Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Rectangular Schur Functions and Fermions

Takeshi Ikeda, Hiroshi Mizukawa, Tatsuhiro Nakjima, and Hiro-Fumi Yamada

Abstract. *We give an expression of the Schur function* $S_{\square(m,n)}$, *indexed by the rectangular partition* $\square(m,n) = (n^m)$ *as a sum of products of certain Schur functions and Schur's Q-functions.*
Résumé. *Nous donnons une expression de la fonction de Schur* $S_{\square(m,n)}$, *indexée par la partition rectangulaire* $\square(m,n) = (n^m)$, *comme une somme de produits de fonctions de Schur et de Q-fonctions de Schur.*

1. Introduction

Let λ be a strict partition. Draw the Young diagram of λ and fill each cell with 0 or 1 in such a way that, in each row the sequence (0110) repeats from the left as long as possible. For a positive integer ℓ, set $A_\ell = (4\ell-3, 4\ell-7, \cdots, 5, 1)$. Let $\mathcal{F}_1^n(A_\ell)$ be the set of strict Young diagrams which are obtained by appending n $\boxed{1}$'s to A_ℓ. Our formula reads

$$\sum_{\mu \in \mathcal{F}_1^n(A_\ell)} \delta(\mu) Q_{\mu^b[0]}(t) S_{\mu^b[1]}(t') = S_{\square(2\ell-n,n)}(t),$$

where $(\mu^b[0], \mu^b[1])$ is the 4-bar quotient of μ, $\delta(\mu)$ is a sign, $S_\nu(t)$ and $Q_\nu(t)$ are the Schur function and the Q-function, respectively, corresponding to the partition ν, expressed as polynomials of the power sum symmetric functions (or the so-called Sato variables) $t = (t_1, t_2, t_3, \cdots)$ and $t' = (t_2, t_4, t_6, \cdots)$.

We understand this formula from the viewpoint of the basic representation $L(\Lambda_0)$ of the affine Lie algebra of type $A_1^{(1)}$, or more suitably, type $D_2^{(2)}$. It is known that the weight vectors of the basic representation of $D_2^{(2)}$ are, in the principal picture, best described by means of the Q-functions ([**6**]). In particular the maximal weight vectors are the Q-functions $Q_\lambda(t)$ with $\lambda = A_\ell = (4\ell-3, 4\ell-7, \cdots, 5, 1)$ or $\lambda = B_\ell = (4\ell-1, 4\ell-5, \cdots, 7, 3, 0)$ $(\ell = 0, 1, 2, \cdots)$. To give the intertwining operator between the principal and homogeneous realizations we make use of 4-bar quotients of the strict partitions, which arise naturally from the parting the neutral fermions into the neutral and charged fermions. Through this intertwining operator, one sees that, in the homogeneous realization, $f_i^n v$ $(i = 0, 1, n = 0, 1, 2, \cdots)$ is expressed as a rectangular Schur function for any maximal weight vector v. For the identification we will employ some fermion calculus.

2. Combinatorics of strict partitions

Let P_n denote the set of all partitions of n, SP_n the set of all strict partitions of n, and OP_n the set of those partitions of n whose parts are odd numbers. For $\lambda \in P_n$, $\ell(\lambda)$ denotes the number of non-zero parts

1991 *Mathematics Subject Classification.* Primary 17B67, 05E05; Secondary 37K30.
Key words and phrases. Schur function, affine Lie algebra, fermion.

of λ. We need the following "4-bar abacus":

$$\begin{array}{ccc} 0 & \textcircled{1} & 3 \\ \textcircled{2} & & \\ 4 & 5 & 7 \\ \textcircled{6} & & \\ 8 & \textcircled{9} & \textcircled{11} \\ 10 & & \\ 12 & 13 & 15 \end{array}$$

For a strict partition λ we put a set of beads on the assigned positions. The above figure is the 4-bar abacus representing the strict partition $\lambda = (11, 9, 6, 2, 1)$. From the 4-bar abacus of the given strict partition λ, we read off a triplet of partitions $(\lambda^{bc}, \lambda^b[0], \lambda^b[1])$ as follows: The strict partition $\lambda^b[0]$ is obtained by reading the halves of even positions of the beads. In the above example we see that $\lambda^b[0] = (3, 1)$. For the right two runners, horizontal levels are numbered as $0, 1, 2, \cdots$ from the top. We mark (by writing 1) the levels of the beads on the central runner and unmark (by writing 0) the vacancies. In the above example we get $(1, 0, 1, \underline{0}) = (1, 0, 1, 0, 0, \cdots)$. Also we unmark the levels of the beads on the rightmost runner and mark the vacancies. In the example we get $(1, 1, 0, \underline{1}) = (1, 1, 0, 1, 1, \cdots)$. Arrange the two obtained infinite $(0, 1)$-sequences:

$$\underline{1}011|101\underline{0}$$

On the right of the bar "|", the sequence of the central runner comes, and on the left the reversed sequence of the rightmost runner comes. Counting the 0's to the left of each 1, we get the partition $\lambda^b[1]$. The above $(0, 1)$-sequence shows that $\lambda^b[1] = (2, 1, 1, 1)$. Finally, if $\ell \in \mathbb{Z}$ is the number of beads on the central runner minus that on the rightmost runner, we set $\lambda^{bc} = A_\ell = (4\ell - 3, 4\ell - 7, \cdots, 5, 1)$ for $\ell \geq 0$ ($A_0 = \emptyset$), and $\lambda^{bc} = B_{|\ell|} = (4|\ell| - 1, 4|\ell| - 5, \cdots, 7, 3, 0)$ for $\ell < 0$. In the above example we see that $\lambda^{bc} = A_1 = (1)$. Note that $|\lambda^{bc}| + 2|\lambda^b[0]| + 4|\lambda^b[1]| = |\lambda|$. The above procedure is invertible and the correspondence between SP_n and the set $\{(\lambda^{bc}, \lambda^b[0], \lambda^b[1]); |\lambda^{bc}| + 2|\lambda^b[0]| + 4|\lambda^b[1]| = n\}$ is shown to be one-to-one. The strict partition λ^{bc} is called the "4-bar core" of λ and the pair $(\lambda^b[0], \lambda^b[1])$ is called the "4-bar quotient" of λ (cf. [**7**]).

For a strict partition λ we draw the Young diagram and fill each cell with 0 or 1 in such a way that, in each row the sequence (0110) repeats from the left as long as possible. Let $\mathcal{F}_i^n(\lambda)$ $(i = 0, 1)$ denote the set of the strict partitions obtained by appending n $\boxed{i}$'s to the Young diagram λ. It is easy to see that the cardinality of $\mathcal{F}_1^n(A_\ell)$ is the coefficient of x^n of $(1 + x + x^2)^\ell$, and that of $\mathcal{F}_0^n(B_\ell)$ is the sum of coefficients of x^n and x^{n-1} of the same polynomial.

Each strict partition μ in $\mathcal{F}_1^n(A_\ell)$ or $\mathcal{F}_0^n(B_\ell)$ has its own sign $\delta'(\mu) = (-1)^g$, where g is, in the 4-bar abacus of μ, the number of beads on the central runner at the positions bigger than that of each bead on the leftmost runner. For example, for $\mu = (9, 7, 2) \in \mathcal{F}_1^3(A_3)$, whose 4-bar abacus looks

$$\begin{array}{ccc} 0 & 1 & 3 \\ \textcircled{2} & & \\ 4 & 5 & \textcircled{7} \\ 6 & & \\ 8 & \textcircled{9} & 11 \\ 10 & & \\ 12 & 13 & 15 \end{array}$$

the sign is $\delta'(\mu) = -1$, since $g = 1$.

3. The formula

Let χ_ρ^λ be the irreducible character of the symmetric group S_n, indexed by $\lambda \in P_n$ and evaluated at the conjugacy class $\rho \in P_n$. And let ζ_ρ^λ be the irreducible negative character of the double cover $\tilde{S_n}$ of the symmetric group, indexed by $\lambda \in SP_n$ and evaluated at the conjugacy class $\rho \in OP_n$. In our context the Schur function indexed by $\lambda \in P_n$ is defined, as a polynomial of $u = (u_1, u_2, u_3, \cdots)$, by

$$S_\lambda(u) = \sum_{\rho \in P_n} \chi^\lambda_\rho \frac{u_1^{m_1} u_2^{m_2} \cdots}{m_1! m_2! \cdots},$$

where the summation runs over the partitions $\rho = (1^{m_1} 2^{m_2} \cdots) \in P_n$ (cf. [**5**]). Schur's Q-function indexed by $\lambda \in SP_n$ appears, as a polynomial of $t = (t_1, t_3, t_5, \cdots)$, in the form

$$Q_\lambda(t) = \sum_{\rho \in OP_n} 2^{\frac{\ell(\lambda) - \ell(\rho) + \epsilon}{2}} \zeta^\lambda_\rho \frac{t_1^{m_1} t_3^{m_3} \cdots}{m_1! m_3! \cdots},$$

where the summation runs over the partitions $\rho = (1^{m_1} 3^{m_3} \cdots) \in OP_n$, and $\epsilon = 0$ or 1 according to that $n - \ell(\lambda)$ is even or odd (cf. [**2**]). It is sometimes convenient to normalize Q-functions as

$$P_\lambda(t) = 2^{-\ell(\lambda)} Q_\lambda(t).$$

These functions are called Schur's P-functions. We can now state our formula.

Theorem 3.1. *For non-negative integers ℓ and n, we have*

$$\sum_{\mu \in \mathcal{F}_1^n(A_\ell)} \delta'(\mu) Q_{\mu^b[0]}(t) S_{\mu^b[1]}(t') = S_{\square(2\ell - n, n)}(t),$$

$$\sum_{\mu \in \mathcal{F}_0^n(B_\ell)} \delta'(\mu) Q_{\mu^b[0]}(t) S_{\mu^b[1]}(t') = S_{\square(n, 2\ell+1-n)}(t),$$

where $t = (t_1, t_2, t_3, \cdots)$ and $S_\nu(t') = S_\nu(u)|_{u_j \mapsto t_{2j}}$.

4. Basic representation

In this section we connect our formula with the basic representation of the affine Lie algebra of type $A_1^{(1)}$. It turns out to be that our formula describes certain weight vectors in the homogeneous realization of the basic representation $L(\Lambda_0)$.

Associated with the Cartan matrix

$$(a_{ij})_{i,j=0,1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

the Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$ is generated by e_i, f_i, h_i and d subject to the relations

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$
$$[e_i, f_j] = \delta_{i,j} h_i, \qquad (\mathrm{ad} e_i)^{1 - a_{ij}} e_j = (\mathrm{ad} f_i)^{1 - a_{ij}} f_j = 0 \quad (i \neq j),$$

and

$$[d, h_i] = 0, \qquad [d, e_j] = \delta_{j,0} e_j, \qquad [d, f_j] = -\delta_{j,0} f_j.$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by h_0, h_1 and d. Choose the basis $\{\alpha_0, \alpha_1, \Lambda_0\}$ for the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by the pairing

$$< h_i, \alpha_j > = a_{ij}, \qquad < h_i, \Lambda_0 > = \delta_{i,0},$$
$$< d, \alpha_j > = \delta_{0,j}, \qquad < d, \Lambda_0 > = 0.$$

The fundamental imaginary root is $\delta = \alpha_0 + \alpha_1$.

The basic representation $L(\Lambda_0)$ of $\mathfrak{g}$ is by definition the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ_0 (cf. [**4**]). The weight system of $L(\Lambda_0)$ is well-known:

$$P(\Lambda_0) = \{\Lambda_0 - p\delta + q\alpha_1; p, q \in \mathbb{Z}, p \geq q^2\}.$$

A weight Λ on the parabola $\Lambda_0 - q^2\delta + q\alpha_1 \quad (q \in \mathbb{Z})$ is said to be maximal in the sense that $\Lambda + \delta$ is no longer a weight.

We discuss a twisted version of the principal realization of $L(\Lambda_0)$, or more suitably, the basic representation of the affine Lie algebra of type $D_2^{(2)}$, which is isomorphic to $A_1^{(1)}$.

The basic representation in principal grading is realized on the space

$$V^{(2)} = \mathbb{C}[t_j; j \geq 1, \text{odd}]$$

of the P-functions ([**6**]). In fact, the P-functions form a weight basis of $L(\Lambda_0) = V^{(2)}$. Given a strict partition λ, fill the Young diagram with 0 or 1 as in Section 2. If the number of $\boxed{i}$'s is m_i $(i = 0, 1)$, then the weight of the corresponding P-function $P_\lambda(t)$ equals $\Lambda_0 - m_0\alpha_0 - m_1\alpha_1$. In particular the weight of $P_{A_\ell}(t)$ (resp. $P_{B_\ell}(t)$) equals $\Lambda_0 - \ell^2\delta + \ell\alpha_1$ (resp. $\Lambda_0 - \ell^2\delta - \ell\alpha_1$), which is maximal for any $\ell \geq 0$. The action of $f_i \in \mathfrak{g} \quad (i = 0, 1)$ to the P-function $P_\lambda(t)$ is easily described:

$$f_i P_\lambda = \sum_{\mu \in \mathcal{F}_i^1(\lambda)} P_\mu.$$

Strict partitions in $\mathcal{F}_i^n(\lambda)$ occur in the expression of $f_i^n P_\lambda$.

Another realization of the basic representation is known, one in the homogeneous grading. The representation space turns out $\mathcal{B} = V \otimes \mathbb{C}[q, q^{-1}]$, where $V = \mathbb{C}[t_j; j \geq 1]$. Define the linear isomorphism Ψ by

$$\Psi : V^{(2)} \longrightarrow \mathcal{B}$$
$$P_\lambda(t) \mapsto 2^{\frac{\ell(\lambda^b[0])}{2}} \delta(\lambda) P_{\lambda^b[0]}(t) S_{\lambda^b[1]}(t') \otimes q^{m(\lambda)}$$

where $m(\lambda)$ is determined by drawing the 4-bar abacus of λ:

$$\begin{aligned} m(\lambda) = &\ (\text{number of beads on the central runner of } \lambda) \\ &- (\text{number of beads of the rightmost runner of } \lambda) \end{aligned}$$

and $\delta(\lambda)$ is certain sign which is naturally determined by arranging the fermion operators corresponding to λ. Here we only remark that $\delta(\lambda)$ coincides with $\delta'(\lambda)$ for λ in $\mathcal{F}_1^n(A_\ell)$.

The representation of $\mathfrak{g}$ on $\mathcal{B}$, which is induced by Ψ, is the basic representation in the homogeneous grading. In fact, if we define the degree in $\mathcal{B}$ by

$$\deg(f(t) \otimes q^m) = \deg f(t) + m^2,$$

then $\deg\Psi(P_\lambda)$ is equal to the number of $\boxed{0}$'s in λ.

Now our first formula can be translated into

$$\Psi(\frac{1}{n!} f_1^n P_{A_\ell}) = 2^{-\frac{n}{2}} S_{\square(2\ell - n, n)} \otimes q^{\ell - n}.$$

As for the second formula we need to extend Ψ to a superspace $V^{(2)} \oplus V^{(2)}\theta$.

5. Fermionic Fock space

In this section we look at the formula from the fermionic point of view. Although we will not give a proof of the formula in this extended abstract, we emphasize that the discussion of this section is essential for our proof.

We first recall how to realize the basic representation of $\mathfrak{g}$ in terms of free fermions. Let $\mathbb{B}$ be the $\mathbb{C}$-algebra generated by $\beta_n \quad (n \in \mathbb{Z})$ subject to the relations

$$\beta_n\beta_m + \beta_m\beta_n = (-1)^n \delta_{n+m,0} \quad (n, m \in \mathbb{Z}).$$

Note that $\beta_0^2 = 1/2$. These generators are often called the neutral free fermions. Define the degree on $\mathbb{B}$ by $\deg\beta_n = 1$. If we let $\mathbb{B}_0$ (resp. $\mathbb{B}_1$) be the subspace consisting of the elements of even (resp. odd) degree, then $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_1$ has a structure of a superalgebra. Let $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = \mathbb{B}_0|0\rangle \oplus \mathbb{B}_1|0\rangle$ be the fermionic Fock space, where the vacuum $|0\rangle$ is defined by $\beta_n|0\rangle = 0 \quad (n < 0)$. The vacuum expectation value $\langle 0|a|0\rangle \quad (a \in \mathbb{B})$ is uniquely determined by putting $\langle 0|1|0\rangle = 1, \langle 0|\beta_0|0\rangle = 0$. The normal ordering for the quadratic elements is defined by $: \beta_n\beta_m := \beta_n\beta_m - \langle 0|\beta_n\beta_m|0\rangle$.

Set

$$e_0 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n}\beta_{-4n-1}, \quad e_1 = \sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n+2}\beta_{-4n-3}$$

$$f_0 = -\sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n}\beta_{-4n+1}, \quad f_1 = -\sqrt{2}\sum_{n\in\mathbb{Z}} \beta_{4n+2}\beta_{-4n-1}$$

$$h_1 = 2\sum_{n\in\mathbb{Z}} : \beta_{4n+3}\beta_{-4n-3} :$$

and $h_0 = 1 - h_1$. These elements generate a Lie algebra inside $\mathbb{B}_0$, which is known to be isomorphic to the affine Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$. The representation of $\mathfrak{g}$ on $\mathcal{F}$ via the action of $\mathbb{B}_0$ turns out to be the direct sum $V^{(2)} \oplus V^{(2)}\theta$ of the basic representation, where θ is a symbol satisfying $\theta^2 = 1$. One often identifies θ with $\sqrt{2}\beta_0$. The isomorphism Φ_P from $\mathcal{F}$ to $V^{(2)} \oplus V^{(2)}\theta$ is given by

$$\Phi_P : a|0\rangle \mapsto \langle 0|e^{H_B(t)}a|0\rangle + \langle 0|\sqrt{2}\beta_0 e^{H_B(t)}a|0\rangle\theta \quad (a \in \mathbb{B}),$$

where $H_B(t) = \frac{1}{2}\sum_{j\geq 1,\mathrm{odd}}\sum_{n\in\mathbb{Z}}(-1)^{n+1}t_j\beta_n\beta_{-n-j}$. This type of isomorphism is often called the boson-fermion correspondence (cf [**1**] or [**6**]). A standard fermion calculus shows that, putting $\beta_\lambda|0\rangle = \beta_{\lambda_1}\cdots\beta_{\lambda_\ell}|0\rangle$, $\Phi_P(\beta_\lambda|0\rangle) = \sqrt{2}^{-\ell}Q_\lambda(t)\theta^\epsilon$ for a strict partition $\lambda = (\lambda_1, \cdots, \lambda_\ell) \quad (\lambda_1 > \cdots > \lambda_\ell > 0)$, where $\epsilon = 0$ or 1 according to that ℓ is even or odd. In order to give a boson-fermion correspondence in the homogeneous grading, we make parting of the neutral free fermions into three groups: $\{\psi_n; n \in \mathbb{Z}\}, \{\psi_n^*; n \in \mathbb{Z}\}$ and $\{\phi_n; n \in \mathbb{Z}\}$, where

$$\psi_n = i\beta_{4n+1}, \psi_n^* = i\beta_{-4n-1}, \phi_{2n} = \beta_{4n}, \phi_{2n+1} = i\beta_{4n+2} \quad (i = \sqrt{-1}).$$

A product of β's is rewritten as a word of ψ, ψ^* and ϕ's.

For an integer m and for $\epsilon = 0, 1$, we set

$$\langle m, \epsilon| = \begin{cases} \langle 0|\psi_0^*\cdots\psi_{m-1}^*\theta^\epsilon & (m \geq 1) \\ \langle 0|\theta^\epsilon & (m = 0) \\ \langle 0|\psi_{-1}\cdots\psi_m\theta^\epsilon & (m \leq -1). \end{cases}$$

The homogeneous boson-fermion correspondence

$$\Phi_H : \mathcal{F} \longrightarrow (V \oplus V\theta) \otimes \mathbb{C}[q, q^{-1}] \cong \mathcal{B} \oplus \mathcal{B}\theta$$

is given by

$$\Phi_H(a|0\rangle) = \sum_{m\in\mathbb{Z},\epsilon=0,1} \langle m, \epsilon|e^{H_A(t)+H_B(t)}a|0\rangle\theta^\epsilon \otimes q^m \quad (a \in \mathbb{B}),$$

where $H_A(t) = \frac{1}{2}\sum_{j,n\in\mathbb{Z}} t_{2j} : \psi_n\psi_{n+j}^* :$. It is easily seen that

$$\Phi_H(\mathcal{F}_0) = V \otimes \mathbb{C}[q^2, q^{-2}] \oplus V\theta \otimes \mathbb{C}[q^2, q^{-2}]q,$$

$$\Phi_H(\mathcal{F}_1) = V\theta \otimes \mathbb{C}[q^2, q^{-2}] \oplus V \otimes \mathbb{C}[q^2, q^{-2}]q.$$

Both are isomorphic to $V \otimes \mathbb{C}[q, q^{-1}] = \mathcal{B}$.

Here we give an example illustrating how to associate a polynomial with an element of the fermionic Fock space $\mathcal{F}$. Take $A_3 = (9,5,1)$ and $\mathcal{F}_1^1(A_3) = \{(10,5,1),(9,6,1),(9,5,2)\}$. For $\mu = (9,5,2)$, we consider $\beta_\mu = \beta_9\beta_5\beta_2|0\rangle = \phi_1\psi_2\psi_1|0\rangle \in \mathcal{F}$ and see that

$$\begin{aligned}\Phi_H(\beta_9\beta_5\beta_2|0\rangle) &= \Phi_H(\phi_1\psi_2\psi_1|0\rangle)\\ &= \langle 2,1|e^{H_A(t)+H_B(t)}\phi_1\psi_2\psi_1|0\rangle\theta\otimes q^2\\ &= \langle 0|\psi_0^*\psi_1^*\sqrt{2}\phi_0 e^{H_A(t)+H_B(t)}\phi_1\psi_2\psi_1|0\rangle\theta\otimes q^2\\ &= \sqrt{2}\langle 0|e^{H_B(t)}\phi_1\phi_0|0\rangle\theta\langle 0|e^{H_A(t)}\psi_0^*\psi_1^*\psi_2\psi_1|0\rangle\otimes q^2\\ &= \sqrt{2}^{-1}Q_{(1)}(t)S_{(1,1)}(t')\theta\otimes q^2.\end{aligned}$$

Likewise one computes

$$\Phi_H(\beta_{10}\beta_5\beta_1|0\rangle) = \sqrt{2}^{-1}Q_{(5)}(t)\theta\otimes q^2,$$
$$\Phi_H(\beta_9\beta_6\beta_1|0\rangle) = -\sqrt{2}^{-1}Q_{(3)}(t)S_{(1)}(t')\theta\otimes q^2.$$

A combinatorial calculation shows that, for a strict partition λ,

$$\Phi_H(\beta_\lambda|0\rangle) = cQ_{\lambda^b[0]}(t)S_{\lambda^b[1]}(t')\theta^\epsilon\otimes q^{m(\lambda)},$$

where $\epsilon = 0$ or 1, and $c = \pm\sqrt{2}^g$ with $g \in \mathbb{Z}$. The sign ($\pm$) comes from the arranging the fermions to the normal form, which we shall discuss in the next section. The action of the Lie algebra $\mathfrak{g}$ on $\mathcal{F}$ (and on $\mathcal{B}\oplus\mathcal{B}\theta$ via Φ_H) is best described in terms of the vertex operator. To prove the formula we employ a calculation of the vertex operators (cf. [**3**]), which we shall omit here.

6. Determining the sign

We see that the sign which appears in our formula can be easily determined by looking at the 4-bar abacuses. We explain this fact through an example. Take the partition

$$\lambda = (35,31,25,23,18,15,11,6,1) \in \mathcal{F}_1^{12}(A_9)$$

and write the corresponding state

$$\beta_\lambda|0\rangle = \psi_{-9}^*\psi_{-8}^*\psi_6\psi_{-6}^*\phi_9\psi_{-4}^*\psi_{-3}^*\phi_3\psi_0|0\rangle.$$

We rewrite this state into the normal form as follows.

Step 1. Equate the number of ψ's and ψ^*'s by "shifting the vacuum".
Step 2. Move ϕ's to the left of ψ's and ψ^*'s.
Step 3. Make pairs $\psi^*\psi$:

$$\begin{aligned}&\psi_{-9}^*\psi_{-8}^*\psi_6\psi_{-6}^*\phi_9\psi_{-4}^*\psi_{-3}^*\phi_3\psi_0|0\rangle\\ &\underset{\text{step1}}{=} (-1)^4\psi_{-9}^*\psi_{-8}^*\psi_6\psi_{-6}^*\phi_9\psi_{-4}^*\phi_3\psi_0\psi_{-1}\psi_{-2}|-3\rangle\\ &\underset{\text{step2}}{=} (-1)^{4+9}\phi_9\phi_3\psi_{-9}^*\psi_{-8}^*\psi_6\psi_{-6}^*\psi_{-4}^*\psi_0\psi_{-1}\psi_{-2}|-3\rangle\\ &\underset{\text{step3}}{=} (-1)^{4+9+a'}\phi_9\phi_3(\psi_{-9}^*\psi_6)(\psi_{-8}^*\psi_0)(\psi_{-6}^*\psi_{-1})(\psi_{-4}^*\psi_{-2})|-3\rangle,\end{aligned}$$

where the shifted vacuum is, by definition,

$$|m\rangle = \begin{cases}\psi_{m-1}\cdots\psi_0|0\rangle & (m\geq 1)\\ |0\rangle & (m=0)\\ \psi_m^*\cdots\psi_{-1}^*|0\rangle & (m\leq -1).\end{cases}$$

We observe that step 1 does not change the sign. Therefore we only have to consider the sign change which comes from step 2 and step 3. We express the state by the following (modified) 4-bar abacus and attach a number to each bead:

$$\begin{array}{ccc} \psi & \phi & \psi^* \\ & & \\ & & \\ & & \\ \textcircled{0} & 1 & -1 \\ 1 & \textcircled{3} & -2 \\ 2 & 5 & \textcircled{-3} \\ 3 & 7 & \textcircled{-4} \\ 4 & \textcircled{9} & -5 \\ 5 & 11 & \textcircled{-6} \\ \textcircled{6} & 15 & -7 \\ 7 & 17 & \textcircled{-8} \\ 8 & 19 & \textcircled{-9} \end{array} \underset{\text{step1}}{\longrightarrow} \begin{array}{ccc} \psi & \phi & \psi^* \\ -3 & & \\ \textcircled{-2}_{10} & & \\ \textcircled{-1}_{8} & & \\ \textcircled{0}_{6} & 1 & \\ 1 & \textcircled{3}_{2} & \\ 2 & 5 & \\ 3 & 7 & \textcircled{-4}_{9} \\ 4 & \textcircled{9}_{1} & -5 \\ 5 & 11 & \textcircled{-6}_{7} \\ \textcircled{6}_{4} & 15 & -7 \\ 7 & 17 & \textcircled{-8}_{5} \\ 8 & 19 & \textcircled{-9}_{3} \end{array} \underset{\text{word}}{\longrightarrow} \begin{array}{c} w = 3\ 5\ 4\ 7\ 1\ 9\ 2\ 6\ 8\ 10, \\ \text{number of inversions of } w = 13. \end{array}$$

The numbering of the beads is given in the following way.

1. Number ϕ's from bottom to top.
2. Number ψ^*'s and ψ's according to the *layers*.

We read the numbers by rows from bottom to top and count the inversions involved in the obtained word.

One can see that this inversion number gives the number $a(= a' + 9)$ of the interchanges of fermions.

Next we consider the sign which comes from the boson-fermion correspondence, i.e.,

$$\Phi_H(\phi_9\phi_3(\psi^*_{-9}\psi_6)(\psi^*_{-8}\psi_0)(\psi^*_{-6}\psi_{-1})(\psi^*_{-4}\psi_{-2})|-3\rangle)$$
$$= (-1)^{1+3+5+6}\frac{1}{2}Q_{(9,3)}S_{(10,5^3,3,2)}.$$

We read $1, 3, 5, 6$ by renumbering beads on the rightmost runner.

$$\begin{array}{c} \textcircled{-4}_1 \\ -5_2 \\ \textcircled{-6}_3 \\ -7_4 \\ \textcircled{-8}_5 \\ \textcircled{-9}_6 \end{array} \to (1,3,5,6) \to b = 1+3+5+6.$$

Finally we get the desired sign $\delta(\lambda) = (-1)^{a+b}$. A more careful case-by-case check shows that $\delta(\lambda) = \delta'(\lambda)$(see Section 2) for $\lambda \in \mathcal{F}_1^n(A_\ell) \cup \mathcal{F}_0^n(B_\ell)$.

References

[1] E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation groups for soliton equations, in Nonlinear Integrable Systems—Classical Theory and Quantum Theory, ed. by M. Jimbo and T. Miwa, 39-119, World Sci. Publishing, Singapore, 1983.

[2] P. N. Hoffman and J. F. Humphreys, *Projective Representations of the Symmetric Groups,* Oxford, 1992.

[3] T. Ikeda and H.-F. Yamada, Polynomial τ-functions of the NLS-Toda hierarchy and the Virasoro singular vectors, Lett. Math. Phys. 60 (2002), 147-156.

[4] V. G. Kac, *Infinite Dimensional Lie Algebras, 3rd. ed.* , Cambridge, 1990.

[5] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd. ed.* , Oxford, 1995.

[6] T. Nakajima and H.-F. Yamada, Schur's Q-functions and twisted affine Lie algebras, Adv. Stud. in Pure Math. 28 (2000), 241-259.

[7] J. B. Olsson, Combinatorics and Representations of Finite Groups, Essen, 1993.

Takeshi IKeda, Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan
E-mail address: ike@xmath.ous.ac.jp

Hiroshi Mizukawa, Department of Mathematics, National Defense Academy in Japan, Yokosuka 239-8686, Japan
E-mail address: mzh@math.okayama-u.ac.jp

Tatsuhiro Nakajima, Faculty of Economics, Meikai University, Urayasu 279-8550, Japan
E-mail address: nakajima@phys.metro-u.ac.jp

Hiro-Fumi Yamada, Department of Mathematics, Okayama University, Okayama 700-8530, Japan
E-mail address: yamada@math.okayama-u.ac.jp