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Descents, Major Indices, and Inversions in Permutation Groups

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Abstract. *We give a new proof of a multivariate generating function involving the descent, major index, and inversion statistic due to Gessel. We then show how one can easily modify this proof to give new generating functions involving these three statistics over Young's hyperoctahedral group, the Weyl group of type D, and multiples of permutations. All of our proofs are combinatorial in nature and exploit fundamental relationships between the elementary and homogeneous symmetric functions.*

1. Introduction

Let $\sigma = \sigma_1 \cdots \sigma_n$ be an element of the symmetric group S_n written in one line notation. The descent, major index, and inversion statistics are defined by

$$des(\sigma) = \sum_{i=1}^{n-1} \chi(\sigma_{i+1} < \sigma_i),\ maj(\sigma) = \sum_{i=1}^{n-1} i\,\chi(\sigma_{i+1} < \sigma_i), \quad \text{and} \quad inv(\sigma) = \sum_{j<i} \chi(\sigma_i < \sigma_j),$$

where for any statement A, $\chi(A)$ is 1 if A is true and 0 if A is false. These definitions also hold for any finite sequence. The past century has witnessed a beautiful theory develop from the study of these (and other) permutation statistics. To this day, new generalizations and variations of these statistics are investigated. In this work, we will create multivariate generating functions involving the three statistics defined above. They will follow from the combinatorial manipulation of objects arising from fundamental relationships between bases of symmetric functions.

Standard notation from hypergeometric function theory will be used. For $n \geq 1$, $\lambda \vdash n$, and an indeterminate q, let

$$[n]_q = q^0 + \cdots + q^{n-1},\ [n]_q! = [n]_q \cdots [1]_q, \quad \text{and} \quad \begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}$$

be the q-analogues of n, $n!$, and $\binom{n}{\lambda}$, respectfully. Let $(x;q)_n = (1 - xq^0) \cdots (1 - xq^{n-1})$. Finally, define a q-analogue of the exponential function such that

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!} q^{\binom{n}{2}}.$$

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In Gessel's thesis and in a paper by Garsia and Gessel [**Ga, Ge**], it was shown that

$$\sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in S_n} x^{des(\sigma)} r^{maj(\sigma)} q^{inv(\sigma)} = \sum_{k\geq 0} \frac{x^k}{\exp_q(-tr^0)\cdots\exp_q(-tr^k)}. \tag{1.1}$$

This will be our starting point. First, we give a new, combinatorial proof of 1.1. Then, we show how similar proofs indicate a systematic approach to finding more generating functions for permutation statistics. To this end, we highlight some basic facts about the ring of symmetric functions needed for the journey.

The n^{th} elementary symmetric function e_n and the n^{th} homogeneous symmetric function h_n are polynomials in the variables $x_1, x_2, \ldots$ defined to satisfy

$$\sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1-x_i t} \quad \text{and} \quad \sum_{n\geq 0} e_n t^n = \prod_i (1+x_i t).$$

Therefore,

$$\sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1-x_i t} = \left(\prod_i 1 + x_i(-t)\right)^{-1} = \left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1}. \tag{1.2}$$

Multiply both sides of 1.2 by the reciprocal of the right hand side and then compare the coefficient of t^n on both sides of the result to see that

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0, \quad \text{or equivalently,} \quad h_n = (-1)^{n-1} e_n + \sum_{i=1}^{n-1} (-1)^{i-1} e_i h_{n-i}. \tag{1.3}$$

For a partition λ, e_λ is defined to be $e_{\lambda_1}\cdots e_{\lambda_\ell}$. It is well known that $\{e_\lambda : \lambda \text{ a partition}\}$ is a basis for the ring of symmetric functions [**S**]. A combinatorial interpretation of the expansion of h_n in terms of this basis was first given by Egecioglu and Remmel [**E**]. It is now described as it will be of great use to us.

A rectangle of height 1 and length n chopped into "bricks" of lengths found in the partition λ is known as a brick tabloid of shape (n) and type λ. For example, Figure 1 shows one brick tabloid of shape (12) and type $(2,3,7)$. Let $B_{\lambda,n}$ be the number of such objects. Note that $(-1)^{n-1}B_{(n),(n)} = (-1)^{n-1}$. Furthermore,

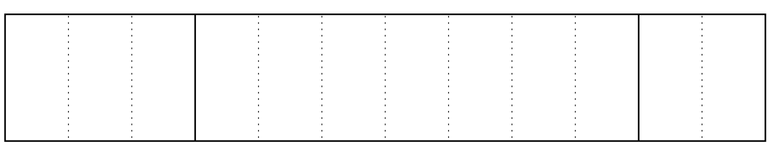

FIGURE 1. A brick tabloid of shape (12) and type $(2,3,7)$.

sorting by the length of the first brick, it may be seen that

$$(-1)^{n-\ell(\lambda)} B_{\lambda,(n)} = \sum_{i=1}^{n-1} (-1)^{i-1} \left((-1)^{(n-i)-(\ell(\lambda)-1)} B_{\lambda\backslash i,(n-i)}\right)$$

where $B_{\lambda\backslash i,(n-i)}$ is defined to be zero if λ does not have a part of size i. These two facts completely determine the numbers $(-1)^{n-\ell(\lambda)}B_{\lambda,(n)}$ recursively. By using 1.3, the coefficient of e_λ in h_n may be shown to satisfy the exact same recursions. Therefore,

$$h_n = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{1.4}$$

We have now established enough terminology and basic facts to commence our discussion of methods to find generating functions involving the descent, major index, and inversion statistics.

2. A new proof of Gessel's generating function

For $k \geq 0$, define a homomorphism ξ_k on the ring of symmetric functions such that

$$\xi_k(e_n) = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q!\cdots[i_k]_q!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}$$

for indeterminates q and r. Since products of n^{th} elementary symmetric functions form a basis, the definition of ξ_k extends to all other elements in the ring of symmetric functions. In particular, we may apply ξ_k to the n^{th} homogeneous symmetric function.

Theorem 2.1. *For $k, n \geq 0$,*

$$[n]_q!\xi_k(h_n) = \frac{1}{(x;r)_{n+1}} \sum_{\sigma \in S_n} x^{des(\sigma)} r^{maj(\sigma)} q^{inv(\sigma)} \Bigg|_{x^k}$$

where expression$|_x$ denotes the coefficient of x in expression.

PROOF. Expand h_n in terms of the elementary symmetric functions by 1.4:

$$\begin{aligned} [n]_q!\xi_k(h_n) &= [n]_q! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi_k(e_\lambda) \\ &= [n]_q! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0},\ldots,i_{j,k} \geq 0 \\ i_{j,0}+\cdots+i_{j,k}=\lambda_j}} \frac{r^{0i_{j,0}+\cdots+ki_{j,k}}}{[i_{j,0}]_q!\cdots[i_{j,k}]_q!} q^{\binom{i_{j,0}}{2}+\cdots+\binom{i_{j,k}}{2}}. \end{aligned}$$

Rewriting q-analogues, the right hand side of the above is equal to

$$(2.1) \qquad \sum_{\lambda \vdash n} \begin{bmatrix} n \\ \lambda \end{bmatrix}_q (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0},\ldots,i_{j,k} \geq 0 \\ i_{j,0}+\cdots+i_{j,k}=\lambda_j}} \begin{bmatrix} \lambda_j \\ i_{j,0},\ldots,i_{j,k} \end{bmatrix}_q r^{0i_{j,0}+\cdots+ki_{j,k}} q^{\binom{i_{j,0}}{2}+\cdots+\binom{i_{j,k}}{2}}.$$

2.1 may be interpreted as a sum of signed, weighted brick tabloids. After combinatorial objects are described, a sign-reversing, weight-preserving involution will be applied to leave only objects with positive sign. Then, the fixed points will help count the number of permutations with k descents by the major index and inversion statistics.

Start creating combinatorial objects from 2.1 by using the "$\sum_{\lambda \vdash n}$" and the factor of $B_{\lambda,n}$ to give a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$. Let us call this brick tabloid T. The factor of $(-1)^{n-\ell(\lambda)}$ allows for the labeling of each cell not terminating a brick in T with a "-1". In each terminal cell in a brick, place a "1".

For each brick in T, choose nonnegative integers $i_0, \ldots, i_k$ that sum to the total length of the brick. This accounts for the product and second sum in 2.1. Using the power of r, these choices for $i_0, \ldots, i_k$ can be recorded in T. In each brick, place a power of r in each cell such that the powers weakly increase from left to right such that the number of occurrences of r^j will be equal to i_j. At this point, we have constructed T which may look something like Figure 2 below.

-1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
r^1	r^1	r^3	r^0	r^0	r^0	r^0	r^2	r^3	r^3	r^1	r^1

FIGURE 2. One possible T when $k = 3$ and $n = 12$.

The only components in 2.1 which have not been used involve powers of q. We will explain how these powers of q will fill the cells of T with a permutation of n such that a decrease must occur between consecutive cells labeled with the same power of r. Along with this permutation of n, a power of q will be recorded in each cell counting the number of smaller integers in the permutation to the right (given an integer i in a permutation, the number of smaller integers to the right of i is sometimes referred to as the number of inversions caused by i).

In [**C**], Carlitz shows that if $\mathcal{R}(0^{i_0}, \dots, k^{i_k})$ is the number of rearrangements of i_0 0's, i_1 1's, etc., then

$$\begin{bmatrix} n \\ i_0, \dots, i_k \end{bmatrix}_q = \sum_{r \in \mathcal{R}(0^{i_0}, \dots, k^{i_k})} q^{inv(r)}.$$

Thus, the $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q$ term in 2.1 gives a rearrangement of λ_1 0's, λ_2 1's, etc. We will use this to select which integers in a permutation of n will appear in each brick. Start with a brick tabloid T of shape (n) such that the size of the bricks read from left to right are $b_0, \dots, b_k$. For example, if T is the brick tabloid in Figure 2, then $b_0 = 3$, $b_1 = 7$ and $b_2 = 2$. Then consider a rearrangement r of $0^{b_0}, \dots, k^{b_k}$ and construct a permutation $\sigma(r)$ by labeling the 0's from left to right with $1, 2, \dots, b_0$, the 1's from right to left with $b_0 + 1, \dots, b_0 + b_1$ and in general the i's from right to left with $1 + \sum_{j=1}^{i-1} b_j, \dots, b_i + \sum_{j=1}^{i-1} b_j$. In this way, $\sigma(r)^{-1}$ starts with the positions of the 0's in r increasing order, followed by the positions of the 1's in r in increasing order, etc. For example, for T pictured in Figure 2, one possible rearrangement to consider is $r = 1\,0\,1\,1\,2\,0\,1\,2\,1\,0\,1\,1$. Below we picture $\sigma(r)$ and $\sigma(r)^{-1}$.

$$\begin{array}{lcllllllllllll} r & = & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ \sigma(r) & = & 4 & 1 & 5 & 6 & 11 & 2 & 7 & 12 & 8 & 3 & 9 & 10 \\ \sigma(r)^{-1} & = & 2 & 6 & 10 & 1 & 3 & 4 & 7 & 9 & 11 & 12 & 5 & 8. \end{array}$$

This tells us that when selecting a permutation of 12 to place in T, the integers $2, 6, 10$ should appear in the brick of size 3, the integers $1, 3, 4, 7, 9, 11, 12$ should appear in the brick of size 7, and the integers $5, 8$ should appear in the brick of size 2. It is easy to see that $inv(r) = inv(\sigma(r))$ and $inv(\sigma(r)) = inv(\sigma(r)^{-1})$. Thus, the theorem of Carlitz tells us that $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q$ is the sum of the the number of inversions of all sequences that are the result of placing a permutation of numbers $1, \dots, n$ in the cells of T such that the numbers in each brick increase from left to right.

For each brick of length λ_j in T, there is an unused term of the form $\begin{bmatrix} \lambda_j \\ i_0, \dots, i_k \end{bmatrix}_q q^{\binom{i_0}{2} + \cdots + \binom{i_k}{2}}$ where $i_0 + \cdots + i_k = \lambda_j$. The theorem of Carlitz enables us to start with a rearrangement a of i_0 0's, i_1 1's, etc. to use the q-multinomial coefficient. Record from right to left the 0's in a with $1, \dots, i_0$. Then record the 1's in a from right to left with $i_0 + 1, \dots, i_1$. Continue this process k times to form a permutation of λ_j from a, τ_a^{-1}. The inverse, τ_a, records the places of the 0's, 1's, etc., and therefore must have decreasing sequences of length $i_0, \dots, i_k$. Let $\overline{\tau_a}$ be the permutation τ_a where the integers $1, \dots, \lambda_j$ have been replaced with whatever integers the factor $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q$ dictates should appear in the j^{th} brick.

For example, if $k = 3$ and $i_0 = 4$, $i_1 = 0$, $i_2 = 1$, and $i_3 = 2$ as found in the second brick in Figure 2, a permutation of 7 may be formed from 0 2 0 3 3 0 0. Continuing our example from above, the brick of size 7 should contain the integers $1, 3, 4, 7, 9, 11$, and 12. The permutations τ_a^{-1}, τ_a, and $\overline{\tau_a}$ can be found:

	1	2	3	4	5	6	7
a	0	2	0	3	3	0	0
τ_a^{-1}	4	5	3	7	6	2	1
τ_a	7	6	3	1	2	5	4
$\overline{\tau_a}$	12	11	4	1	3	9	7

.

By construction, we have that

$$inv(\overline{\tau_a}) = inv(\tau_a) = inv(\tau_a^{-1}) = inv(r) + \binom{i_0}{2} + \cdots + \binom{i_k}{2}.$$

Therefore, for each brick of size λ_j, we may associate a permutation of λ_j such that the permutation must have a descent if two consecutive cells have the same power of r. By taking along a power of $q^{inv(\overline{\tau_a})}$, we are able to account for the factors in 2.1 of the form $\left[{\lambda_j \atop i_0,\ldots,i_k}\right]_q q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}$. Every term in 2.1 has now been used.

Let $\mathcal{T}$ be the set of all possible brick tabloids decorated in this way. Figure 3 gives one example of such an object. We have shown how each $T \in \mathcal{T}$ has the following five properties:

-1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
r^1	r^1	r^3	r^0	r^0	r^0	r^0	r^2	r^3	r^3	r^1	r^1
q^9	q^1	q^4	q^8	q^7	q^2	q^0	q^0	q^3	q^1	q^1	q^0
10	2	6	12	11	4	1	3	9	7	8	5

FIGURE 3. An object coming from 2.1 when $k = 3$ and $n = 12$.

(1) T is a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$,
(2) the cells in each brick contain -1 except for the final cell which contains 1,
(3) each cell contains a power of r such that the powers weakly increase within each brick,
(4) T contains a permutation of n which must have a decrease between consecutive cells within a brick if the cells are marked with the same power of r, and
(5) each cell contains a power of q recording the number of inversions caused by the integer entry.

Define the sign of $T \in \mathcal{T}$, $sgn(T)$, as the product of all -1 labels in T. Define the weight of $T \in \mathcal{T}$, $w(T)$, as the product of all r, and q labels. In this way, the T in Figure 3 has sign $(-1)^9$ and weight $r^{15}q^{36}$. From our development, we have

$$[n]_q!\xi_k(h_n) = \sum_{T \in \mathcal{T}} sgn(T)w(T).$$

At this point, we will introduce a sign-reversing weight-preserving involution I on $\mathcal{T}$ to rid ourselves of any $T \in \mathcal{T}$ with $sgn(T) = -1$. Scan the cells of $T \in \mathcal{T}$ from left to right looking for the first of two situations:

(1) a cell containing a -1, or
(2) two consecutive cells c_1 and c_2 such that c_1 ends a brick and either the powers of r increase from c_1 to c_2 or the powers of r are the same and the permutation decreases from c_1 to c_2.

If situation 1 is scanned first, let $I(T)$ be T where the brick containing the -1 is broken into two immediately after the violation and the -1 is changed to a 1. If situation 2 is scanned first, let $I(T)$ be T where bricks containing c_1 and c_2 are glued together and the 1 on c_1 is changed to -1. If when scanning from left to right neither case happens, let $I(T) = T$. For example, the image of the element of $\mathcal{T}$ in Figure 3 under I is displayed in Figure 4.

1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
r^1	r^1	r^3	r^0	r^0	r^0	r^0	r^2	r^3	r^3	r^1	r^1
q^9	q^1	q^4	q^8	q^7	q^2	q^0	q^0	q^3	q^1	q^1	q^0
10	2	6	12	11	4	1	3	9	7	8	5

FIGURE 4. The image of Figure 3 under I.

By definition, if $T \neq I(T)$, then $sgn(I(T)) = -sgn(T)$, $w(I(T)) = w(T)$, and $I(I(T)) = T$. Thus I is a sign-reversing weight-preserving involution on $\mathcal{T}$. The fixed points under I have the properties that

(1) there are no bricks with -1 in them,
(2) the powers of r weakly decrease, and
(3) if two consecutive bricks have the same power of r, then the permutation must increase there.

Since every brick of length greater than 1 contains a -1, a fixed point can only have bricks of length 1. One example of a fixed point may be found in Figure 5. We now have

1	1	1	1	1	1	1	1	1	1	1	1
r^3	r^3	r^3	r^2	r^2	r^1	r^1	r^1	r^1	r^1	r^0	r^0
q^3	q^4	q^5	q^1	q^1	q^1	q^1	q^1	q^2	q^2	q^0	q^0
4	6	8	2	3	5	7	9	11	12	1	10

FIGURE 5. A fixed point in $\mathcal{T}$ under I when $k = 3$ and $n = 12$.

$$[n]_q!\xi_k(h_n) = \sum_{T \in \mathcal{T}} sgn(T)w(T) = \sum_{\substack{T \in \mathcal{T} \text{ is a} \\ \text{fixed point under } I}} w(T),$$

so counting fixed points is the only remaining task.

Suppose that the powers of r in a fixed point are $r^{p_1}, \ldots, r^{p_n}$ when read from left to right. It must be the case that $k \geq p_1 \geq \cdots \geq p_n$. Let $a_1, \ldots, a_n$ be the nonnegative numbers defined by $a_i = p_i - p_{i+1}$ for $i = 1, \ldots, n-1$ and $a_n = p_n$. It follows that $p_1 + \cdots + p_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = p_1 \leq k$, and if σ is the permutation in a fixed point, $a_i \geq \chi(\sigma_i > \sigma_{i+1})$. In this way, we have the sum of weights over all fixed points under I is equal to

$$\begin{aligned}
&\sum_{\substack{\sigma \in S_n \\ k \geq p_1 \geq \cdots \geq p_n \geq 0}} q^{inv(\sigma)} r^{p_1 + \cdots + p_n} \\
&= \sum_{\sigma \in S_n} q^{inv(\sigma)} \sum_{\substack{a_1 + \cdots + a_n \leq k \\ a_i \geq \chi(\sigma_i > \sigma_{i+1})}} r^{a_1 + 2a_2 + \cdots + na_n} \\
&= \sum_{\sigma \in S_n} q^{inv(\sigma)} \sum_{a_1 \geq \chi(\sigma_1 > \sigma_2)} \cdots \sum_{a_n \geq \chi(\sigma_n > n+1)} x^{a_1 + \cdots + a_n} r^{a_1 + 2a_2 + \cdots + na_n} \Bigg|_{x^{\leq k}}
\end{aligned}$$

where the notation $expression|_{x^{\leq k}}$ means to sum the coefficients of x up to and including x^k in $expression$. Rewriting the above equation, we have

$$\begin{aligned}
&\sum_{\sigma \in S_n} q^{inv(\sigma)} \sum_{a_1 \geq \chi(\sigma_1 > \sigma_2)} (xr)^{a_1} \cdots \sum_{a_n \geq \chi(\sigma_n > n+1)} (xr^n)^{a_n} \Bigg|_{x^{\leq k}} \\
&= \sum_{\sigma \in S_n} q^{inv(\sigma)} \frac{(xr)^{\chi(\sigma_1 > \sigma_2)} \cdots (xr^n)^{\chi(\sigma_n > n+1)}}{(1 - xr) \cdots (1 - xr^n)} \Bigg|_{x^{\leq k}} \\
&= \frac{\sum_{\sigma \in S_n} x^{des(\sigma)} r^{maj(\sigma)} q^{inv(\sigma)}}{(1 - xr) \cdots (1 - xr^n)} \Bigg|_{x^{\leq k}}.
\end{aligned}$$

Dividing by $(1 - x)$ allows for $x^{\leq k}$ to be changed to x^k in the above expression, thereby arriving at the statement of the theorem. □

1.1 is a corollary. We have

$$\begin{aligned}
&\sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in S_n} x^{des(\sigma)} r^{maj(\sigma)} q^{inv(\sigma)} \\
&\qquad = \sum_{k\geq 0} x^k \sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in S_n} x^{des(\sigma)} r^{maj(\sigma)} q^{inv(\sigma)} \Bigg|_{x^k} \\
&\qquad = \sum_{k\geq 0} x^k \sum_{n\geq 0} t^n \xi_k(h_n),
\end{aligned}$$

which by an application of 1.2 is equal to

$$\begin{aligned}
\sum_{k\geq 0} x^k \xi_k \left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1} &= \sum_{k\geq 0} x^k \left(\sum_{n\geq 0} (-t)^n \sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q!\cdots[i_k]_q!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}\right)^{-1} \\
&= \sum_{k\geq 0} x^k \left(\sum_{n\geq 0} \frac{(-tr^0)^n}{[n]_q!} q^{\binom{n}{2}}\right)^{-1} \cdots \left(\sum_{n\geq 0} \frac{(-tr^k)^n}{[n]_q!} q^{\binom{n}{2}}\right)^{-1} \\
&= \sum_{k\geq 0} \frac{x^k}{\exp_q(-tr^0)\cdots\exp_q(-tr^k)}.
\end{aligned}$$

The proof we have given for 1.1, although elementary and combinatorial, is not any "easier" than that given by Garsia and Gessel. However, there are at least two distinct advantages of our methods. First, the techniques in the proof of theorem 2.1 may be slightly modified to give a wide swath of seemingly unrelated generating functions for the permutation enumeration of the symmetric group, Weyl groups of type B and D, subsets of the symmetric group, and more [**M**]. Second, the ideas in the proof of Theorem 2.1 may be generalized to give new generating functions involving the descent, major index, and inversion statistics.

3. Generating functions for the Weyl groups of type B and D

Let us turn our attention to applying this machinery to the hyperoctahedral group B_n and its subgroup D_n. The hyperoctahedral group B_n may be considered the set of permutations of n where each integer in the permutation is assigned either a + or − sign. For $\sigma\in B_n$, let $neg(\sigma)$ count the total number of negative signs in σ. The subgroup D_n of B_n contains those $\sigma\in B_n$ with $neg(\sigma)$ an even number. These are Weyl groups appearing in the study of root systems and Lie algebras.

Define a linear order Θ on $\{\pm 1,\ldots,\pm n\}$ such that

$$1 <_\Theta \cdots <_\Theta n <_\Theta -n <_\Theta \cdots <_\Theta -1$$

and define $des_B(\sigma)$ on B_n such that

$$des_B(\sigma) = \chi(n <_\Theta \sigma_n) + \sum_i \chi(\sigma_{i+1} <_\Theta \sigma_i).$$

This definition and the linear order Θ arises from an interpretation of B_n as a Coxeter group. For $\sigma\in B_n$, let $maj_B(\sigma)$ and $inv_B(\sigma)$ be the major index and inversion statistics with respect to the linear order Θ.

Using the methods of Garsia and Gessel and the study of upper binomial posets, Reiner found a generalization of 1.1 for B_n involving the statistic counting the number of negative signs in B_n and versions of descents, inversions, and the major index [**R**]. In this Section we will indicate (without a formal proof) how a simple modification of the method given in Section 2 to prove 1.1 can do the same.

Define a homomorphism $\xi_{B,k}$ on the ring of symmetric functions by defining it on e_n such that

$$\xi_{B,k}(e_n) = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q!\cdots[i_k]_q!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} [i_0+1]_y \cdots [i_k+1]_y.$$

Then it may be proved that

$$[n]_q!\xi_{B,k}(h_n) = \frac{1}{(x;r)_{n+1}} \sum_{\sigma \in B_n} x^{des_B(\sigma)} r^{maj_B(\sigma)} q^{inv_B(\sigma)} y^{neg(\sigma)} \Bigg|_{x^k}. \tag{3.1}$$

The proof of this fact may be found by first expanding h_n via 1.4 to form combinatorial objects like Figure 6 below. The $[i_0+1]_y \cdots [i_k+1]_y$ term in the definition of the ring homomorphism $\xi_{B,k}$ allows for

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
r^0	r^0	r^1	r^1	r^1	r^2	r^3	r^1	r^3	r^3	r^0	r^1
q^{11}	q^0	q^8	q^7	q^0	q^1	q^0	q^1	q^1	q^0	q^1	q^0
12	1	10	9	2	4	3	6	7	5	11	8
y		y	y			y		y		y	

FIGURE 6. An example of a combinatorial object coming from the application of $\xi_{B,k}$ to $[n]_q!h_n$.

the bottom row of T to contain some number of y's in cells marked with the same power of r so that these objects have the exact same properties as the T found in Figures 3, 4, and 5 with the addition of powers of y recorded in the bottom of the object. These powers of y will be interpreted to mean that the integer in the permutation in the cell marked with y is negative.

Suppose we have j consecutive cells within a brick with the same power of r and marked with a y. Instead of writing these integers in decreasing order as prescribed in the proof of Theorem 2.1, let us reverse the order of these j integers in T so that they are in increasing order. An example of this may be found in Figure 7. This is done so that cells in a brick marked with the same power of r are in descending order according

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
r^0	r^0	r^1	r^1	r^1	r^2	r^3	r^1	r^3	r^3	r^0	r^1
q^{11}	q^0	q^8	q^7	q^0	q^1	q^0	q^1	q^1	q^0	q^1	q^0
12	1	9	10	2	4	3	6	7	5	11	8
y		y	y			y		y		y	

FIGURE 7. Reversing the order of two integers in Figure 6.

to the linear order Θ. The same brick breaking/combining sign-reversing weight-preserving involution as in Theorem 2.1 may be now applied to leave a set of fixed points which may be counted to yield 3.1.

A generating function may be found employing 1.2. We have

$$\sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in B_n} x^{des_B(\sigma)} r^{maj_B(\sigma)} q^{inv_B(\sigma)} y^{neg(\sigma)}$$

$$= \sum_{k\geq 0} x^k \left(\sum_{n\geq 0} (-t)^n \sum_{\substack{i_0,\ldots,i_k\geq 0 \\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q!\cdots[i_k]_q!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} [i_0+1]_y \cdots [i_k+1]_y \right)^{-1}$$

which in turn may be simplified to look like

$$\sum_{k\geq 0} \frac{(x-xy)^k}{\left(\exp_q(-tr^0) - y\exp_q(-tyr^0)\right)\cdots\left(\exp_q(-tr^k) - y\exp_q(-tyr^k)\right)}. \tag{3.2}$$

Let $A(t,x,r,q,y)$ denote the generating function in 3.2 above. Notice that $A(t,x,r,q,0)$ is equal to the generating function in 1.1 as it should.

For any series $f(x) = \sum_{n\geq 0} c_n x^n$ for $c_i \in \mathbb{C}$, we have

$$\frac{f(x)+f(-x)}{2} = \sum_{n\geq 0} c_{2n}x^{2n}.$$

Therefore,

$$\sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in D_n} x^{des_B(\sigma)} r^{maj_B(\sigma)} q^{inv_B(\sigma)} y^{neg(\sigma)} = \frac{A(t,x,r,q,y)+A(t,x,r,q,-y)}{2},$$

giving a multivariate generating function for the Weyl group of type D.

4. A Generating function for pairs of permutations

Let us find a generating function for two copies of the symmetric group S_n. Given $\sigma^1,\sigma^2\in S_n$, define $comdes(\sigma^1,\sigma^2)$ as the number of times $\sigma^j_i > \sigma^j_{i+1}$ for all $j=1,2$—this is known as the number of common descents. Let $commaj(\sigma^1,\sigma^2)$ register i for every time $\sigma^j_i > \sigma^j_{i+1}$ for $j=1,2$. These type of statistics have been studied and a multivariate generating function for descents and inversions was found in [**F**] by Fedou and Rawlings. In this Section, we will apply our methods to finding a generating function involving the descent, major index, and inversion statistics by altering the ring homomorphism ξ_k.

Define $\xi_{k,2}$ as a homomorphism on the ring of symmetric functions by defining it on the n^{th} elementary symmetric functions such that

$$\xi_{k,2}(e_n) = \sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q![i_0]_p!\cdots[i_k]_q![i_k]_p!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} p^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}.$$

The difference between ξ_k and $\xi_{k,2}$ is that all the terms involving q in ξ_k have written down twice in the indeterminates q and p. Using $\xi_{k,2}$, it may be shown that

$$\sum_{n\geq 0} \frac{t^n}{[n]_q![n]_p!(x;r)_{n+1}} \sum_{\sigma^1,\sigma^2\in S_n} x^{comdes(\sigma^1,\sigma^2)} r^{commaj(\sigma^1,\sigma^2)} q^{inv(\sigma^1)} p^{inv(\sigma^2)}$$

$$= \sum_{k\geq 0} x^k \left(\sum_{n\geq 0} (-t)^n \sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q![i_0]_p!\cdots[i_k]_q![i_k]_p!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} p^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} \right)^{-1}. \tag{4.1}$$

The proof of 4.1 follows in the same way that we have proved 1.1. First, it may be shown that

$$[n]_q![n]_p!\xi_{k,2}(h_n) = \frac{1}{(x;r)_{n+1}} \sum_{\sigma^1,\sigma^2\in S_n} x^{comdes(\sigma^1,\sigma^2)} r^{commaj(\sigma^1,\sigma^2)} q^{inv(\sigma^1)} p^{inv(\sigma^2)} \Bigg|_{x^k}. \tag{4.2}$$

This is analogous to our Theorem 2.1. The combinatorial objects we are able to create based on brick tabloids are the same as those found in Figures 3, 4, and 5 with one slight change. The q and p analogues give rise to two different permutations in a brick tabloid instead of one. The powers of q and p register the inversions of each permutation.

For example, one such combinatorial object which may be formed starting with 4.2 and using the techniques in the proof of Theorem 2.1 is found in Figure 5 below. The combinatorial objects may be

-1	-1	-1	-1	1	-1	1	-1	-1	1	1	1
r^0	r^0	r^1	r^1	r^1	r^2	r^3	r^1	r^3	r^3	r^0	r^1
q^{11}	q^0	q^8	q^7	q^0	q^1	q^0	q^1	q^1	q^0	q^1	q^0
12	1	10	9	2	4	3	6	7	5	11	8
p^{10}	p^3	p^9	p^8	p^3	p^0	p^0	p^2	p^2	p^1	p^0	p^0
11	4	12	10	5	1	2	7	8	6	3	9

FIGURE 8. An example of T arising from 4.2

constructed to have the following properties:

(1) T is a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$,
(2) the cells not at the end of a brick are marked with -1 and cells at the end a brick are marked with 1,
(3) each cell contains a power of r such that the powers weakly increase within each brick,
(4) T contains *two* permutations of n which both must have a decrease between consecutive cells within a brick if the cells are marked with the same power of r, and
(5) each cell contains a power of q and p recording the number of integers in each of the permutations to the right which are smaller.

The sign of such an object is the product of the -1 signs in the objects and the weight is defined to be the product of all indeterminates in the object. Once these combinatorial objects are defined, a very similar sign-reversing weight-preserving involution I as given in the proof of Theorem 2.1 may be employed. That is, to define I, scan the cells of $T \in \mathcal{T}$ from left to right looking for the first of two situations:

(1) a cell containing a -1, or
(2) two consecutive cells c_1 and c_2 such that c_1 ends a brick and either the powers of r increase from c_1 to c_2 or the powers of r are the same and the both permutations decrease from c_1 to c_2.

If situation 1 is scanned first, let $I(T)$ be T where the brick containing the -1 is broken into two immediately after the violation and the -1 is changed to a 1. If neither (1) or (2) applies, then we define $I(T) = T$. By definition, if $T \neq I(T)$, then $sgn(I(T)) = -sgn(T)$, $w(I(T)) = w(T)$, and $I(I(T)) = T$. Thus I is a sign-reversing weight-preserving involution on $\mathcal{T}$. The fixed points under I have the properties that

(1) there are no bricks with -1 in them,
(2) the powers of r weakly decrease, and
(3) if two consecutive bricks have the same power of r, then at least one of the permutations must increase there.

Using the same techniques as in Section 2, one can show that the fixed points under this involution may then be counted to prove 4.2. 4.1 follows by an application of the simple relationship between the elementary and homogeneous symmetric functions in 1.2.

Instead of defining a "double" version of ξ_k in $\xi_{k,2}$, one may define a "m-tuple" version of ξ_k to help record generating functions for m copies of the symmetric group. The process is no more difficult than the method we have outlined for two copies of the symmetric group except for the fact that there are m indeterminates to keep track of instead of two.

Furthermore, we can keep track of two elements in B_n or D_n using a combination of the ideas behind the definitions of $\xi_{B,k}$ and $\xi_{k,2}$. That is, by defining a homomorphism by mapping e_n to

$$\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}} \frac{r^{0i_0+\cdots+ki_k}}{[i_0]_q![i_0]_p!\cdots[i_k]_q![i_k]_p!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} p^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} [i_0+1]_y[i_0+1]_z\cdots[i_k+1]_y[i_k+1]_z,$$

then we can find a generating function registering common B_n descents and major indices together with inversions and the number of negative signs over pairs of signed permutations. This technique, in general, can provide many different generating functions for permutation statistics.

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