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Littelmann Paths for Affine Lie Algebras

Peter Magyar

Abstract. *We give a new combinatorial model for the crystal graphs of an affine Lie algebra $\widehat{\mathfrak{g}}$, unifying Littelmann's path model with the Kyoto path model. The vertices of the crystal graph are represented by certain infinitely looping paths which we call skeins.*

We apply this model to the case when the corresponding finite-dimensional algebra $\mathfrak{g}$ has a minuscule representation (classical type and E_6, E_7). We prove that the basic level-one representation of $\widehat{\mathfrak{g}}$, when considered as a representation of $\mathfrak{g}$, is an infinite tensor product of fundamental representations of $\mathfrak{g}$.

This is the infinite limit of a finer result: that the finite-dimensional Demazure submodules of the basic representation are finite tensor products. The corresponding Demazure characters give generalizations of the Hall-Littlewood polynomials.

This paper is an extended abstract of [**Mag**].

1. Littelmann's path model

Littelmann's combinatorial model [**Lit1**],[**Lit2**],[**LLM2**] for the representations of a Kac-Moody algebra $\mathfrak{g}$ is a vast generalization of Young tableaux. Littelmann's paths and path operators give a flexible construction of the crystal graphs associated to quantum $\mathfrak{g}$-modules by Kashiwara [**K1**] and Lusztig [**Lus**] (see also [**Jos**],[**HK**]). We briefly sketch Littelmann's theory.

For concreteness, let $\mathfrak{g}$ be a complex simple Lie algebra. For our purposes, we define a $\mathfrak{g}$-*crystal* as a *set* $\mathcal{B}$ with a *weight function*, wt : $\mathcal{B}\tilde{\Omega} \oplus_{i=1}^{r} \mathbb{Z}\varpi_i$, as well as partially defined *crystal operators* $e_1, \ldots, e_r, f_1, \ldots, f_r$: $\mathcal{B}\tilde{\Omega}\mathcal{B}$ satisfying:

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{and} \qquad e_i(b) = b' \Longleftrightarrow f_i(b') = b\,.$$

Here $\varpi_1, \ldots, \varpi_r$ are the fundamental weights and $\alpha_1, \ldots, \alpha_r$ are the roots of $\mathfrak{g}$. A *dominant* element is a $b \in B$ such that $e_i(b)$ is not defined for any i. We say that a crystal $\mathcal{B}$ is a *model* for a $\mathfrak{g}$-module V if the formal character of $\mathcal{B}$ is equal to the character of V, and the dominant elements of $\mathcal{B}$ correspond to the highest-weight vectors of V. That is:

$$\mathrm{char}(V) = \textstyle\sum_{b\in B} e^{\mathrm{wt}(b)} \qquad \text{and} \qquad V \cong \bigoplus_{b \text{ dom}} V(\mathrm{wt}(b))\,,$$

where the second sum is over the dominant elements of $\mathcal{B}$. Clearly, a $\mathfrak{g}$-module V is determined up to isomorphism by any model $\mathcal{B}$.

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We construct such $\mathfrak{g}$-crystals $\mathcal{B}$ consisting of polygonal paths in the vector space of weights, $\mathfrak{h}^*_{\mathbb{R}} := \oplus_{i=1}^r \mathbb{R}\varpi_i$. Specifically:

- The *elements* of $\mathcal{B}$ are certain continuous piecewise-linear mappings $\pi : [0,1]\tilde{\Omega}\mathfrak{h}^*_{\mathbb{R}}$, up to reparametrization, with initial point $\pi(0) = 0$. We use the notation $\pi = (v_1 \star v_2 \star \cdots \star v_k)$, where $v_1, \ldots, v_k \in \mathfrak{h}^*_{\mathbb{R}}$ are vectors, to denote the polygonal path starting at 0 and moving linearly to v_1, then to $v_1{+}v_2$, etc.
- The *weight* of a path is its endpoint:
$$\mathrm{wt}(\pi) := \pi(1) = v_1 + \cdots + v_k\,.$$
- The *crystal lowering operator* f_i is defined as follows (and there is a similar definition of the raising operator e_i). Let $\star$ denote the natural associative operation of concatenation of paths, and let any linear map $w : \mathfrak{h}^*_{\mathbb{R}}\tilde{\Omega}\mathfrak{h}^*_{\mathbb{R}}$ act pointwise on paths: $w(\pi) := (w(v_1) \star \cdots \star w(v_k))$. We will divide a path π into three well-defined sub-paths, $\pi = \pi_1 \star \pi_2 \star \pi_3$, and reflect the middle piece by the simple reflection s_i:
$$f_i\pi := \pi_1 \star s_i\pi_2 \star \pi_3\,.$$
The pieces π_1, π_2, π_3 are determined according to the behavior of the i-height function $h_i(t) = h_i^\pi(t) := \langle \pi(t), \alpha_i^\vee \rangle$. As the point $\pi(t)$ moves along the path from $\pi(0) = 0$ to $\pi(1) = \mathrm{wt}(\pi)$, this function may attain its minimum value $h_i(t) = M$ several times. If, after the *last* minimum point, $h_i(t)$ never rises to the value $M{+}1$, then $f_i\pi$ is *undefined*. Otherwise, we define π_2 as the last sub-path of π on which $M \le h_i(t) \le M{+}1$, and π_1, π_3 as the remaining initial and final pieces of π.

A key advantage of the path model is that the crystal operators, while complicated, are universally defined for all paths. Hence a path crystal is completely specified by giving its set of paths $\mathcal{B}$.

Also, the dominant elements have a neat pictorial characterization, as the paths π which never leave the fundamental Weyl chamber: that is, $\mathfrak{h}_i^\pi(t) \ge 0$ for all $t \in [0,1]$ and all $i = 1, \ldots, r$. For simplicity we restrict ourselves to *integral* dominant paths, meaning that all the steps are integral weights: $v_1, \ldots, v_k \in \oplus_{i=1}^r \mathbb{Z}\varpi_i$. (For arbitrary dominant paths, see [**Lit2**].)

Littelmann's Character Theorem [**Lit2**] states that if π is any integral dominant path with weight λ, then the set of paths $\mathcal{B}(\pi)$ generated from π by $f_1, \ldots, f_r$ is a model for the irreducible $\mathfrak{g}$-module $V(\lambda)$. (This $\mathcal{B}(\pi)$ is also closed under $e_1, \ldots, e_r$.) Note that we can choose *any* integral path π which stays within the Weyl chamber and ends at λ, and each such choice gives a different (but isomorphic) path crystal modelling $V(\lambda)$. In principle, any reasonable indexing set for a basis of $V(\lambda)$ should be in natural bijection with $\mathcal{B}(\pi)$ for some choice of π. For example, classical Young tableaux correspond to choosing the steps v_j to be coordinate vectors in $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^n$.

Furthermore, we have Littelmann's Product Theorem [**Lit2**]: if $\pi_1, \ldots, \pi_m$ are dominant integral paths of respective weight $\lambda_1, \ldots, \lambda_m$, then $\mathcal{B}(\pi_1) \star \cdots \star \mathcal{B}(\pi_m)$, the set of all concatenations, is a model for the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_m)$.

Everything we have said also holds for the corresponding affine algebra [**Kac**, Ch. 6 and 7]:
$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \,\oplus\, \mathbb{C}K \,\oplus\, \mathbb{C}d,$$
provided we replace the roots $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{g}$ by the roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ of $\widehat{\mathfrak{g}}$; and the weights $\varpi_1, \ldots, \varpi_r$ of $\mathfrak{g}$ by the weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_r$ of $\widehat{\mathfrak{g}}$. We also replace the vector space $\mathfrak{h}^*_{\mathbb{R}}$ by $\widehat{\mathfrak{h}}^*_{\mathbb{R}} := \oplus_{i=0}^r \mathbb{R}\Lambda_i \oplus \mathbb{R}\delta$, where δ is the non-divisible positive imaginary root of $\widehat{\mathfrak{g}}$. (Indeed, Littelmann's theory works uniformly for all symmetrizable Kac-Moody algebras.) We denote representations and path crystals of $\mathfrak{g}$ as $V(\lambda)$ and $\mathcal{B}$, and the corresponding objects for $\widehat{\mathfrak{g}}$ as $\hat{V}(\Lambda)$ and $\hat{\mathcal{B}}$.

We can also model the affine Demazure module $\hat{V}_z(\Lambda) := U(\widehat{\mathfrak{n}}_+) \cdot v_{z\Lambda}$, where $\widehat{\mathfrak{n}}_+$ is the algebra spanned by the positive weight-spaces of $\widehat{\mathfrak{g}}$, $z \in \widehat{W}$ is a Weyl group element, and $v_{z\Lambda}$ is a non-zero vector of extremal weight $z\Lambda$ in $\hat{V}(\Lambda)$. Demazure modules are always finite-dimensional vector spaces. If $z = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition and π is an integral dominant path of weight Λ, we define the *Demazure path crystal*:

$$\hat{\mathcal{B}}_z(\pi) := \{ f_{i_1}^{k_1} \cdots f_{i_m}^{k_m} \pi \mid k_1, \ldots, k_m \geq 0 \} .$$

Because of the local nilpotence of the lowering operators, this is always a finite set.

Then the formal character of $\hat{\mathcal{B}}_z(\pi)$ is equal to the character of $\hat{V}_z(\Lambda)$, and π is the unique dominant path [**Lit1**]. Now suppose $z = t_{-\lambda^\vee}$, an anti-dominant translation in $\widehat{W}$, so that $\hat{V}_{\lambda^\vee}(\Lambda) := \hat{V}_z(\Lambda)$ is a $\mathfrak{g}$-submodule of $\hat{V}(\Lambda)$; and consider $\hat{\mathcal{B}}_{\lambda^\vee}(\pi) := \hat{\mathcal{B}}_z(\pi)$ as a $\mathfrak{g}$-crystal by forgetting the action of f_0, e_0 and projecting the affine weight function to $\mathfrak{h}_{\mathbb{R}}^*$. Then Littelmann's Restriction Theorem [**Lit2**] implies that the $\mathfrak{g}$-crystal $\hat{\mathcal{B}}_{\lambda^\vee}(\pi)$ is a model for the $\mathfrak{g}$-module $\hat{V}_{\lambda^\vee}(\Lambda)$.

2. The Skein model

For the case of an affine algebra $\widehat{\mathfrak{g}}$, we introduce a generalization of Littelmann's model by allowing certain infinite paths.

Let us introduce a notation for a path π which emphasizes the vector steps going toward the endpoint $\Lambda = \mathrm{wt}(\pi)$ rather than away from the starting point 0. Define

$$\pi = (\star\, v_k \star \cdots \star v_1 \vdash \Lambda) := (v' \star v_k \star \cdots \star v_1) ,$$

the path with endpoint Λ, last step v_1, etc, and first step $v' := \Lambda - (v_k + \cdots + v_1)$, a makeweight to assure that the steps add up to Λ.

A *skein* is an infinite list:

$$\pi = (\cdots \star v_2 \star v_1 \vdash \Lambda) ,$$

where $\Lambda \in \oplus_{i=0}^r \mathbb{Z}\Lambda_i$ and $v_j \in \mathfrak{h}_{\mathbb{R}}^*$ (not $\widehat{\mathfrak{h}}_{\mathbb{R}}^*$), subject to conditions (i) and (ii) below. For $i = 0, \ldots, r$ and $k > 0$, define:

$$h_i[k] := \langle \Lambda - (v_1 + \cdots + v_k),\, \alpha_i^\vee \rangle .$$

We require:
(i) For each i and all $k \gamma 0$, we have $h_i[k] \geq 0$.
(ii) For each i, there are infinitely many k such that $h_i[k] = 0$.

We think of the skein π as a "projective limit" of the paths

$$\pi[k] := (\star\, v_k \star \cdots \star v_1 \vdash \Lambda) \quad \text{as} \quad k \tilde{\Omega} \infty .$$

The conditions on π assure that only a finite number of steps of π lie outside the fundamental chamber $\hat{C}$, and that π touches each wall of $\hat{C}$ infinitely many times. Note that π stays always at the level $\ell = \langle \Lambda, K \rangle$.

Lemma 2.1. *For a skein π and $i = 0, \ldots, r$, one of the following is true:*

(i) $f_i(\pi[k])$ is undefined for all $k \gamma 0$;

(ii) there is a unique skein π' such that $\pi'[k] = f_i(\pi[k])$ for all $k \gamma 0$.

In the second case, we define $f_i \pi := \pi'$.

Proof. Recall that a path π is i-neutral if $h_i^\pi(t) \geq 0$ for all t and $h_i^\pi(1) = 0$. For a fixed i, divide π into a concatenation: $\pi = (\cdots \star \pi_2 \star \pi_1 \star \pi_0 \vdash \Lambda)$, where each π_j is an i-neutral finite path except for π_0, which is an arbitrary finite path. Now it is clear that if $f_i(\pi_0)$ is undefined, then (i) holds. Otherwise (ii) holds and

$$f_i \pi = (\cdots \star \pi_2 \star \pi_1 \star f_i(\pi_0) \vdash \Lambda - \alpha_i) .$$

□

We can immediately carry over the definitions of the path model to skeins, including that of (Demazure) path crystals. For example, we say that π is an integral dominant skein if $\pi[k]$ is integral dominant for $k\gamma 0$, and hence for all k. There exist integral dominant skeins of level $\ell = 1$ only when $\mathfrak{g}$ has a minuscule coweight. We cannot concatenate two skeins, but we can concatenate a skein π_1 and a path π_0: that is, $\pi_1 \star \pi_0 := (\, \pi_1 {\star} \pi_0 \vdash \mathrm{wt}(\pi_1) + \mathrm{wt}(\pi_0)\,)$.

Proposition 2.2. *For an integral dominant skein π of weight Λ, the crystal $\hat{\mathcal{B}}(\pi)$ is a model for $\hat{V}(\Lambda)$, and $\hat{\mathcal{B}}_z(\pi)$ is a model for the Demazure module $\hat{V}_z(\Lambda)$.*

Proof. Given an integral dominant skein π and a Weyl group element $z \in \widetilde{W}$, we can divide $\pi = \pi_1 \star \pi_0$ in such a way that the Demazure operator $\hat{\mathcal{B}}_z$ acts on π by reflecting intervals in π_0 rather than π_1. This gives an isomorphism between the Demazure crystals generated by the path $\mathrm{wt}(\pi_1) \star \pi_0$ and by the skein π:

$$\begin{array}{c} \hat{\mathcal{B}}_z(\mathrm{wt}(\pi_1) \star \pi_0) \overset{\sim}{\tilde{\Omega}} \hat{\mathcal{B}}_z(\pi_1 \star \pi_0) = \hat{\mathcal{B}}_z(\pi) \\ \mathrm{wt}(\pi_1) \star \pi' \mapsto \pi_1 \star \pi' \end{array} .$$

This proves the assertion about Demazure modules.

Now, given an infinite chain of Weyl group elements $z_1 < z_2 < \cdots$, we have the morphisms of $\widehat{\mathfrak{g}}$-crystals:

$$\begin{array}{ccccc} \hat{\mathcal{B}}_{z_1}(\Lambda) & \overset{\sim}{\leftarrow} & \hat{\mathcal{B}}_{z_1}(\mathrm{wt}(\pi_1) \star \pi_0) & \overset{\sim}{\tilde{\Omega}} & \hat{\mathcal{B}}_{z_1}(\pi) \\ \cap & & & & \cap \\ \hat{\mathcal{B}}_{z_2}(\Lambda) & \overset{\sim}{\leftarrow} & \hat{\mathcal{B}}_{z_2}(\mathrm{wt}(\pi_1') \star \pi_0') & \overset{\sim}{\tilde{\Omega}} & \hat{\mathcal{B}}_{z_2}(\pi) \\ \cap & & & & \cap \\ \vdots & & \vdots & & \vdots \\ \hat{\mathcal{B}}(\Lambda) & & & & \hat{\mathcal{B}}(\pi) \end{array}$$

Here $\hat{\mathcal{B}}_z(\Lambda)$ denotes the canonical path crystal of Lakshmibai-Seshadri paths, generated from the straight-line path (Λ). Since the $\widehat{\mathfrak{g}}$ crystals at the bottom are the unions of their Demazure crystals, they are isomorphic: $\hat{\mathcal{B}}(\Lambda) \cong \hat{\mathcal{B}}(\pi)$. $\square$

3. Product theorems

As before, we let $\widehat{\mathfrak{g}}$ be the untwisted affine Kac-Moody algebra corresponding the to the complex simple algebra $\mathfrak{g}$. The basic representation $\hat{V}(\Lambda_0)$, the fundamental representation corresponding to the distinguished node of the extended Dynkin diagram, is the simplest and most important $\widehat{\mathfrak{g}}$-module (cf. [**Kac**, Ch. 14],[**PS**, Ch. 10]).

One of its remarkable properties is the Tensor Product Phenomenon. In many cases, the Demazure modules $\hat{V}_z(\Lambda_0) \subset \hat{V}(\Lambda_0)$ are representations of the finite-dimensional algebra $\mathfrak{g}$, and they factor into a tensor product of many small $\mathfrak{g}$-modules. Hence the full $\hat{V}(\Lambda_0)$ could be constructed by extending the $\mathfrak{g}$-structure on the semi-infinite tensor power $V \otimes V \otimes \cdots$ of a small $\mathfrak{g}$-module V.

The Kyoto school of Jimbo, Kashiwara, et al. has established this phenomenon in many cases (and for a large class of $\widehat{\mathfrak{g}}$-modules $\hat{V}(\Lambda)$) via the theory of perfect crystals [**KKMMNN**], [**KMOTU1**], [**KMOTU2**], [**HK**], [**K2**] a development of their earlier theory of semi-infinite paths [**DJKMO**]. See especially [**HKKOT**]. Pappas and Rapoport [**PR**] have given a geometric version of the phenomenon for type A: they construct a flat deformation of Schubert varieties of the affine Grassmannian into a product of finite Grassmannians.

We extend the Tensor Product Phenomenon for $\hat{V}(\Lambda_0)$ to the non-classical types E_6 and E_7 by a uniform method which applies whenever $\mathfrak{g}$ possesses a minuscule representation, or more precisely a minuscule coweight. We shall rely on a key property of such coweights which may be taken as the definition. Let $\hat{X}$ be the extended Dynkin diagram (the diagram of $\widehat{\mathfrak{g}}$). A coweight $\varpi^\vee$ of $\mathfrak{g}$ is *minuscule* if and only if it is a fundamental coweight $\varpi^\vee = \varpi_i^\vee$ and there exists an automorphism σ of $\hat{X}$ taking the node i to the distinguished node 0. Such automorphisms exist in types A, B, C, D, E_6, E_7.

We let $V(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight λ, and $V(\lambda)^*$ its dual module. Our main representation-theoretic result is:

Theorem 3.1. *Let $\lambda^\vee$ be an element of the coroot lattice of $\mathfrak{g}$ which is a sum:*

$$\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_m^\vee,$$

where $\lambda_1^\vee, \cdots, \lambda_m^\vee$ are minuscule fundamental coweights (not necessarily distinct), with corresponding fundamental weights $\lambda_1, \cdots, \lambda_m$. Let $\hat{V}_{\lambda^\vee}(\Lambda_0) \subset \hat{V}(\Lambda_0)$ be the Demazure module corresponding to the anti-dominant translation $t_{-\lambda^\vee}$ in the affine Weyl group.

Then there is an isomorphism of $\mathfrak{g}$-modules:

$$\hat{V}_{\lambda^\vee}(\Lambda_0) \cong V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_m)^* .$$

Now fix a minuscule coweight $\varpi^\vee$ and its corresponding fundamental weight ϖ. Let N be the smallest positive integer such that $N\varpi^\vee$ lies in the coroot lattice of $\mathfrak{g}$. Then we have the following characterization of the basic irreducible $\widehat{\mathfrak{g}}$-module:

Theorem 3.2. *The tensor power $V_N{:=}V(\varpi)^{\otimes N}$ possesses non-zero $\mathfrak{g}$-invariant vectors. Fix such a vector v_N, and define the $\mathfrak{g}$-module $V^{\otimes\infty}$ as the direct limit of the sequence:*

$$V_N \hookrightarrow V_N^{\otimes 2} \hookrightarrow V_N^{\otimes 3} \hookrightarrow \cdots$$

where each inclusion is defined by: $v \mapsto v_N \otimes v$.

Then $\hat{V}(\Lambda_0)$ is isomorphic as a $\mathfrak{g}$-module to $V^{\otimes\infty}$.

It would be interesting to define the action of the full algebra $\widehat{\mathfrak{g}}$ on $V^{\otimes\infty}$, and thus give a uniform "path construction" of the basic representation (cf. [**DJKMO**]): that is, to define the raising and lowering operators E_0, F_0, as well as the energy operator d. Combinatorial definitions of the energy for $\mathfrak{g}$ of classical type produce generalizations of the Hall-Littlewood and Kostka-Foulkes polynomials (c.f. [**Oka**]), with connections to Macdonald polynomials [**San**], [**Ion**].

4. Crystal theorems

We prove Theorem 3 by reducing it to an identity of paths: we construct a path crystal for the affine Demazure module which is at the same time a path crystal for the tensor product.

For λ a dominant weight, define its dual weight λ^* by the dual $\mathfrak{g}$-module: $V(\lambda^*) = V(\lambda)^*$.

Theorem 4.1. *Let $\lambda^\vee$ be as in Theorem 3, and let $\mathcal{B}(\lambda)$ denote the path crystal generated by the straight-line path (λ). Then the set of concatenated paths $\Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$ is a path crystal for the Demazure module $\hat{V}_{\lambda^\vee}(\Lambda_0)$. In fact, there is a unique $\widehat{\mathfrak{g}}$-dominant path π with weight Λ_0 such that:*

$$\hat{\mathcal{B}}_{\lambda^\vee}(\pi) = \Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*) \mod \mathbb{R}\delta .$$

This is to be understood as an equality of sets of paths in $\widehat{\mathfrak{h}}_{\mathbb{R}}^*$ mod $\mathbb{R}\delta$, and hence an isomorphism of $\widehat{\mathfrak{g}}$-crystals.

Proof. Let σ_j be the automorphism of the diagram $\hat{X}$ corresponding to the minuscule coweight $\lambda_j^\vee$ for $j = 1, \ldots, m$. This also defines an automorphism of $\widehat{\mathfrak{h}}^*$ by $\sigma(\Lambda_i) = \Lambda_{\sigma(i)}$. We define π_m inductively as the last of a sequence of paths $\pi_0, \pi_1, \ldots, \pi_m$:

$$\pi_0 := \Lambda_0, \qquad \pi_j := \sigma_j^{-1}(\pi_{j-1} \star \lambda_j^*) .$$

We may picture the path π_m as jumping from 0 up to level Λ_0, winding horizontally around the fundamental alcove $A \subset \mathfrak{h}_{\mathbb{R}}^* {+} \Lambda_0$, and ending at Λ_0.

We prove the Theorem by showing that the Demazure operator $\hat{\mathcal{B}}_{\lambda^\vee} = \hat{\mathcal{B}}_{\lambda_1^\vee}\hat{\mathcal{B}}_{\lambda_2^\vee}\cdots\hat{\mathcal{B}}_{\lambda_m^\vee}$ "unwinds" π_m starting from its endpoint. The dual weights enter because $\lambda_j^* = -\sigma_j(\lambda_j)$.

The key fact is that the linear mapping σ_i preserves the set of paths $\mathcal{B}(\lambda_j^*)$ for all i, j. This is obvious if $V(\lambda_j^*)$ is a minuscule representation, but the general case requires some work using results of Stembridge [**Ste**]. □

Theorem 3 now follows immediately. Indeed, $s_i\Lambda_0 = \Lambda_0$ for $i = 1, \ldots, r$, so $f_i(\Lambda_0 \star \pi') = \Lambda_0 \star f_i(\pi')$ for any path π'. Thus the right-hand side of the equation in the Theorem is isomorphic as a $\mathfrak{g}$-crystal to $\mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$, which models $V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_r)^*$. See [**GM**] for methods of enumerating the paths in this crystal (and hence computing the dimension of the corresponding representation).

Theorem 4 follows as a corollary. We describe the crystal graph of the semi-infinite tensor product by the appropriate skein-crystal. We thus recover the Kyoto path model for classical $\mathfrak{g}$, and our results are equally valid for E_6, E_7.

Theorem 4.2. *Let $\varpi^\vee$, N be as in Theorem 4. Define the m-fold concatenation $\mathcal{B}_m = \mathcal{B}(\varpi^*) \star \cdots \star \mathcal{B}(\varpi^*)$. Then $\Lambda_0 \star \mathcal{B}_N$ contains a unique $\widehat{\mathfrak{g}}$-dominant path $\Lambda_0 \star \pi_N$.*

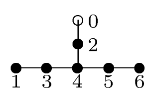
Define the skein $\pi := (\cdots \star \pi_N \star \pi_N \star \pi_N \vdash \Lambda_0)$, which satisfies $\pi \star \pi_N = \pi$. Then the $\widehat{\mathfrak{g}}$-crystal of $\hat{V}(\Lambda_0)$ is given by the skein-crystal:

$$\hat{\mathcal{B}}(\pi) = \bigcup_{m \geq 1} \pi \star \mathcal{B}_m\,.$$

That is, $\hat{\mathcal{B}}(\pi)$ is the set of all semi-infinite paths which are equal to π except for a finite length near the end, and all of whose vector steps lie in $\mathcal{B}(\varpi^*)$.

5. Example: E_6

Referring to Bourbaki [**Bour**], we write the extended Dynkin diagram $\hat{X} = \hat{E}_6$:



The simple roots are defined inside $\mathbb{R}^6$ with standard basis $\epsilon_1, \ldots, \epsilon_6$. (Our ϵ_6 is $\frac{1}{\sqrt{3}}(-\epsilon_6 - \epsilon_7 + \epsilon_8)$ in Bourbaki's notation.) They are:

$$\alpha_1 = \tfrac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5) + \tfrac{\sqrt{3}}{2}\epsilon_6, \quad \alpha_2 = \epsilon_1+\epsilon_2,$$
$$\alpha_3 = \epsilon_2-\epsilon_1, \quad \alpha_4 = \epsilon_3-\epsilon_2, \quad \alpha_5 = \epsilon_4-\epsilon_3, \quad \alpha_6 = \epsilon_5-\epsilon_4\,.$$

Since E_6 is simply laced, the coroots and coweights may be identified with the roots and weights, with the natural pairing given by the standard dot product on $\mathbb{R}^6$.

We focus on the minuscule coweight $\varpi_1^\vee$ corresponding to the diagram automorphism σ with $\sigma(1) = 0$ and $\sigma(0) = 6$. In this case, the corresponding fundamental representation $V(\varpi_1)$ is also minuscule, meaning that all of its weights are extremal weights $\lambda \in W(E_6)\cdot\varpi_1$. The roots $\alpha_2, \cdots, \alpha_6$ generate the root subsystem $D_5 \subset E_6$, and the reflection subgroup $W(D_5) = \mathrm{Stab}_{W(E_6)}(\varpi_1)$ acts by permuting $\epsilon_1, \ldots, \epsilon_5$ (the subgroup $W(A_4) = S_5$) and by changing an even number of signs $\pm\epsilon_1, \ldots, \pm\epsilon_5$. We have $\dim V(\varpi_1) = |W(E_6)/W(D_5)| = 27$. The weights are:

$$\varpi_1 = \tfrac{2\sqrt{3}}{3}\epsilon_6,$$
$$S_5\cdot\tfrac{1}{2}(-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5) + \tfrac{\sqrt{3}}{6}\epsilon_6,$$
$$S_5\cdot\tfrac{1}{2}(-\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4+\epsilon_5) + \tfrac{\sqrt{3}}{6}\epsilon_6,$$
$$-\tfrac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5) + \tfrac{\sqrt{3}}{6}\epsilon_6,$$
$$\pm S_5\cdot\epsilon_1 - \tfrac{\sqrt{3}}{3}\epsilon_6\,.$$

The lowest weight is $-\varpi_6 = -\epsilon_5 - \frac{\sqrt{3}}{3}\epsilon_6$, so that $V(\varpi_1)^* = V(\varpi_6)$ and $\varpi_1^* = \varpi_6$.

The simplest path crystal for $V(\varpi_1^*)$ is the set of 27 straight-line paths from 0 to the negatives of the above extremal weights:

$$\mathcal{B}(\varpi_1^*) = \{\, (v) \mid v \in -W(E_6)\cdot\varpi_1 \,\}$$

We have $3\varpi_1^\vee \in \oplus_{i=1}^6 \mathbb{R}\alpha_i^\vee$ the coroot lattice, so that $N = 3$ in Theorem 4, and this N is also the order of the automorphism σ. The path crystal $\mathcal{B}_3 := \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*)$, the set of all 3-step walks with steps chosen from the 27 weights of $V(\varpi_1^*)$, is a model for $V(\varpi_1^*)^{\otimes 3}$.

By Theorem 5, $\Lambda_0 \star \mathcal{B}_3$ contains a unique $\widehat{\mathfrak{g}}$-dominant path $\Lambda_0 \star \pi_3$, where

$$\pi_3 := (\varpi_6) \star (\varpi_1 - \varpi_6) \star (-\varpi_1)\,.$$

In this case, π_3 has the even stronger property that it is the unique $\mathfrak{g}$-dominant path of weight 0, so that it corresponds to the one-dimensional space of $\mathfrak{g}$-invariant vectors in $V(\varpi_1^*)^{\otimes 3}$.

Now Theorem 5 states that the affine Demazure module $\hat{V}_{3m\varpi_1^\vee}(\Lambda_0)$ is modelled by the $\widehat{\mathfrak{g}}$-path crystal:

$$\mathcal{B}_{3m} = \{(\Lambda_0 \star v_1 \star \cdots \star v_{3m}) \mid v_j \in -W(E_6)\cdot\varpi_1\}\,,$$

the set of all $3m$-step walks in $\Lambda_0 \oplus \mathbb{R}^6$ starting at Λ_0, with steps chosen from the 27 weights of $V(\varpi_1^*)$. This path crystal is generated from its unique $\widehat{\mathfrak{g}}$-dominant path $\Lambda_0 \star \pi_3 \star \cdots \star \pi_3$. Considering it as a $\mathfrak{g}$-crystal, we have $\mathcal{B}_{3m} \cong \mathcal{B}_3^{\star m}$, which shows that $\hat{V}_{3m\varpi_1^\vee}(\Lambda_0) \cong V(\varpi_1^*)^{\otimes 3m}$ as $\mathfrak{g}$-modules.

By Theorem 6, the $\widehat{\mathfrak{g}}$-crystal of the basic $\widehat{\mathfrak{g}}$-module $\hat{V}(\Lambda_0)$ is given by the set of all infinite walks (skeins) of the form:

$$\pi = \Lambda_0 \star \underbrace{\pi_3 \star \cdots \star \pi_3}_{\text{infinite}} \star v_1 \star \cdots \star v_{3m}\,,$$

with $m > 0$ and $v_j \in -W(E_6)\cdot\varpi_1$. The endpoint of such a skein is $\mathrm{wt}(\pi) := \Lambda_0 + v_1 + \cdots + v_{3m}$. The crystal operators f_i are defined just as for finite paths. Acting near the end of the skein, they unwind the coils π_3 one at a time, right-to-left. As a $\mathfrak{g}$-module, $\hat{V}(\Lambda_0)$ is an infinite tensor power of $V(\varpi_1^*)$.

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Department of Mathematics, Michigan State University, East Lansing, MI, 48824
E-mail address: magyar@math.msu.edu