Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Alternating Sign Matrices With One -1 Under Vertical Reflection

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Abstract. We define a bijection that transforms an alternating sign matrix A with one -1 into a pair (N, E) where N is a (so called) *neutral* alternating sign matrix (with one -1) and E is an integer. The bijection preserves the classical parameters of Mills, Robbins and Rumsey as well as three new parameters (including E). It translates vertical reflection of A into vertical reflection of N. A hidden symmetry allows the interchange of E with one of the remaining two new parameters. A second bijection transforms (N, E) into a configuration of lattice paths called "mixed configuration".

Résumé. On définit une bijection qui transforme une matrice à signes alternants A ayant un seul -1 en une paire (N, E) constituée d'une matrice à signes alternants dite *neutre* N (elle aussi à un seul -1) et d'un paramètre entier E. La bijection préserve les paramètres classiques de Mills, Robbins et Rumsey ainsi que trois nouveaux paramètres (dont E). Elle transforme la réflexion verticale de A en la réflexion verticale de N. Une symétrie cachée permet l'échange de E avec un des deux autres nouveaux paramètres. Une seconde bijection transforme (N, E) en une configuration de chemins dite "configuration mixte".

1. Introduction

Recall that a square matrix $A = (a_{ij})_{1\le i,j\le n}$ is an order n *alternating sign matrix* if $a_{ij} \in \{1, 0, -1\}$ and if, in each row and each column, the non-zero entries alternate in sign, beginning and ending with a 1. Thus, the entries of each row and of each column add up to 1.

The entries in the first row of an alternating sign matrix are all 0 except for one, which must be a 1. It will be called the *first* 1.

In their paper [**MRR**], Mills, Robbins and Rumsey defined the following parameters on order n alternating sign matrices $A = (a_{ij})$:

- $r(A)$ is the number of entries to the left of the first 1. We have $0 \le r(A) \le n - 1$.
- $s(A)$ is the number of entries that are equal to -1.
- $i(A) = \sum_{k>i,\ell<j} a_{ij}a_{k\ell} = \sum_{i,j} a_{ij}\left(\sum_{k>i,\ell<j} a_{k\ell}\right)$ is the *number of inversions* of A. If A is a permutation matrix, $i(A)$ reduces to the usual number of inversions.

We will use the following notation: $\mathcal{A}_n$ denotes the set of order n alternating sign matrices and $\mathcal{A}_{n,s}$ the set of order n alternating sign matrices A with $s(A) = s$.

One of the Mills, Robbins and Rumsey conjectures asserts that $|\mathcal{A}_n|$ is also the number of order n descending plane partitions. In this form, the conjecture was solved by Zeilberger (see [**Ze1**], [**Ze2**])

1991 *Mathematics Subject Classification.* Primary 05A15.
Key words and phrases. alternating sign matrices, lattice paths.
Supported by a grant from NSERC..

with subsequent simplifications by Kuperberg (see [**Ku**]). Bressoud (see [**Br**]) gives an historical and mathematical account of the whole subject.

Stronger forms of the conjectures involve the parameters (defined above), which should translate into known combinatorially significant parameters on descending plane partitions. In that direction, only special cases of the conjectures are solved. This is well known, of course, for $\mathcal{A}_{n,0}$ (permutation matrices). The conjectures are also true for $\mathcal{A}_{n,1}$ (see [**La1**]). This was done by encoding descending plane partitions into configurations of non-intersecting paths (so called TB-configurations), which allows enumeration by a determinant. After application of an algebraic transformation, the determinant is reinterpreted as enumerating another kind of lattice paths (mixed configurations), the set of which follows the same recurrences that describe $\mathcal{A}_{n,1}$. Mixed configurations are seen to generalize inversion tables of permutations.

In the present paper, we will give a bijective version of the last step, transforming $A \in \mathcal{A}_{n,1}$ into a pair (N, E), where $N \in \mathcal{A}_{n,1}$ is "neutral" (to be defined in the next section) and E is an integer (that can thus be seen as a measure of the "difference" between A and N). A second bijection will transform the pair (N, E) into a mixed configuration Ω. The bijections translate the already defined parameters (as well as three new ones) in a way that is coherent with the Mills, Robbins and Rumsey conjectures. Morover, the bijective link $A \leftrightarrow \Omega$ generalizes the encoding of permutations into inversion tables.

Let $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{A}_n$. We write $\overline{A} = (a_{i,n+1-j})_{1\leq i,j\leq n}$ to denote the matrix obtained from A by vertical reflection. The classical parameters r, i and s applied to A and to $\overline{A}$ are easily related (see [**MRR**]):

- $r(A) + r(\overline{A}) = n - 1$,
- $i(A) + i(\overline{A}) = \binom{n}{2} + s(A)$,
- $s(\overline{A}) = s(A)$.

Vertical reflection can be included in the conjectures. It is then believed to correspond to an operation that can be interpreted as a kind of "complementation" operation on descending plane partitions. In [**La2**], it is shown that this operation takes a simple form in terms of Gessel-Viennot path duality (see [**GV**]) on TB-configurations. (Krattenthaler (see [**Kr**]) has an even simpler interpretation in terms of rhombus tilings.) Our bijections behave similarly: if $A \in \mathcal{A}_{n,1}$ is sent to (N, E) and then to the mixed configuration Ω, then $\overline{A}$ is sent to $(\overline{N}, -E)$, which is sent to $\overline{\Omega}$, the Gessel-Viennot dual of Ω.

This is of course a first step toward an eventual general bijection between unrestricted alternating sign matrices and mixed configurations. The results suggest that the three new parameters will play an important role in the general bijection. The natural guess is that each -1 of an alternating sign matrix will be associated to three similar parameters collectively encoded into a second neutral alternating sign matrix. Moreover, the bijections introduced here are, in some sense, the simplest possible and thus should appear in some form in the general bijection.

A few words on the organization of the paper: In section 2, we will define the sign (positive, neutral, negative) of an alternating sign matrix with one -1, define the new parameters and describe the various parts of the matrix that intervene in the bijections. The first bijection (to neutral matrices) is introduced in section 3. In section 4, the equivalence of two of the new parameters is shown. Section 5 describe the second bijection (to mixed configurations). Finally, in section 6 we show the translation of vertical reflection (on matrices) into path duality (on mixed configurations).

2. Three new parameters

In what follows, we will introduce the three new parameters defined for a matrix $A \in \mathcal{A}_{n,1}$. These parameters are related to various sub-matrices of A, which we describe below (see also figure 1).

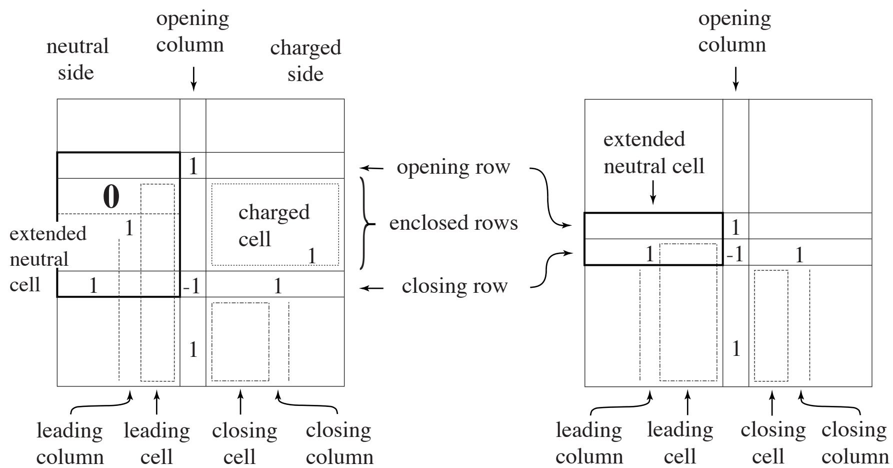

FIGURE 1. Schematic view of a positive matrix (left) and a neutral one (right), with some of the related regions as defined in this section. Only the significant non-zero entries are depicted. The 0 region contains only 0's.

- The *opening column* of A is the column of its (unique) -1. The highest 1 in this column is the *opening* 1 and the corresponding row, the *opening row*. The *closing row* is the row of the -1. The opening column divides A into a *left side* and a *right side* (both excluding the opening column).
- The closing row is the only row that contains two 1, one in each side. These 1 will be referred to as the *left* 1 and the *right* 1.
- If any, the rows between the opening and the closing rows are the *enclosed rows*. If there are no enclosed rows, A is said to be *neutral*; otherwise A is *charged*. In the latter case, define the *charged side* to be the side (left or right) where we find the 1 of the lowest enclosed row, the other side being the *neutral side*. If the charged side is the right side (respectively: left side), we say that A is *positive* (respectively: *negative*).

In fact, we can define more generally $A = (a_{ij}) \in \mathcal{A}_n$ to be *neutral* if $a_{ij} = 1$ when $a_{i+1\,j} = -1$.

Let $\mathcal{A}^+_{n,1}$ (respectively: $\mathcal{A}^0_{n,1}$, $\mathcal{A}^-_{n,1}$) be the set of positive (respectively: neutral, negative) matrices $A \in \mathcal{A}_{n,1}$. These sets are mutually disjoint and form a partition of $\mathcal{A}_{n,1}$. Moreover, $\mathcal{A}^+_{n,1}$ and $\mathcal{A}^-_{n,1}$ are mirror-images of one another: $A \in \mathcal{A}^+_{n,1}$ iff $\overline{A} \in \mathcal{A}^-_{n,1}$.

We further define the following for $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$:

- The intersections of the enclosed rows with the right (respectively: left) side define the *charged* (respectively: *neutral*) *cell*. The *extended* neutral cell includes the intersection of the opening and of the closing rows with the left side. If $A \in \mathcal{A}^0_{n,1}$, the charged and the neutral cells are empty.
- The highest 1 in the left side below the opening row is the *leading* 1. Its column is the *leading column*. The sub-matrix between the leading and the opening column and below the opening row is the *leading cell*. The sum of the entries of the leading cell is denoted $\ell(A)$.
- Finally, the right 1 (in the closing row) is also called the *closing* 1. Its column is the *closing column*. The sub-matrix of A between the closing and the opening column and below the closing row is the *closing cell*. The *extended* closing cell includes the parts of opening and of the closing columns that are below the closing row. The sum of the entries of the closing cell is denoted $c(A)$.

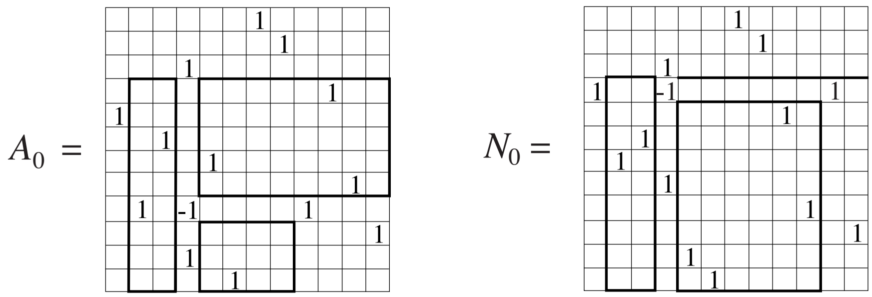

FIGURE 2. In each of the above matrices, the leading, charged and closing cells are emphasized. Matrix A_0 is positive, with $E(A_0) = 3$, $B(A_0) = -1$ and $J(A_0) = 7$. Matrix N_0 is neutral, with $E(N_0) = 0$, $B(N_0) = 2$ and $J(N_0) = 7$. The classical parameters are: $r(A_0) = r(N_0) = 6$ and $i(A_0) = i(N_0) = 30$.

REMARK 2.1. It should be observed that $\ell(\overline{A}) = c(A)$ and $c(\overline{A}) = \ell(A)$ when $A \in \mathcal{A}^0_{n,1}$.

We can now define the new parameters (see figure 2):

- If $A \in \mathcal{A}^+_{n,1}$, its *electric charge*, $E(A)$, is the sum of the entries of the charged cell of A. In that case, $E(A) > 0$. Define $E(A) = 0$ if $A \in \mathcal{A}^0_{n,1}$ and $E(A) = -E(\overline{A})$ if $A \in \mathcal{A}^-_{n,1}$. Thus A is positive, neutral or negative according to the sign of $E(A)$.
- If $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$, define its *magnetic charge* by $B(A) = c(A) - \ell(A)$. If $A \in \mathcal{A}^0_{n,1}$, we clearly have $B(\overline{A}) = -B(A)$. Extend this property to define $B(A)$ for $A \in \mathcal{A}^-_{n,1}$.
- If $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$, define $J(A) = c(A) + \ell(A) + |E(A)| + 1$. Notice that $J(A) = J(\overline{A})$ if $A \in \mathcal{A}^0_{n,1}$. Extend this property to define $J(A)$ for $A \in \mathcal{A}^-_{n,1}$.

Clearly, with respect to vertical reflection, E and B are anti-invariants, while J is invariant. Algebraically:

$$E(M) + E(\overline{M}) = 0, \qquad B(M) + B(\overline{M}) = 0 \qquad \text{and} \qquad J(\overline{M}) = J(M)\,.$$

3. Neutralizing alternating sign matrices

Our first task will be to learn how to “neutralize” a given matrix $A \in \mathcal{A}^+_{n,1}$. This requires many steps based on the *horizontal/vertical displacement* procedure, which applies to some of the entries of a $(0,1)$-matrix.

Horizontal displacement (H): Let $P = (p_{ij})_{1\le i\le m, 1\le j\le n}$ be a $(0,1)$-matrix. Suppose that the non-zero columns occupy positions $j_1 < j_2 < \cdots < j_k$ with $j_1 = 1$ and $j_k < n$.

Its horizontal displacement, $H(P)$, is the matrix obtained from P by displacing column j_i to column j_{i+1} (for $1 \le i \le k$), where $j_{k+1} = n$. Column j_1 is replaced by a column of 0's. Clearly, $H(P)$ is a $(0,1)$-matrix of the same dimension as P, with non-zero columns in positions $j_2 < \cdots j_k < j_{k+1} = n$. The procedure is obviously injective.

We define similarly the *vertical displacement* $V(P)$ for $(0,1)$-matrices P such that the first row is 0 and the last, non-zero. (The rows are displaced from bottom to top.)

We will apply the horizontal/vertical displacement to some of the cells of a given matrix $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$ (or to some modifications of A). This will give the *discharging procedure* which essentially transforms A into a permutation matrix P of the same dimension. The opening column, closing

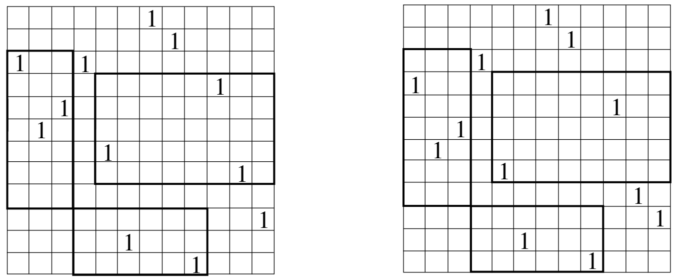

FIGURE 3. The rectangles in these matrices enclose the extended neutral cell, the charged cell and the extended closing cell. The first matrix result from A_0 (figure 2) after applying the first three steps of δ. The last matrix is $P_0 = \delta(A_0)$.

cell,... of any transformation of A refer to sub-matrices of the transformed matrix that occupies the same position as in A.

DEFINITION 3.1. **(Partial discharging procedure δ)** Let $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$. We obtain $\delta(A)$ from A by:

(1) Erasing the -1 and the closing 1 of A.
(2) Applying H to the extended closing cell of A.
(3) Applying V to the extended neutral cell. (See figure 3 (left).)
(4) Lower the 1's in the extended neutral and in the charged cells by one row (erasing or writing 0's when necessary). (See figure 3 (right).)

The resulting matrix is $\delta(A)$.

REMARK 3.2. Carefully keeping track of the number of 1's in each row and each column after each step, we see that the resulting matrix $\delta(A)$ is always a permutation matrix. It is clear that $r(\delta(A)) = r(A)$ since the procedure never affects (permanently) the first row. A fine analysis will show that $\ell(\delta(A)) = \ell(A)$ and that $i(\delta(A)) = i(A) - 1 - (c(A) + E(A))$.

DEFINITION 3.3. **(Complete discharging procedure Δ)** Let $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$. We define

$$\Delta(A) = (k, \delta(A), c(A), E(A)),$$

where k is the position of the opening row of A ($1 \le k \le n$).

EXAMPLE 3.4. For instance, referring to figure 2, we have $\Delta(A_0) = (3, P_0, 1, 3)$ and $\Delta(N_0) = (3, P_0, 4, 0)$ where P_0 is the last matrix of figure 3.

Notice that we can recover A from its image $\Delta(A) = (k, \delta(A), c(A), E(A))$. In fact, we can apply δ backward to the permutation $P = \delta(A)$ provided that:

(1) we can identify the opening column and the opening 1. (The latter being unaffected by δ.) This is determined by k, the position of the opening row.
(2) we determine the closing row. This is given by $E(A)$: the closing row is the highest row below the opening row such that the elements between (and including) these rows in the right side sum up to $E(A)$.

(3) we determine the closing column. But it is the leftmost column to the right of the opening column such that the elements between (and including) these columns and below (strictly) the closing row sum up to $c(A)+1$.

It is always possible to do so since, by construction, $\delta(A)$ contains at least $E(A)+c(A)+1$ non-zero elements below the opening row and to the right of the opening column.

This shows that δ is injective. More generally, let $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$ with $\Delta(A) = (k, \delta(A), c(A), E(A))$ then, for any $c, E \geq 0$ such that $c + E \leq c(A) + E(A)$, there is a unique $B \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$ such that $\Delta(B) = (k, \delta(A), c, E)$ (we will write $B = \Delta^{-1}(k, \delta(A), c, E)$). In particular, we can take $N = \Delta^{-1}(k, \delta(A), c(A) + E(A), 0)$, which will be neutral by construction. In that case, N is quite easy to find from $\delta(A)$ by applying δ backward: steps 4 and 3 cancel each other.

Definition 3.5. **(Neutralizing procedure Λ)** Let $A \in \mathcal{A}_{n,1}$.

(1) If $A \in \mathcal{A}^+_{n,1} \cup \mathcal{A}^0_{n,1}$, let $(k, P, c, E) = \Delta(A)$, define $\Lambda(A) = (\Delta^{-1}(k, P, c + E, 0), E)$.

(2) If $A \in \mathcal{A}^-_{n,1}$, notice that $\overline{A} \in \mathcal{A}^+_{n,1}$. Writing $(\overline{N}, -E) = \Lambda(\overline{A})$, define $\Lambda(A) = (N, E)$.

Example 3.6. Referring to figures 2 (and 3), we have: $\Lambda(A_0) = (N_0, 3)$.

Theorem 3.7. *The neutralizing procedure is a bijection*

$$\Lambda : \mathcal{A}_{n,1} \longrightarrow \mathcal{N}_{n,1} := \{(N, E) \mid N \in \mathcal{A}^0_{n,1}, E \in \mathbb{Z}, -\ell(N) \leq E \leq c(N)\}.$$

Moreover, let $A \in \mathcal{A}_{n,1}$ and $\Lambda(A) = (N, E)$, then:

(1) $A \in \mathcal{A}^0_{n,1}$ *iff* $N = A$.

(2) $\Lambda(\overline{A}) = (\overline{N}, -E)$.

(3) *The matrices A and N are the same, from the first row to the opening row (included).*

(4) *The following relations hold:*
   (a) $r(N) = r(A)$,
   (b) $i(N) = i(A)$,
   (c) $E = E(A)$,
   (d) $B(N) = B(A) + E(A)$,
   (e) $J(N) = J(A)$.

Indeed, if A is positive (or neutral), we have $r(N) = r(A)$, $\ell(N) = \ell(A)$ and $i(N) = i(A)$, by remark 3.2. By construction, $c(N) = c(A) + E(A)$; thus $B(N) = B(A) + E(A)$ and $J(N) = J(A)$. In particular, $0 \leq E = E(A) \leq c(N)$. If A is negative, use remark 2.1 and the formulae of section 1.

4. Exchanging the electric charge and the magnetic charge

Using the neutralizing procedure, we define an involution on $\mathcal{A}_{n,1}$ that exchanges E and B. Thus the two charges play the same role and are completely interchangeable.

Lemma 4.1. *Let $A \in \mathcal{A}_{n,1}$ and $(N, E) = \Lambda(A)$. Then $-\ell(N) \leq B(A) \leq c(N)$.*

Theorem 4.2. *Let $\Xi = \Lambda^{-1} \circ \xi \circ \Lambda$ where ξ is defined by $\xi(N, E) = (N, c(N) - \ell(N) - E)$. Then ξ is an involution on $\mathcal{N}_{n,1}$ and Ξ an involution on $\mathcal{A}_{n,1}$. Moreover, if $A \in \mathcal{A}_{n,1}$ and $\Xi(A) = A'$, we have:*

(1) $\Xi(\overline{A}) = \overline{A'}$.

(2) *The matrices A and A' are the same, from the first row to the opening row (included).*

(3) *The involution Ξ exchanges the charges; namely: $E(A') = B(A)$ and $B(A') = E(A)$.*

(4) *All other defined parameters (r, i and J) take the same value on A as on A'.*

Of course, this leads to another bijection, $\xi \circ \Lambda : \mathcal{A}_{n,1} \longrightarrow \mathcal{N}_{n,1}$, which focuses on the parameter B instead of E. In fact $\xi \circ \Lambda(A) = (N, B(A))$.

5. Encoding elements of $\mathcal{N}_{n,1}$ into mixed configurations

It is well known that a permutation matrix $P = (p_{ij}) \in \mathcal{A}_{n,0}$ can be bijectively encoded by a sequence $(a_i)_{i=1}^n$ of non-negative integers called its inversion table. In fact, a_i is the sum of the entries of P that are below row $n+1-i$ and to the left of the (unique) 1 in that row. With this convention, we have $0 \le a_i < i$ for $1 \le i \le n$. The classical parameters are easily recovered: $r(P) = a_n$ and $i(P) = a_1 + \cdots + a_n$. Moreover, if $(\overline{a}_i)_{i=1}^n$ is the inversion table of $\overline{P}$, then $\overline{a}_i = i - 1 - a_i$ (for $1 \le i \le n$). We define a generalization of inversion table that applies to $\mathcal{A}_{n,1}$.

Definition 5.1. Let $(N, E) \in \mathcal{N}_{n,1}$. Let $n + 1 - k$ be the position of the opening row of N (thus the position of the closing row is $n + 2 - k$). For $1 \le i \le n$, define a_i as the sum of the entries of N that are below row $n + 1 - i$ and to the left of the unique 1 (or the leftmost 1 if $i = k - 1$) in row $n + 1 - i$. Let $b = c(N)$ and $\beta = E + \ell(N)$. The sequence of integers $(k; a_1, \ldots, a_n; b, \beta)$ is called the *generalized inversion table* of (N, E).

Remark 5.2. Clearly, $\ell(N) = a_k - 1 - a_{k-1}$, an observation that we will often use later.

Example 5.3. For instance, the generalized inversion table of $(N_0, 3)$ (from figure 2) is:

$$(10;\ 0, 0, 2, 2, 0, 0, 1, 5, 0, 3, 6, 6;\ 4, 5)\,.$$

Inversion tables are encoded as sequences of non-intersecting lattice paths called *mixed configuration* (introduced in [**La1**]).

We consider lattice-paths on the strict half-grid $\mathcal{G}_n = \{(k, \ell) \,|\, 0 \le k < \ell \le n\}$. (For more symmetry in the figures, the grid will be slightly shifted so that its boundary forms a reversed equilateral triangle.) *Mixed paths* on $\mathcal{G}_n$ are composed of two consecutive parts: Left and Right, where:

- the Left part is composed of South steps (S) and East steps (E).
- the Right part is composed of (another kind of) East steps (F) and North-East steps (N).

Notice that each path contains a vertex that belongs both to the Left part and to the Right part. Such vertices are called *junctions*.

An order n *mixed configuration* is a sequence of mixed paths $\Omega = (\omega_1, \ldots, \omega_n)$ on $\mathcal{G}_n$ such that:

- There is a permutation σ such that ω_i starts from $(0, i)$ and ends at $(\sigma(i) - 1, \sigma(i))$.
- The sub-configuration obtained by deleting the Right part of paths is non-intersecting (no common vertex).
- The sub-configuration obtained by deleting the Left part of paths is non-intersecting.

Let $\mathcal{M}_{n,s}$ be the set of order n mixed configurations with s N-steps.

Given a generalized inversion table $(k; a_1, \ldots, a_n; b, \beta)$, we define the corresponding mixed configuration $\Omega = (\omega_1, \ldots, \omega_n) \in \mathcal{M}_{n,1}$. In that case, Ω has two consecutive special paths ω_{k-1} (which contains the N-step) and ω_k (which contains a S-step. The paths, written as sequences of steps, are:

- $\omega_i = E^{a_i} F^{i-1-a_i}$, for i such that $1 \le i \le n$ and $i \ne k - 1, k$. This path joins $(0, i)$ to $(i - 1, i)$.
- $\omega_{k-1} = E^{a_{k-1}} F^{\beta} N F^{k-2-a_{k-1}-\beta}$. This path joins $(0, k - 1)$ to $(k - 1, k)$.
- $\omega_k = E^{a_k} S E^b F^{k-2-a_k-b}$. This path joins $(0, k)$ to $(k - 2, k - 1)$.

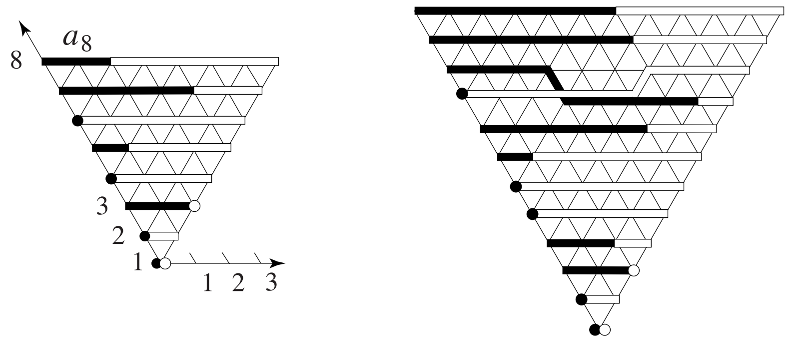

FIGURE 4. A mixed configuration with no N-step (left). A mixed configuration with one N-step (right). The Left part of each is colored black; the Right part, white.

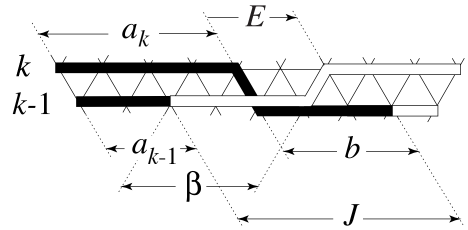

FIGURE 5. The combinatorial interpretations of the parameters E and J on mixed configurations.

(see figure 4 (right)). It is easy to check that this defines a mixed configuration.

THEOREM 5.4. *The encoding of the generalized inversion table of an element* $(N, E) \in \mathcal{N}_{n,1}$ *into a mixed configuration* $\Omega \in \mathcal{M}_{n,1}$ *defines a bijection* $\Phi : \mathcal{N}_{n,1} \longrightarrow \mathcal{M}_{n,1}$.

Moreover, let $A \in \mathcal{A}_{n,1}$, $(N, E) = \Lambda(A)$, $(k; a_1, \ldots, a_n; b, \beta)$ *its generalized inversion table and* $\Omega = (\omega_1, \ldots, \omega_n) = \Phi(N, E)$. *Then*

(1) $r(A) = r(N)$ *is the number of E-steps of* Ω *that are at level* n *(all occurring in path* ω_n*).*
(2) $i(A) = i(N)$ *is the total number of E-steps and of N-steps of* Ω.
(3) $E(A) = E = a_{k-1} + \beta + 1 - a_k$ *is the signed distance from the beginning of the S-step to the end of the N-step of* Ω *(see figure 5).*
(4) $B(A) = B(N) - E = b - \beta$.
(5) $J(A) = J(N) = a_k - a_{k-1} + b$ *is the (non-signed) distance between the junctions of path* ω_{k-1} *and of path* ω_k.

COROLLARY 5.5. *Let* $A \in \mathcal{A}_{n,1}$, $(k; a_1, \ldots, a_n; b, \beta)$ *the generalized inversion table of* $\Lambda(A)$ *and* $\Omega = (\omega_1, \ldots, \omega_n) = \Phi(\Lambda(A))$. *Let* $A' = \Xi(A)$ *then* $(k; a_1, \ldots, a_n; b, \beta')$ *is the generalized inversion table of* $\Lambda(A')$ *(where* $\beta' = b - \beta + a_k - a_{k-1} - 1$*).*

Thus $\Omega' = (\omega'_1, \ldots, \omega'_n) = \Phi(\Lambda(A'))$ is obtained from Ω by replacing β by β' (this only changes the Right part of ω_{k-1}).

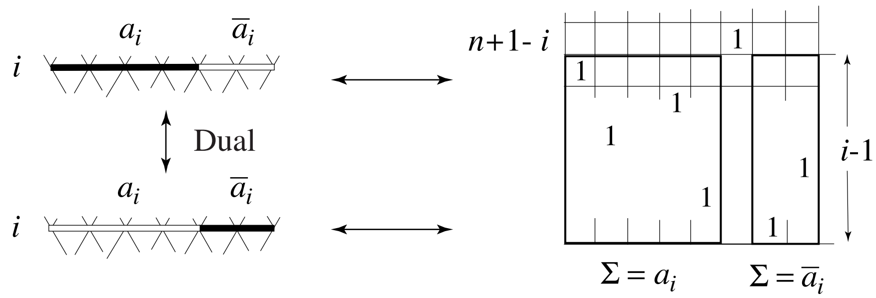

FIGURE 6. Duality on $\mathcal{M}_{n,0}$ (before the final reflection) and its relation with vertical reflection on matrices.

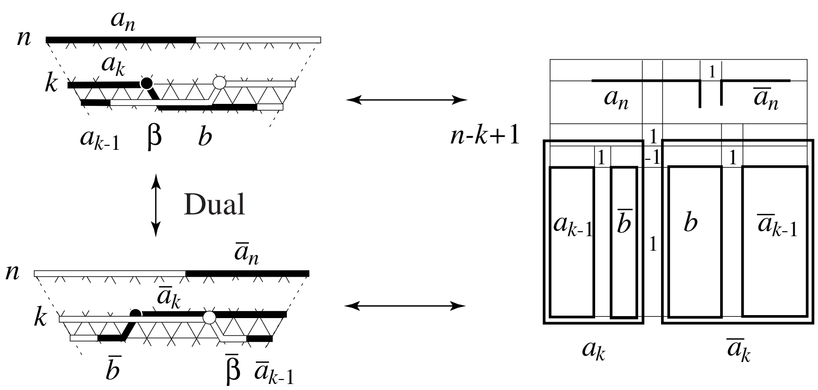

FIGURE 7. Duality on $\mathcal{M}_{n,1}$ (before the final reflection) and its relation with vertical reflection on matrices.

6. Duality and Mixed Configurations

First, we examine path duality for mixed configurations $\Omega = (\omega_1, \dots, \omega_n) \in \mathcal{M}_{n,0}$. We saw how to extract the inversion table $(a_i)_{i=1}^n$, (which then corresponds to a unique permutation matrix P). The dual $\overline{\Omega}$ of Ω is obtained by "complementing" the Left and the Right parts (separately) of each path, leading to the sequence $(\overline{a}_i)_{i=1}^n = (i-1-a_i)_{i=1}^n$ which is the inversion table of $\overline{P}$. Graphically, we obtain $\overline{\Omega}$ from Ω by starting from the right edge of the grid $\mathcal{G}_n$, putting (reversed) E-steps until we reach a junction. We then continue by putting (reversed) F-steps until we touch the left edge of the grid. We get a reversed mixed configuration; a vertical reflection gives the (ordinary) mixed configuration $\overline{\Omega}$ (see figure 6).

For mixed configurations $\Omega = (\omega_1, \dots, \omega_n) \in \mathcal{M}_{n,1}$ (or more generally $\mathcal{M}_{n,s}$), the graphical procedure is similar, with the added rules:

- replace every S-step by a reversed S-step with the same starting vertex.
- replace every N-step by a reversed N-step with the same ending vertex.

Figure 7 (left) shows how this is done. Observe that duality (before the final vertical reflection) preserves the positions of the starting vertex of the S-step, of the ending vertex of the N-steps and of the junctions.

THEOREM 6.1. *Let $A \in \mathcal{A}_{n,1}$ and $\Omega = (\omega_1, \dots, \omega_n) = \Phi(\Lambda(A))$. Then $\overline{\Omega} = \Phi(\Lambda(\overline{A}))$.*

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