons with oriented matroids. Let Γ be a cell complex whose face poset is isomorphic to $B(P)$ for some P. Let m be the number of minimal elements of P. The number of maximal cells of Γ is clearly 2^m, which equals

$$\sum_{x \in J(P)} |\mu_{J(P)}(\hat{0}, x)|, \tag{4.6}$$

where $\mu_{J(P)}$ is the Möbius function of $J(P)$. This is easily proved using well-known properties of the Möbius function of a distributive lattice; see, e.g., [**Sta4**, Example 3.9.6].

Formula (4.6) is reminiscent of a famous result of Zaslavsky's expressing the f-vector of a hyperplane arrangement in terms of its intersection lattice [**Z**]. He showed in particular that the number of regions in a hyperplane arrangement is $\sum_{x \in L} |\mu_L(\hat{0}, x)|$, where L is the intersection lattice. This result holds more generally in the setting of oriented matroids, where the intersection lattice is now replaced by the geometric lattice of flats. We refer the reader to [**BLSWZ**] for background and references in this area. Note that whereas a signed Birkhoff poset is completely determined by its underlying distributive lattice, an oriented matroid is not necessarily determined by its geometric lattice. In this respect, Zaslavsky's formula is more surprising, and indeed more subtle, than (4.6). Bayer and Sturmfels [**BS**] extended Zaslavsky's result by showing that the entire flag f-vector of an oriented matroid depends only on the underlying geometric lattice. The dependency is formulated explicitly in [**BLSWZ**, Proposition 4.6.2] in terms of the zero map, which "forgets the signs" of covectors. Using φ in place of the zero map, we have an essentially identical formula:

Proposition 4.10. *Let $A_k < A_{k-1} < \cdots < A_0 = \emptyset$ be a chain in $J(P)$. The number of chains in the preimage of c under the map $\varphi : B(P) \to J(P)$ is*

$$\#\varphi^{-1}(c) = \prod_{i=1}^{k} \sum_{\substack{B \in J(P) \\ A_i \leq B \leq A_{i-1}}} |\mu_{J(P)}(A_i, B)|.$$

Billera, Ehrenborg, and Readdy described explicitly the **cd**-index of an oriented matroid in terms of the flag f-vector of the underlying geometric lattice [**BER**]. To state their result, let us define a linear map $\vartheta : \mathcal{Q} \to \Pi$ on the basis $\{L_S\}$ by

$$\vartheta(L_{Des(\sigma)}) = \theta_{Peak(\sigma)}$$

for any fixed $n \geq 1$ and any sequence of $\sigma = (\sigma_1, \ldots, \sigma_n)$. We set $\vartheta(1) = 1$. It is easy to see that ϑ is well-defined. Stembridge [**Ste**] introduced ϑ as a means of relating the weight enumerator of P-partitions to that of enriched P-partitions. A basic consequence of the definition of ϑ is that

$$\vartheta(K_P) = \widetilde{K}_P \tag{4.7}$$

for any poset P. It is also possible to view ϑ as a specialization of a family of maps on noncommutative symmetric functions defined by Krob, Leclerc, and Thibon [**KLT**]. Many properties about these maps are proved in their work, and connections to the peak algebra are explained in [**BHT**].

The following is [**BER**, Theorem 3.1], stated in the present form in [**BHW**, Proposition 3.5]:

Theorem 4.11 (Billera, Ehrenborg, and Readdy)**.** *For the geometric lattice L of an oriented matroid $\mathcal{O}$,*

$$2F_{T^*} = \vartheta(F_{L_0}),$$

where T is the face lattice of $\mathcal{O}$.

By comparison, using (4.2) and (4.7) we can restate Theorem 4.1 as follows:

Theorem 4.12. *For any poset P,*

$$2F_{\widehat{B}(P)^*} = \vartheta(F_{J(P_0)}).$$

Theorem 4.12 summarizes the relationship between the flag enumerative invariants of a signed Birkhoff poset and its underlying distributive lattice.

Remark 4.13. It is possible to prove Theorem 4.12 (and hence Theorem 4.1) by adapting Billera, Ehrenborg, and Readdy's proof of [**BER**, Theorem 3.1], with Proposition 4.10 now playing the role of [**BLSWZ**, Proposition 4.6.2].

5. An analog of the Distributive Lattice Conjecture

The *chain polynomial* of a graded poset Q of rank n is defined by $C(Q,t) := \sum_{i=0}^{n} c_i t^i$, where c_i is the number of chains in Q of length i from $\hat{0}$ to $\hat{1}$. We state a well-known reformulation of a conjecture of Neggers [**N**] from 1978:

Conjecture 5.1 (The Distributive Lattice Conjecture)**.** *The chain polynomial of a distributive lattice has only real zeros.*

For a poset P on $[n]$, $n > 0$, define $W(P,t) := \sum_{\sigma \in \mathcal{L}(P)} t^{\#Des(\sigma)+1}$. It is a standard exercise to show that if P is naturally labeled then $(1-t)^n C(J(P), t/(1-t)) = W(P,t)$. Thus $C(J(P),t)$ has only real zeros if and only if $W(P,t)$ does. More generally, the Neggers-Stanley Poset Conjecture predicts that $W(P,t)$ has only real zeros for any labeling of P. It is a classical result that a polynomial with non-negative coefficients has only real zeros if and only if its coefficients form a *Pólya frequency sequence.* This would imply, in the case of $W(P,t)$, that the coefficients form a log-concave, unimodal sequence. We refer the reader to [**Br1**], [**Br2**], and [**RW**] for results and references related to the Neggers-Stanley Conjecture and Pólya frequency sequences.

The following is a signed analog of the Distributive Lattice Conjecture:

Conjecture 5.2. *For any poset P, $C(\widehat{B}(P),t)$ has only real zeros.*

We make some observations in support of this conjecture. In the enumerative theory of P-partitions, $W(P,t)$ arises as the numerator of the rational generating function $\sum_{m \geq 0} \Omega(P,m)t^m$ [**Sta1**]. Likewise, in the enumerative theory of enriched P-partitions, one has the identity [**Ste**, Theorem 4.1]

$$\sum_{m \geq 0} \Omega'(P,m)t^m = \frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}} \cdot W'\left(P, \frac{4t}{(1+t)^2}\right), \tag{5.1}$$

where $W'(P,t) := \sum_{\sigma \in \mathcal{L}(P)} t^{\#Peak(\sigma)+1}$ and P has n elements. Stembridge's Enriched Poset Conjecture [**Ste**, Conjecture 4.3] predicts that $W'(P,t)$ has only real zeros for any labeled poset P. This is known to be true when P is a disjoint union of labeled chains [**Ste**, Corollary 4.6] and has been verified for all labeled posets of size up to 7 and all naturally labeled posets of size 8. The relevance to our work is explained by the following:

Proposition 5.3. *For a naturally labeled poset P, $W'(P_0,t)$ has only real zeros if and only if $C(\widehat{B}(P),t)$ has only real zeros.*

Remark 5.4. Proposition 5.3 shows that Conjecture 5.2 is a special case of the Enriched Poset Conjecture. Brenti's work [**Br1**] indicates the usefulness of the distributive-lattice approach to the Neggers-Stanley Conjecture for naturally labeled posets. We hope that some progress can be made on the Enriched Poset Conjecture in light of Proposition 5.3.

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