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Tutte Meets Poincaré

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Abstract. *Let G be a graph and $\mathcal{X}^d(G)$ the space of all "pictures" of G in complex projective d-space. We prove that $\mathcal{X}^d(G)$ has no torsion or odd-dimensional integral homology, and that its Poincaré series is a specialization of the Tutte polynomial of G. As an application to combinatorial rigidity theory, we give a criterion for d-parallel independence in terms of the Tutte polynomial. In the case that $\mathcal{X}^d(G)$ is smooth (which is equivalent to the condition that G is an orchard), we give a presentation of its cohomology ring, and relate the intersection theory on $\mathcal{X}^d(G)$ to the Schubert calculus on flag varieties.*

Résumé. *Soient G un graphe et $\mathcal{X}^d(G)$ l'espace de toutes les "figures" de G dans l'espace complexe projectif d-dimensionnel. Nous prouvons que $\mathcal{X}^d(G)$ ne présente ni de torsion, ni d'homologie entière en dimension impaire, et que sa série de Poincaré est une spécialisation du polynôme de Tutte de G. Comme application à la théorie combinatoire de la rigidité, nous développons un critère pour l'indépendance d-parallel en termes du polynôme de Tutte. Dans le cas où $\mathcal{X}^d(G)$ est lisse (ce qui est équivalent à la condition que G soit un verger), nous donnons une présentation de son anneau de cohomologie, et relions la théorie d'intersection de $\mathcal{X}^d(G)$ au calcul de Schubert sur les variétés de drapeaux.*

1. Introduction

Let G be a graph with vertices V and edges E, and let $d \geq 2$ be an integer. A *picture* of G in complex projective d-space $\mathbb{P}^d = \mathbb{P}^d_{\mathbb{C}}$ consists of a point in $\mathbb{P}^d$ for each vertex of G and a line for each edge, subject to containment conditions inherited from incidence in G. The set of all pictures of G is a projective algebraic set, the *picture space* $\mathcal{X}^d(G)$. In Section 2, we state our main result (Theorem 2.3) which expresses the Poincaré series of $\mathcal{X}^d(G)$ as a specialization of the Tutte polynomial of G.

In Section 3, we apply this result to the theory of combinatorial rigidity. Briefly, a graph G is *d-parallel independent* if there are no constraints on the direction vectors of the lines in a generic picture of G in d-space. In fact, this is a *matroid* independence condition; see [**11**]. Generalizing a result of [**8**], we show that G is d-parallel independent if and only if $\mathcal{X}^d(G)$ is irreducible and $\dim \mathcal{X}^d(G) = d|V|$, where $\mathbf{v}(G)$ is the number of vertices of G. Whether these conditions hold can be determined from the Poincaré series of $\mathcal{X}^d(G)$, which implies that d-parallel independence is a function of the Tutte polynomial.

In section 4, we study the cohomology ring $H^*(\mathcal{X}^d(G); \mathbb{Z})$ in the case that $\mathcal{X}^d(G)$ is smooth. It turns out that smoothness is equivalent to the property that G is an "orchard"; that is, every edge is either a loop

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or an isthmus (an edge whose deletion increases the number of connected components). In this case, $\mathcal{X}^d(G)$ is an iterated projectivized vector bundle, so its cohomology ring may be presented in terms of Chern classes of line bundles, just as for Grassmannians and flag varieties (see, e.g., [**2**] or [**4**]). Using this presentation (Theorem 4.3), we apply the classical Schubert calculus of partial flag varieties to solve enumerative geometry problems in the picture space of an orchard.

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2. The Main Theorem

We assume familiarity with elementary graph theory (see, e.g., [**10**]) but will briefly mention a few key terms and notations. A *graph* is a pair $G = (V, E)$, where $V = V(G)$ is a finite nonempty set of vertices and $E = E(G)$ is a finite set of edges. An edge whose endpoints are equal is called a *loop*. A graph is *simple* if it has no loops or multiple edges; that is, an edge may be specified by its pair of endpoints. The numbers of vertices, edges and connected components of G will be denoted $\mathbf{v}(G)$, $\mathbf{e}(G)$, $\mathbf{c}(G)$ respectively.

For $e \in E$, the *deletion* $G - e$ is the graph $(V, E \setminus \{e\})$. In general, either $\mathbf{c}(G-e) = \mathbf{c}(G)$ or $\mathbf{c}(G-e) = \mathbf{c}(G) + 1$; in the latter case, e is called an *isthmus* (or *bridge* or *coloop*). If e is not a loop, the *contraction* G/e is obtained by removing e from G and identifying its endpoints with each other. An *isthmus* (or *bridge*) is an edge e such that $\mathbf{c}(G-e) = \mathbf{c}(G) + 1$; otherwise, $\mathbf{c}(G-e) = \mathbf{c}(G)$.

Definition 2.1. Let $G = (V, E)$ be a graph. The *Tutte polynomial* $\mathbf{T}_G(x, y)$ is defined as follows. If $\mathbf{e}(G) = 0$, then $\mathbf{T}_G(x, y) = 1$. Otherwise, $\mathbf{T}_G(x, y)$ is defined recursively as

$$\mathbf{T}_G(x,y) \;=\; \begin{cases} x \cdot \mathbf{T}_{G/e}(x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot \mathbf{T}_{G-e}(x,y) & \text{if } e \text{ is a loop,} \\ \mathbf{T}_{G-e}(x,y) + \mathbf{T}_{G/e}(x,y) & \text{otherwise.} \end{cases} \tag{2.1}$$

for any $e \in E(G)$. (It is a standard fact, albeit not immediate from the definition, that the choice of e does not matter.)

Many isomorphism invariants of graphs, such as the number of acyclic orientations and the chromatic polynomial, satisfy deletion-contraction recurrences akin to (2.1). The Tutte polynomial may thus be regarded as the most general deletion-contraction invariant. For a comprehensive treatment of many aspects of the Tutte polynomial, see [**3**].

There is an equivalent (and non-recursive) definition of the Tutte polynomial as a certain generating function for the edge subsets $F \subset E(G)$. Define the *rank* of F, denoted $r(F)$, as the cardinality of a maximal acyclic subset of F. Equivalently, $r(F) = \mathbf{v}(G|_F) - \mathbf{c}(G|_F)$, where $G|_F$ is the subgraph with edges F and vertices

$$\{v \in V(G) : \ v \text{ is an endpoint of at least one edge of } F\}.$$

Then the Tutte polynomial may be defined in closed form as the *corank-nullity generating function*

$$\mathbf{T}_G(x,y) \;=\; \sum_{F \subset E(G)} (x-1)^{r(E)-r(F)}(y-1)^{|F|-r(F)} \tag{2.2}$$

[**3**, eq. 6.13]; this formula will be useful in the study the d-parallel matroid in Section 3.

The main objects of our study are projective algebraic sets which parametrize "pictures" of graphs. (For more details, see [**8**].)

Definition 2.2. Let $G = (V, E)$ be a graph and $d \geq 2$ a positive integer. Denote complex projective d-space by $\mathbb{P}^d$. A *picture* $\mathbf{P}$ of G consists of a point $\mathbf{P}(v) \in \mathbb{P}^d$ for each $v \in V$ and a line $\mathbf{P}(e)$ in $\mathbb{P}^d$ for each $e \in E$, such that $\mathbf{P}(v) \in \mathbf{P}(e)$ whenever v is an endpoint of e. The set of all pictures is called the *d-dimensional picture space* of G, denoted $\mathcal{X}^d(G)$.

Our main theorem concerns the enumeration of the (non-reduced) integral homology groups $H_i(\mathcal{X}^d(G)) = H_i(\mathcal{X}^d(G);\, \mathbb{Z})$.

Theorem 2.3. *Let G be a graph and $d \geq 2$ an integer. Then*

1. *The picture space $\mathcal{X}^d(G)$ is path-connected and simply connected.*
2. *$H_i(\mathcal{X}^d(G))$ is free abelian for i even and zero for i odd.*
3. *The "compressed Poincaré series" defined by*

$$P_G^d(q) \;:=\; \sum_i q^i \;\operatorname{rank}_{\mathbb{Z}} H_{2i}(\mathcal{X}^d(G)) \tag{2.3}$$

(that is, the generating function for the even Betti numbers) is given by the formula

$$P_G^d(q) \;=\; ([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}\; [d+1]_q^{\mathbf{c}(G)}\; \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right)$$

where $[d]_q = (1-q^d)/(1-q)$.

In the remainder of this section, we sketch the proof of Theorem 2.3. We begin with a few elementary observations about picture spaces.

First, $\mathcal{X}^d(G)$ is easily seen to be path-connected: any picture can be deformed continuously to a "maximally degenerate" picture in which all points (resp. lines) coincide, and the set of maximally degenerate pictures is isomorphic to a partial flag variety.

Second, $\mathcal{X}^d(G)$ is the product of the picture spaces of the connected components of G. In particular, if $\mathbf{e}(G) = 0$, then $\mathcal{X}^d(G) \cong (\mathbb{P}^d)^{\mathbf{v}(G)}$. Moreover, if e is a loop, then $\mathcal{X}^d(G)$ is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{X}^d(G-e)$.

At the heart of our methods are two canonical morphisms between picture spaces that correspond to the graph operations of deletion and contraction. First, for every $e \in E(G)$, there is a natural epimorphism

$$\mathcal{X}^d(G) \twoheadrightarrow \mathcal{X}^d(G-e) \tag{2.4}$$

given by forgetting the data for the line $\mathbf{P}(e)$. (In fact, there is a canonical epimorphism $\mathcal{X}^d(G)\tilde{\Omega}\mathcal{X}^d(G')$ for any subgraph G' of G, but this is the most important case for our present purposes.)

Let e be a nonloop edge with endpoints v, w. The *coincidence locus* of e in $\mathcal{X}^d(G)$ is defined as

$$Z_e(G) \;=\; Z_{vw}(G) \;:=\; \left\{\mathbf{P} \in \mathcal{X}^d(G) \;\mid\; \mathbf{P}(v) = \mathbf{P}(w)\right\}. \tag{2.5}$$

The second canonical map is the natural monomorphism

$$\mathcal{X}^d(G/e) \hookrightarrow \mathcal{X}^d(G-e) \tag{2.6}$$

whose image is the coincidence locus $Z_e(G-e)$.

We remark briefly that in light of (2.4) and (2.6), one may regard $\mathcal{X}^d$ as a contravariant functor from the category of graphs to that of projective algebraic sets.

The maps (2.4) and (2.6) form part of a commutative diagram

$$\begin{tikzcd} Z_e(G) \arrow[r, hook] \arrow[d, two heads] & \mathcal{X}^d(G) \arrow[d, two heads] \\ \mathcal{X}^d(G/e) \arrow[r, hook] & \mathcal{X}^d(G-e) \end{tikzcd} \tag{2.7}$$

By a technical but not difficult argument, one can show that the map $Z_e(G)\tilde{\Omega}\mathcal{X}^d(G/e)$ is a $\mathbb{P}^{d-1}$-fibration, and that the diagram (2.7) is a homotopy pushout square. Consequently, there is a Mayer-Vietoris long exact sequence

$$\begin{aligned} \ldots \tilde{\Omega}\ \ H_i(Z_e(G))\ \ \tilde{\Omega}\, H_i(\mathcal{X}^d(G/e)) \oplus H_i(\mathcal{X}^d(G))\, \tilde{\Omega}\, H_i(\mathcal{X}^d(G-e)) \\ \tilde{\Omega}\, H_{i-1}(Z_e(G))\, \tilde{\Omega} \ldots. \end{aligned} \tag{2.8}$$

We first consider two simple cases. If $\mathbf{e}(G) = 0$, then $\mathcal{X}^d(G) \cong (\mathbb{P}^d)^{\mathbf{v}(G)}$, while if $\mathbf{v}(G) = 1$, then $\mathcal{X}^d(G)$ is a $(\mathbb{P}^{d-1})^{\mathbf{e}(G)}$-bundle over $\mathbb{P}^d$, whose Poincaré series is the same as that of $\mathbb{P}^d \times (\mathbb{P}^{d-1})^{\mathbf{e}(G)}$ (see, e.g., Proposition 2.3 of [**5**]). In both cases, $\mathcal{X}^d(G)$ is a simply connected complex manifold with no torsion or odd-dimensional integral homology. Since the compressed Poincaré series of $\mathbb{P}^d_{\mathbb{C}}$ is $[d+1]_q$, we have

$$P^d_G(q) \ = \ \begin{cases} [d+1]_q^{\mathbf{v}(G)} & \text{if } \mathbf{e}(G) = 0, \\ [d]_q^{\mathbf{e}(G)}\ [d+1]_q & \text{if } \mathbf{v}(G) = 1. \end{cases} \tag{2.9}$$

We now consider the general case. To show that $\mathcal{X}^d(G)$ is simply connected and has no torsion or odd-dimensional homology, we proceed inductively, choosing an edge e and assuming these properties for $\mathcal{X}^d(G-e)$ and $\mathcal{X}^d(G/e)$. Since $Z_e(G)$ is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{X}^d(G/e)$, it follows from Proposition 2.3 of [**5**] that $Z_e(G)$ has no torsion or odd-dimensional homology (essentially because the Leray-Serre spectral sequence degenerates quickly), so that (2.8) splits into short exact sequences

$$0\ \tilde{\Omega}\ H_i(Z)\ \tilde{\Omega}\ H_i(\mathcal{X}^d(G/e)) \oplus H_i(\mathcal{X}^d(G))\ \tilde{\Omega}\ H_i(\mathcal{X}^d(G-e))\ \tilde{\Omega}\ 0, \tag{2.10}$$

from which we obtain the desired properties for $\mathcal{X}^d(G)$. Furthermore, the short exact sequences (2.10) lead to recurrences expressing the compressed Poincaré series $P^d_G(q)$ in terms of $P^d_{G-e}(q)$ and $P^d_{G/e}(q)$. By suitable normalizations, these recurrences can be transformed into the Tutte recurrence (2.2).

3. Parallel Independence

Let $\mathbf{P}$ be a d-dimensional picture of a simple graph $G = (V, E)$ (that is, with no loops or multiple edges). Consider a physical model of $\mathbf{P}$ consisting of a "bar" for each edge e and a "joint" for each vertex v. If e has v as an endpoint, then the corresponding bar is attached to the corresponding joint. The bars may cross, and their lengths are allowed to vary, but we fix the angles at which the bars are attached to the joints. Thus, for example, a square framework may be deformed to produce an arbitrary rectangle, but not any other rhombus. Under what conditions on G is such a model rigid? That is, when is the model determined up to congruence by specifying the attaching angles? These and similar questions are the focus of *combinatorial rigidity theory*; for more details, see, e.g., [**6**] and [**11**].

The graph G (or, more properly, its edge set) is said to be *d-parallel independent* if for a generic picture in $\mathcal{X}^d(G)$, the directions of the lines representing edges are mutually unconstrained. This is in fact a *matroid* independence condition on edge sets; for the reader not familiar with matroids, we remark here only that it satisfies certain axioms which abstract the idea of linear independence in a vector space. In particular, loops and multiple edges are dependent sets in t

The Poincaré series formula of Theorem 2.3 can be applied to give the following criterion for independence in the d-parallel matroid:

Theorem 3.1. *Let d be a positive integer and G a simple graph (with no loops or multiple edges). Then $E(G)$ is independent in the generic d-parallel matroid if and only if the polynomial*

$$([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}\ \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right)$$

is monic of degree $d(\mathbf{v}(G) - \mathbf{c}(G))$.

We briefly sketch the proof of Theorem 3.1. The first fact we need is that the leading term of $\mathrm{Poin}(X; q)$ is cq^{2d}, where $d = \dim_{\mathbb{C}} X$ and c is the number of irreducible components of X of dimension d; see [**4**, Appendix A, Lemmas 2 and 4].

Call a picture $\mathbf{P}$ of G *generic* if the points $\mathbf{P}(v)$, for $v \in V(G)$, are all distinct. The *picture variety* $\mathcal{V}^d(G)$ is the closure of the set of generic pictures; in general, $\mathcal{V}^d(G)$ is an irreducible component of $\mathcal{X}^d(G)$ of dimension $2\mathbf{v}(G)$, and all other components have equal or greater dimension (for details, see [**8**]). Furthermore, Theorem 4.5 of [**8**] admits the following generalization: For G a simple graph and $d \geq 2$, $E(G)$ is d-parallel independent if and only if $\mathcal{X}^d(G) = \mathcal{V}^d(G)$.

Combining these observations with Theorem 2.3, one sees that d-parallel independence is equivalent to the condition that the compressed Poincaré series

$$([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)} \; [d+1]_q^{\mathbf{c}(G)} \; \mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right)$$

be monic of degree $d \cdot \mathbf{v}(G)$. On the other hand, the corank-nullity generating function (2.2) says that

$$\mathbf{T}_G\left(\frac{[2]_q\,[d]_q}{[d]_q - 1},\, [d]_q\right) \;=\; \frac{f(q)}{([d]_q - 1)^{r(E)}} \;=\; \frac{f(q)}{([d]_q - 1)^{\mathbf{v}(G)-\mathbf{c}(G)}}$$

where $f(q)$ is a polynomial in q, and r is the rank function on subsets of E (see Section 2). Therefore, we may divide the compressed Poincaré series by $[d+1]_q^{\mathbf{c}(G)}$ yields a polynomial in q to obtain the statement of Theorem 3.1.

4. Orchard Schubert Calculus

4.1. The cohomology ring of an orchard. An edge e in a graph G is an *isthmus* if $\mathbf{c}(G-e) = \mathbf{c}(G)+1$ (otherwise $\mathbf{c}(G-e) = \mathbf{c}(G)$). We denote the number of isthmuses and loops by $\mathbf{i}(G)$ and $\boldsymbol{\ell}(G)$ respectively. In addition, if v is an endpoint of e, we will write $e \in E(v)$ or say that v, e is an *incident pair*.

An *orchard* is a graph G such that every edge is either an isthmus or a loop; that is, $\mathbf{e}(G) = \mathbf{i}(G) + \boldsymbol{\ell}(G)$. In this case, the Tutte polynomial of G is

$$\mathbf{T}_G(x,y) \;=\; x^{\mathbf{i}(G)} y^{\boldsymbol{\ell}(G)},$$

so by Theorem 2.3 the compressed Poincaré series of $\mathcal{X}^d(G)$ is

$$P_G^d(q) \;=\; [d+1]_q^{\mathbf{c}(G)} \; [2]_q^{\mathbf{i}(G)} \; [d]_q^{\mathbf{e}(G)}. \tag{4.1}$$

This polynomial is palindromic, suggesting that the picture space of an orchard is smooth (by Poincaré duality). In fact, more is true.

Proposition 4.1. *Let $G = (V, E)$ be a graph and $d \geq 2$. The picture space $\mathcal{X}^d(G)$ is smooth if and only if G is an orchard.*

Proposition 4.1 is proved as follows. When G is an orchard, $\mathcal{X}^d(G)$ may be realized explicitly as an iterated projective bundle over $\mathbb{P}^d$ with smooth fibers. If G is not an orchard, let $\mathbf{P}$ be a generic picture (where no points coincide) and let $\mathbf{Q}$ be a picture that is "maximally degenerate"—that is, all points $\mathbf{Q}(v)$ coincide, as do all lines $\mathbf{Q}(e)$. Then one can show directly that the tangent space to $\mathcal{X}^d(G)$ at $\mathbf{P}$ has dimension exactly $d \cdot \mathbf{v}(G)$, while the tangent space at $\mathbf{Q}$ has strictly greater dimension; it follows that $\mathbf{Q}$ is a singular point.

Remark 4.2. If G is an orchard then $P_G^d(q)$ is palindromic, by Proposition 4.1 and Poincaré duality. The converse is not true. For instance, let G have two vertices and three nonloop edges. Then $\mathcal{X}^2(G)$ is not smooth, but by Theorem 2.3 its compressed Poincaré series is $1 + 5q + 9q^2 + 9q^3 + 5q^4 + q^5$.

We will need several facts about vector bundles over complex manifolds. For more details, see chapter IV of [**2**], especially pp. 269–271. The main fact is as follows. Let M be a complex manifold and $\mathcal{E}$ a complex vector bundle on M of rank d. The *projectivization* of $\mathcal{E}$ is the fiber bundle $\mathbb{P}(\mathcal{E})\overset{\pi}{\Omega}M$ whose fiber at a point $m \in M$ is $\mathbb{P}(\mathcal{E})_m = \mathbb{P}(\mathcal{E}_m)$, that is, the space of lines through the origin in the fiber of $\mathcal{E}$ at m. Thus $\pi^{-1}\mathcal{E}$ is a rank-d vector bundle over $\mathbb{P}(\mathcal{E})$. The *tautological subbundle* $\mathcal{L}$ is the line bundle on $\mathbb{P}(\mathcal{E})$ defined fiberwise by $\mathcal{L}_p = p$ (regarding p as a line in $\mathcal{E}_{\pi(p)}$). With this setup,

$$H^*(\mathbb{P}(\mathcal{E})) \;\cong\; H^*(M)[x] \,/\, \big(x^d + c_1(\mathcal{E})x^{d-1} + \cdots + c_d(\mathcal{E})\big) \tag{4.2}$$

where $c_i(\mathcal{E})$ denotes the ith Chern class of $\mathcal{E}$, and $x = c_1(\mathcal{L}^*)$, the first Chern class of the dual line bundle $\mathcal{L}^*$.

The idea of the presentation of $H^*(\mathcal{X}^d(G))$ (to follow in Theorem 4.3) is that we can say precisely how the graph-theoretic operations of deletion and contraction correspond to projectivizations of certain vector bundles. Every nontrivial orchard can be "pruned"; that is, we can identify a simpler orchard G' and a vector bundle $\mathcal{E}$ on $\mathcal{X}^d(G')$ such that $\mathcal{X}^d(G) = \mathbb{P}(\mathcal{E})$. Moreover, the fiber of $\mathcal{E}$ has an elementary description in terms of the data for a picture $\mathbf{P}$ of G'. By the aforementioned machinery of Chern classes, in particular (4.2), we can express $H^*(\mathcal{X}^d(G))$ as an algebra over $\mathcal{X}^d(G')$.

Let $e \in E(G)$. We have already seen that if e is a loop and $G' = G - e$, then $\mathcal{X}^d(G)$ is a $\mathbb{P}^{d-1}$-bundle over $\mathcal{X}^d(G')$. More precisely, if v is the unique endpoint of e, then $\mathcal{X}^d(G) = \mathbb{P}(\mathcal{W}/\mathcal{L}_v)$, where $\mathcal{W}$ is the trivial bundle of rank $d+1$ and $\mathcal{L}_v$ is the line bundle whose fiber is $\mathbf{P}(v)$.

Now suppose that e is an isthmus. It suffices to consider the case that e is the "stem of a leaf v"; that is, $v \in V(G)$ and $E(v) = \{e\}$. Let w be the other endpoint of e, and let G' be the graph obtained from G by deleting e and v and attaching a loop e' at the other endpoint of e. Then $\mathcal{X}^d(G) = \mathbb{P}(\mathcal{F}_e)$, where $\mathcal{F}_e$ is the plane bundle on $\mathcal{X}^d(G')$ with fiber $\mathbf{P}(e)$.

Theorem 4.3. *Let $G = (V, E)$ be an orchard, with vector bundles $\mathcal{L}_v$ and $\mathcal{F}_e$ as above. For each $v \in V$, let $x_v = c_1(\mathcal{L}_v^*)$, and for each incident pair v, e, let $y_{v,e} = c_1((\mathcal{F}_e/\mathcal{L}_v)^*)$. Then*

$$H^*(\mathcal{X}^d(G);\, \mathbb{Z}) \cong \mathbb{Z}\,[x_v, y_{v,e} :\; v \in V,\; e \in E(v)] \;\, / \;\, I_G,$$

where I_G is the ideal

$$I_G \;=\; \left\langle \begin{array}{lll} x_v^{d+1} & & \text{for } v \in V, \\ h_d(x_v, y_{v,e}) & & \text{for } v \in V,\; e \in E(v), \\ x_v - x_w + y_{v,e} - y_{w,e}, & x_v y_{v,e} - x_w y_{w,e} & \text{for } e = vw \end{array} \right\rangle .$$

Here $h_d(x,y) = x^d + x^{d-1}y + \cdots + xy^{d-1} + y^d$ is the dth complete homogeneous symmetric function in x and y, and $e = vw$ means that e is an isthmus with endpoints v, w.

Setting $z_e := c_1(\mathcal{F}_e)$, the Whitney product formula for vector bundles gives the relations $z_e = x_v + y_{v,e} = x_w + y_{w,e}$ whenever e is an edge with endpoints v, w. This yields an equivalent and somewhat more concise presentatation of the cohomology ring.

Corollary 4.4. *Let G be an orchard and x_v, z_e as above.*

Then $H^*(\mathcal{X}^d(G)) = \mathbb{Z}\,[x_v, z_e :\; v \in V,\; e \in E] \;/\; J_G$, where

$$J_G \;=\; \left\langle \begin{array}{ll} x_v^{d+1} & \text{for } v \in V, \\ h_d(x_v, z_e - x_v) & \text{for } v \in V,\; e \in E(v), \\ (x_v - x_w)(z_e - x_v - x_w) & \text{for } e = vw \end{array} \right\rangle .$$

4.2. Enumerative geometry. The Schubert calculus (see, e.g., [**7**] or [**4**]) reduces certain enumerative geometry questions to calculations in the cohomology ring of a flag manifold. If G is an orchard, then the presentation of $H^*(\mathcal{X}^d(G))$ given in Theorem 4.3, together with the canonical epimorphism (2.4), allows us to answer similar enumerative geometry questions about pictures of G.

Let L_1 be the graph consisting of a vertex and a loop. A picture of L_1 is a point lying on a line in complex projective d-space, or equivalently a line through the origin lying on a plane in $\mathbb{C}^{d+1}$. That is, $\mathcal{X}^d(L_1)$ is naturally isomorphic to the partial flag manifold $F\ell^{1,2}(d+1)$ (in the notation of [**4**]).

More generally, suppose that G is an orchard, $v \in V(G)$, and $e \in E(v)$. As in (2.4), there is an epimorphism (in fact, a smooth fibration)

$$\pi_{v,e}:\ \mathcal{X}^d(G)\tilde{\Omega}\mathcal{X}^d(L_1) \ \cong\ F\ell^{1,2}(d+1) \tag{4.3}$$

forgetting all data except $\mathbf{P}(v)$ and $\mathbf{P}(e)$. This gives a decomposition of $\mathcal{X}^d(G)$ as a disjoint union of *orchard Schubert cells*

$$\Omega^\circ_\sigma \ = \ \bigcap_{e \in E(v)} \pi^{-1}(\Omega^\circ_{\sigma_{v,e}})$$

indexed by $(2\mathbf{i}(G) - \boldsymbol{\ell}(G))$-tuples σ of permutations $\sigma_{v,e}$ in the symmetric group S_{d+1}.

By induction on $\mathbf{e}(G)$, one can show that each Ω°_σ is isomorphic to an affine space. Moreover, it is not hard to identify permutations $\sigma_{v,e}$ for which Ω°_σ is nonempty. We expect that in general the *orchard Schubert variety* $\Omega_\sigma = \overline{\Omega^\circ_\sigma}$ should be a union of orchard Schubert cells.

Problem 4.5. *Describe the* orchard Bruhat order*, the partial order on tuples of partitions given by $\sigma \preceq \tau$ iff $\Omega^\circ_\sigma \subseteq \Omega_\tau$.*

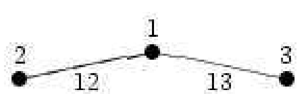
In general, the orchard Bruhat order is weaker than the product of the various strong Bruhat orders: that is, $\sigma \preceq \tau$ implies that $\sigma_{v,e} \leq \tau_{v,e}$ in the strong Bruhat order for all incident pairs v, e. The converse is false in general. For example, if $G = K_2$ is the complete graph on two vertices and $d = 2$, then $(231, 231)$ and $(213, 213)$ are incomparable in the orchard Bruhat order, even though $231 > 213$ in the Bruhat order on S_3.

The fibrations (4.3) induce pullback monomorphisms of cohomology rings

$$\pi^*_{v,e}:\ H^*(F\ell^{1,2}(d+1))\tilde{\Omega}H^*(\mathcal{X}^d(G))$$

for every incident pair v, e. This observation allows us to extend the Schubert calculus of (partial) flag varieties to solve enumerative geometry problems about picture of orchards. We devote the remainder of this section to a typical problem and its solution. (For this and many similar computations, the author used the computer algebra system *Macaulay* [**1**].)

Example 4.6. Let G be the tree with vertices $V = \{1, 2, 3\}$ and edges $E = \{12, 13\}$:



Let $A_1, A_2, A_3 \subset \mathbb{P}^3$ be planes, and let $A_4, \ldots, A_9 \subset \mathbb{P}^3$ be lines, with the collection $\{A_i\}$ in general position. We will calculate the number of pictures of G in $\mathbb{P}^3$ satisfying the conditions

$$\begin{aligned} \mathbf{P}(i) \in A_i &\quad \text{for } i = 1,2,3,\\ \mathbf{P}(12) \cap A_i \neq \emptyset &\quad \text{for } i = 4,5,6,\\ \mathbf{P}(13) \cap A_i \neq \emptyset &\quad \text{for } i = 7,8,9. \end{aligned} \tag{4.4}$$

For $i = 1, \ldots, 9$, let Y_i be the subvariety of $\mathcal{X}^3(G)$ consisting of pictures $\mathbf{P}$ for which the condition involving A_i is satisfied. Then the problem is to determine the cardinality of $Y = \bigcap_i Y_i$. Each Y_i is the pullback of some Schubert variety $\Omega_\sigma \subseteq F\ell^{1,2}(\mathbb{C}^4)$, so its cohomology class is a Schubert polynomial (see [**4**]) in the variables $x_1, x_2, x_3, z_{12}, z_{13}$ (using the presentations of Theorem 4.3 and Corollary 4.4). For instance,

$$\begin{aligned} [Y_1] = \left[{\pi_{1,12}}^{-1}(\Omega_{2134})\right] \ &= \ \mathfrak{S}_{2134}(x_1, z_{12} - x_1) \ = \ x_1 \qquad \text{and}\\ [Y_4] = \left[{\pi_{1,12}}^{-1}(\Omega_{1324})\right] \ &= \ \mathfrak{S}_{1324}(x_1, z_{12} - x_1) \ = \ z_{12}. \end{aligned}$$

By similar calculations, we find that

$$[Y_2] = x_2, \qquad [Y_5] = [Y_6] = z_{12},$$
$$[Y_3] = x_3, \qquad [Y_7] = [Y_8] = [Y_9] = z_{13}.$$

Therefore $[Y] = x_1x_2x_3z_{12}^3z_{13}^3$. Finally, the cohomology class of a point in $\mathcal{X}^3(G)$ is $(x_1x_2x_3)^3$. Since

$$x_1x_2x_3z_{12}^3z_{13}^3 = 4(x_1x_2x_3)^3$$

in $H^*(\mathcal{X}^3(G))$, we conclude that $|Y| = 4$. That is, there exist four pictures of the orchard G satisfying the conditions (4.4).

This cohomological calculation depends on the fact that the subvarieties Y_i meet transversely. For the stated example, this can be verified by solving the enumerative problem directly geometrically; however, the author (who is not an expert on Schubert calculus) does not at present have a more general transversality result.

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