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Decomposition of Green polynomials of type A and DeConcini-Procesi algebras of hook partitions

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Abstract. *A Kraśkiewicz-Weymann type theorem is obtained for the DeConcini-Procesi algebras of hook partitions. The DeConcini-Procesi algebras are graded modules of the symmetric groups, that generalize the coinvariant algebras. Defining the direct sums of the homogeneous components of the algebra in some natural way, we show that these submodules are induced from representations of the corresponding subgroup of the symmetric group. The Green polynomials of type A play an essential role in our argument.*

1. Introduction

Let $R_n = \bigoplus_{d\geq 0} R_n^d$ be the coinvariant algebra of the symmetric group S_n. For each $l \in \{1, 2, \ldots, n\}$ and for each $k = 0, 1, \ldots, l-1$, define a subspace $R_n(k;l)$ of R_n by

$$R_n(k;l) := \bigoplus_{d\equiv k \bmod l} R_n^d \ .$$

In our previous work [MN], we have shown that all $R_n(k;l)$ $(k = 0, \ldots l-1)$ are S_n-submodules of equal dimension and induced by modules of the subgroup $H_n(l)$ of S_n. Here $H_n(l)$ indicates a direct product of a cyclic group of order l generated by

$$(1, \ldots, l)(l+1, \ldots, 2l) \cdots ((d-1)l+1, \ldots, dl) \ .$$

and the symmetric group of order r, where r is a remainder of n divided by l.

In this article we consider the "DeConcini-Procesi algebras" in place of the coinvariant algebras in the preceding result. The DeConcini-Procesi algebra R_μ is a graded S_n-module parameterized by a partition μ of n. The algebra R_μ serves the generalization of R_n, since $R_\mu = R_n$ when $\mu = (1^n)$.

From the geometric point of view, DeConcini-Procesi algebras are isomorphic to the cohomology ring of the fixed point subvarieties of flag varieties. Namely, the coinvariant algebras are isomorphic to the cohomology ring of the flag varieties. Since the fixed point subvarieties are singular, we generally cannot expect nice combinatorial properties of the Betti numbers, such as unimodal symmetry. Hence we may face some difficulty when we try to achieve the results similar to the case of the coinvariant algebras. In fact, even if we collect the homogeneous components of R_μ whose degree is congruent to k modulo n for each $k = 0, \ldots, n-1$, the dimensions of them do not coincide.

However, we find that there exists a positive integer M_μ for a partition μ such that R_μ holds the above mentioned property for the coinvariant algebras for $l = 1, \ldots, M_\mu$. Essential tools to prove our main result

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are the "Green polynomials". Green polynomials were introduced by J. A. Green [**Gr**] for the sake of determining the irreducible characters of the general linear groups over finite fields. They also afford the graded characters of the DeConcini-Procesi algebras. In this article, we construct the standard decomposition for the Green polynomials. By applying this decomposition we prove the properties for the homogeneous components of the DeConcini-Procesi algebras associated to the hook type partitions.

2. DeConcini-Procesi algebras

Let $P_n = \mathbb{C}[x_1, \dots, x_n]$ denote the polynomial ring with n variables over $\mathbb{C}$. Then S_n acts on P_n from the left as permutations of variables as follows:

$$(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where $\sigma \in S_n$ and $f(x_1, \dots, x_n) \in P_n$.

We introduce the homogeneous S_n-stable ideal of P_n associated to a partition $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ of n. Let $\mu' = (\mu'_1, \mu'_2, \dots)$ be the conjugate of μ. Now we designate by I_μ the ideal generated by the collection of symmetric functions

$$\left\{ e_m(x_{i_1}, \dots, x_{i_{n-k+1}}) \middle| \begin{array}{l} k = 1, \dots, \mu_1, \\ n - k + 1 - (\mu'_k + \mu'_{k+1} + \cdots + \mu'_{\mu_1}) < m \le n - k + 1 \end{array} \right\},$$

where $e_m(x_{i_1}, \dots, x_{i_{n-k+1}})$ denotes the m-th elementary symmetric function in the variables $x_{i_1}, \dots, x_{i_{n-k+1}}$. For example, when $\mu = (2,1) \vdash 3$, then

$$I_{(2,1)} = \left\langle \begin{array}{l} e_3(x_1, x_2, x_3), e_2(x_1, x_2, x_3), e_1(x_1, x_2, x_3), \\ e_2(x_1, x_2), e_2(x_1, x_3), e_2(x_2, x_3) \end{array} \right\rangle$$

The DeConcini-Procesi algebra R_μ associated to μ is defined as a quotient algebra

$$R_\mu := P_n / I_\mu$$

[**DP, GP, T**]. It is apparent from the definition of I_μ that R_μ is a graded S_n-module. We write its homogeneous decomposition as

$$R_\mu = \bigoplus_{d \ge 0} R_\mu^d.$$

Let $\operatorname{char}_q R_\mu$ denote the graded S_n-character of R_μ defined by

$$\operatorname{char}_q R_\mu := \sum_{d \ge 0} q^d \operatorname{char} R_\mu^d ,$$

where $\operatorname{char} R_\mu^d$ is the character of S_n-submodule R_μ^d.

For $k = 0, \dots, l-1$, define

$$R_\mu(k; l) := \bigoplus_{d \equiv k \bmod l} R_\mu^d .$$

What we would like to know first is which l makes all the dimensions of $R_\mu(k;l)$ coincide. In order to answer this question, we might consider the Poincaré polynomial of R_μ, which is obtained by evaluating the graded character at the identity. The graded characters are discussed in detail in the next section. In this section we only state the following lemma that is useful for our purpose.

Lemma 2.1. *Let q be an indeterminate and $f(q) = \sum_{i \ge 0} a_i q^i \in \mathbb{C}[q]$ a polynomial in q. Let $\ell \ge 2$ be an integer and ζ_ℓ the primitive ℓ-th root of unity. Then the following conditions are equivalent:*

(a) *$f(\zeta_\ell^k) = 0$ for each $k = 1, \dots, \ell - 1$,*

(b) *The partial sums $c_k = \sum_{i \equiv k \bmod \ell} a_i$ $(k = 0, 1, \dots, \ell - 1)$ of coefficients of the polynomial $f(q)$ are independent of the choice of k.*

3. Green polynomials

The graded characters of the DeConcini-Procesi algebras are known as the Green polynomials (of type A). For $\rho \vdash n$, let X^μ_ρ be the coefficient polynomials of the Hall-Littlewood symmetric functions $P_\mu(x;t)$ in the power sum product $p_\rho(x)$, i.e.,

$$p_\rho(x) = \sum_{\mu \vdash n} X^\mu_\rho(t) P_\mu(x;t).$$

The Green polynomials $Q^\mu_\rho(q)$ [**Gr, Mc**] are defined by

$$Q^\mu_\rho(q) = q^{n(\mu)} X^\mu_\rho(q^{-1}),$$

where $n(\mu) = \sum_{i \geq 1}(i-1)\mu_i$ for $\mu = (\mu_1, \mu_2, \ldots)$.

The Green polynomials have another expression using the modified Kostka polynomials $\tilde{K}_{\lambda\mu}(q)$. [**Mc**]. If we denote by χ^λ_ρ the value of an irreducible character χ^λ for S_n at the element of cycle-type ρ, then

$$Q^\mu_\rho(q) = \sum_{\lambda \vdash n} \chi^\lambda_\rho \tilde{K}_{\lambda\mu}(q).$$

Since the coefficient of q^d in the polynomial $\tilde{K}_{\lambda\rho}$ is also known as the multiplicity of the irreducible component V^λ in R^d_μ (see e.g., [GP]), we find that the Green polynomial $Q^\mu_\rho(q)$ affords the value of the graded character of R_μ at the element of cycle-type ρ, i.e.,

$$Q^\mu_\rho(q) = \operatorname{char}_q R_\mu(\rho).$$

In addition, applying the combinatorial expression of the Kostka polynomials [**LS**]

$$\tilde{K}_{\lambda\mu}(q) = \sum_{T \in \mathrm{SSTab}_\mu(\lambda)} q^{\mathrm{coch}(T)} \ ,$$

it immediately follows that

$$[R^d_\mu : V^\lambda] = \sharp\{T \in \mathrm{SSTab}_\mu(\lambda) \mid \mathrm{coch}(T) = d\}.$$

Here $\mathrm{SSTab}_\mu(\lambda)$ denotes the set of semistandard Young tableaux of shape λ with weight μ, and $\mathrm{coch}(T)$ the cocharge of a tableau T.

Some explicit forms of $Q^\mu_\rho(q)$ have been known when μ takes some special partitions. Before we expose them, we give some symbols that appear in the explicit expressions. For each partition $\rho = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ of n, we define

$$\begin{aligned} M_\rho &= \max\{m_1, \ldots, m_n\} \ , \\ b_\rho(q) &= \prod_{i \geq 1} (1-q)(1-q^2)\cdots(1-q^{m_i}) \ , \\ e_\rho(q) &= (1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^n)^{m_n} \ . \end{aligned}$$

In the case of $\mu = (1^n)$, that corresponds to the graded character of the coinvariant algebra, $Q^{(1^n)}_\rho(q)$ can be expressed as follows (see e.g., [G]).

Proposition 3.1. *For $\rho \vdash n$*

$$\operatorname{char}_q R_n(q) = Q^{(1^n)}_\rho(q) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{e_\rho(q)}.$$

If $\mu = (2, 1^{n-2}) \vdash n$ (a hook type), then A. Morris gives Q^μ_ρ explicitly [**Mr**].

Proposition 3.2 (Morris). *For $\rho = (1^{r_1}2^{r_2}\cdots n^{r_n})$,*

$$Q_\rho^{(2,1^{n-2})}(q) = \frac{(1-q)\cdots(1-q^{n-2})}{e_\rho(q)}\{(r_1-1)q^n - r_1q^{n-1}+1\}.$$

In both of the cases above, we find that Green polynomial $Q_\rho^\mu(q)$ can be decomposed into the rational factor

$$\frac{(1-q)\cdots(1-q^{M_\mu})}{e_\rho(q)}$$

and the polynomial factor. This fact holds for the Green polynomial in general and we expose it in the following theorem. Note that the theorem plays an important role to prove our main result.

Theorem 3.3. *Let μ and ρ be partitions of n. Then there exists a polynomial $G_\rho^\mu(q) \in \mathbf{Z}[q]$ such that*

$$Q_\rho^\mu(q) = \frac{b_{(1^{M_\mu})}(q)}{e_\rho(q)}G_\rho^\mu(q).$$

The theorem is deduced from an expansion of the product of two Hall-Littlewood functions $P_{\bar\mu}P_{(1^r)}$ by P_λ's, where r is the last part of μ' and $\bar\mu = (\mu'\backslash(r))' \vdash n-r$. Applying Lemma 2.1 to Theorem 3.3, we obtain the answer to the question in the previous section.

Corollary 3.4. *Let μ be a partition of n. For any $l \in \{2,\ldots,M_\mu\}$ and for each $k = 0,1,\ldots,l-1$, ζ_l^k is zero of $Q_{(1^n)}^\mu(q)$. Hence the dimensions of $R_\mu(k;l)$ $(k=0,\ldots,l-1)$ are equal each other.*

When the partition $\mu \vdash n$ is of a hook type, we can obtain an explicit expression of $G_\rho^\mu(q)$, that generalizes Morris' formula in Proposition 3.2.

Proposition 3.5. *Let $\mu = (n-h,1^h) \vdash n$ be a hook type partition. Then, for $\rho = (1^{r_1}2^{r_2}\cdots n^{r_n})$,*

$$Q_\rho^\mu(q) = \frac{(1-q)\cdots(1-q^h)}{e_\rho(q)}G_\rho^\mu(q)\ ,$$

where

$$G_\rho^\mu(q) = (1-q^n) - \sum_{k=1}^{n-h-1}\ \sum_{\tau=(1^{t_1}\cdots k^{t_k})\vdash k}\binom{r_1}{t_1}\cdots\binom{r_k}{t_k}q^{n-k}e_\tau(q).$$

4. DeConcini-Procesi algebras of hook type

In this section we show that, for a hook partition μ, the DeConcini-Proseci algebra R_μ holds the property similar to the coinvariant algebra R_n, i.e., there is a subgroup of S_n and its modules with equal rank such that all $R_\mu(k;l)$ are induced by the modules.

Let $\mu = (n-h,1^h)$ be a hook partition of n. We suppose $n-h>1$ below, since R_μ is the coinvariant algebra if $n-h=1$. In this case $M_\mu = h$ and it follows from Corollary 3.4 that for each $l \in \{(1,)2,\ldots h\}$ the dimensions of $R_\mu(k;l)$ $(k=0,\ldots,l-1)$ coincide. When $\mu=(n-h,1^h)$ and $l\in\{2,\ldots h\}$, we denote by d and r the quotient and the remainder of h divided by l. Set $\bar\mu = (n-h,1^r)$ and $\nu = (1^{dl})$.

Let C_l be a cyclic subgroup of S_n generated by an element

$$a = (1,2,\ldots,l)(l+1,l+2,\ldots,2l)\cdots((d-1)l+1,(d-1)l+2,\ldots,dl)\ ,$$

where $(i,i+1,\ldots,i+l)$ is a cyclic permutation of length l and S_{n-dl} a subgroup of S_n defined by

$$S_{n-dl} := \{\sigma\in S_n | \sigma(i) = i \quad \text{for } i = 1,2,\ldots,dl\}.$$

We should prove the following theorem to obtain the result mentioned above. This is our main result in this article.

Theorem 4.1. *Let $\mu = (n-h, 1^h)$ $(n-h > 1)$ be a hook type partition of n. For an integer $l \in \{2, 3, \ldots, h\}$, let $h = dl + r$ $(0 \geq r \geq l-1)$. We set $\bar{\mu} = (n-h, 1^r)$. Then there is an $S_n \times C_l$-module isomorphism*

$$R_\mu \cong R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} .$$

In the theorem above, the $S_n \times C_l$-module structures are defined as follows. The algebra R_μ is regarded as an $S_n \times C_l$-module. The group C_l acts on R_μ by

$$ax = \zeta_l^d x \quad (x \in R_\mu^d).$$

The induced module

$$R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} = \bigoplus_{\sigma \in S_n / S_{n-dl}} \sigma \otimes R_{\bar{\mu}}$$

is also regarded as an $S_n \times C_l$-module. The action of C_l is defined by

$$a(\sigma \otimes x) = \sigma a^{-1} \otimes ax = \zeta_l^d \sigma a^{-1} \otimes x \quad (x \in R_{\bar{\mu}}^d).$$

Comparing the ζ_l^k-eigenspaces of $a \in C_l$ in both sides, we have the very thing that we would like to prove.

Corollary 4.2. *Let $\mu = (n-h, 1^h)$ $(n-h > 1)$ be a hook type partition of n. If we choose an integer $l \in \{2, 3, \ldots, h\}$, then there is an S_n-module isomorphism*

$$R_\mu(k; l) \cong Z_\mu(k; l) \Big\uparrow_{C_l \times S_{n-dl}}^{S_n}$$

for each $k = 0, 1, \ldots, l-1$. We define here the representation $Z_\mu(k; l)$ of $C_l \times S_{n-dl}$ by

$$Z_\mu(k; l) := \bigoplus_{\lambda \vdash n-dl} \bigoplus_{T \in \mathrm{SSTab}_{\bar{\mu}}(\lambda)} \psi^{(k - \mathrm{coch}(T))} \otimes V^\lambda ,$$

where $\psi^{(s)} : a \mapsto \zeta_l^s$ denotes an irreducible representation of $C_l = \langle a \rangle$, and V^λ a irreducible representation of S_{n-dl} associated to the partition λ of $n - dl$.

5. Outline of the proof of Theorem 4.1

We will show that

$$\mathrm{char}\, R_\mu(w, a^j) = \mathrm{char}\, R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} (w, a^j)$$

for each $(w, a^j) \in S_n \times C_l$.

First we calculate the right-hand side of the above identity. If $\mathrm{char}\, R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} (w, a^j) \neq 0$, then there exists a basis element $\sigma \otimes x$ of $\mathrm{char}\, R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} (w, a^j)$ such that $(w, a^j)(\sigma \otimes x)|_{\sigma \otimes x} \neq 0$. Since $(w, a^j)(\sigma \otimes x) = w\sigma a^{-j} \otimes a^j x = \zeta_l^{dj} w\sigma a^{-j} \otimes x$, it should follow that $w\sigma a^{-j} \equiv \sigma \bmod S_{n-dl}$ if $\mathrm{char}\, R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} (w, a^j) \neq 0$. Thus we have the following lemma:

Lemma 5.1. *If $\mathrm{char}\, R_{\bar{\mu}} \Big\uparrow_{S_{n-dl}}^{S_n} (w, a^j) \neq 0$, then $w \sim a^j v$ for some $v \in S_{n-dl}$*

If $w\sigma a^{-j} \equiv \sigma \bmod S_{n-dl}$, then $w\sigma a^{-j} = \sigma\tau$ for some $\tau \in S_{n-dl}$. Setting

$$\mathcal{S}_\tau^{(j)}(w) := \{\sigma \in S_n / S_{n-dl} | w\sigma a^{-j} = \sigma\tau\}$$
$$\mathcal{S}^{(j)}(w) := \{\sigma \in S_n / S_{n-dl} | w\sigma a^{-j} \equiv \sigma \bmod S_{n-dl}\},$$

we have

$$\begin{aligned}\operatorname{char} R_{\bar\mu}\Big\uparrow_{S_{n-dl}}^{S_n}(w,a^j) &= \sum_{\tau\in S_{n-dl}} \sharp\mathcal{S}^{(j)}_\tau(w)\operatorname{char} R_{\bar\mu}(\tau)\big|_{q=\zeta_l^j}\\ &= \sum_{\tau\in S_{n-dl}} \sharp\mathcal{S}^{(j)}_\tau(w)\operatorname{char} R_{\bar\mu}(v)\big|_{q=\zeta_l^j}\\ &= \sharp\mathcal{S}^{(j)}(w)\operatorname{char} R_{\bar\mu}(v)\big|_{q=\zeta_l^j}.\end{aligned}$$

(Note that $v\sim\tau$ if $w=a^jv$ and $w\sigma a^{-j}=\sigma\tau$.) We can enumerate the permutations belonging to the set $\mathcal{S}^{(j)}(w)$:

Proposition 5.2. *For $w=a^jv\in S_n$,*

$$\sharp\mathcal{S}^{(j)}(w)=p^e e!\binom{z_p+e}{e},$$

where $\lambda(a^j)=(p^e)$, $\lambda(v)=(1^{z_1}2^{z_2}\cdots)$.

Let us summarize the above arguments:

Proposition 5.3. *Suppose $\lambda(a^j)=(p^e)$. Then*

$$\operatorname{char} R_{\bar\mu}\Big\uparrow_{S_{n-dl}}^{S_n}(w,a^j)=\begin{cases} p^e e!\binom{z_p+e}{e}\operatorname{char}_q R_{\bar\mu}(v)\big|_{q=\zeta_l^j}, & \text{if } w\sim a^jv \text{ for some } v\in S_{n-dl}\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, for the left-hand side of the identity that we are now proving, it follows from Theorem 3.3 the following lemma:

Lemma 5.4. *If $Q^\mu_{\lambda(w)}(\zeta_l^j)\neq 0$ for some $w\in S_n$, then $w\sim a^jv$ for some $v\in S_{n-dl}$.*

Applying decomposition of the Green polynomials in Theorem 3.3 to some recursive relations of them, we can obtain

$$\operatorname{char}_q R_\mu(w)\Big|_{q=\zeta_l^j}=Q^\mu_{\lambda(w)}(\zeta_l^j)=p^e e!\binom{z_p+e}{e}Q^{\bar\mu}_{\lambda(v)}(\zeta_l^j).$$

We have thus shown the identity of two characters, thereby completing the proof of our main theorem.

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