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An Arctic Circle Theorem For Groves

T. Kyle Petersen and David Speyer

Abstract. *In earlier work, Jockusch, Propp, and Shor proved a theorem describing the limiting shape of the boundary between the uniformly tiled corners of a random tiling of an Aztec diamond and the more unpredictable 'temperate zone' in the interior of the region. The so-called arctic circle theorem made precise a phenomenon observed in random tilings of large Aztec diamonds.*

Here we examine a related combinatorial model called groves. Created by Carroll and Speyer as combinatorial interpretations for Laurent polynomials given by the cube recurrence, groves have observable frozen regions which we describe precisely via asymptotic analysis of generating functions, in the spirit of Pemantle and Wilson. Our methods also provide another way to prove the arctic circle theorem for Aztec diamonds.

Résumé. *Dans leurs travaux, Jockusch, Propp, et Shor ont prouvé un théorème décrivant la forme limite de la frontière entre les coins uniformement pavés ("gelés") d'un pavage aléatoire d'un diamant aztèque et la zone "temperee" moins prévisible a l'intérieur du diamant. Le théorème du cercle arctique a rendu précis un phénomène observé dans les pavages aléatoires de grands diamants aztèques.*

Nous examinons un modèle combinatoire relie appelé les bosquets. Créé par Carroll et Speyer en tant qu'interprétation combinatoires pour des polynômes de Laurent donnés par la récurrence du cube, les bosquets laissent apparaître des régions gelées que nous décrivons avec précision par l'intermediaire de l'analyse asymptotique de fonctions génératrices, dans l'esprit de Pemantle et de Wilson. Nos méthodes fournissent également une autre manière de prouver le théorème du cercle arctique pour les diamants aztèques.

1. Introduction

Groves came into existence as combinatorial interpretations of rational functions generated by the *cube recurrence*:

$$f_{i,j,k}f_{i-1,j-1,k-1} = f_{i-1,j,k}f_{i,j-1,k-1} + f_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j,k-1}f_{i-1,j-1,k},$$

where some initial functions are specified. Typically, $f_{i,j,k} := x_{i,j,k}$ for some choice of $(i,j,k) \in \mathbb{Z}^3$ called the *initial conditions*. Fomin and Zelevinsky [**FZ**] were able to show that for arbitrary initial conditions the rational functions generated by the cube recurrence were in fact Laurent polynomials in the $x_{i,j,k}$. The introduction of groves by Carroll and Speyer [**CS**] gave a combinatorial proof of the surprising fact that each

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term of these polynomials has coefficient +1. In this paper we will only examine the family of groves on *standard initial conditions* as described in Section 1.1.[1]

Before getting into the details of groves, let us first describe the motivation for this paper: random domino tilings of large Aztec diamonds. An Aztec diamond of order n consists of the union of all unit squares with integer vertices contained in the planar region $\{ (x,y) \mid |x| + |y| \leq n+1 \}$. A *domino tiling* of an Aztec diamond is an arrangement of 2×1 rectangles, or *dominoes*, that cover the diamond without any overlapping. A random domino tiling of a large Aztec diamond consists of two qualitatively different regions.[2] As seen in the random tiling in Figure 1, the dominoes in the corners of the diamond are *frozen* in a brickwork pattern, whereas the dominoes in the interior have a more random, *temperate* behavior. It was shown in [**JPS**] and [**CEP**] that asymptotically, the boundary between the frozen and temperate regions in a random tiling is given by the circle inscribed in the Aztec diamond. Since everything outside the circle is expected to be frozen, it is referred to as the *arctic* circle.

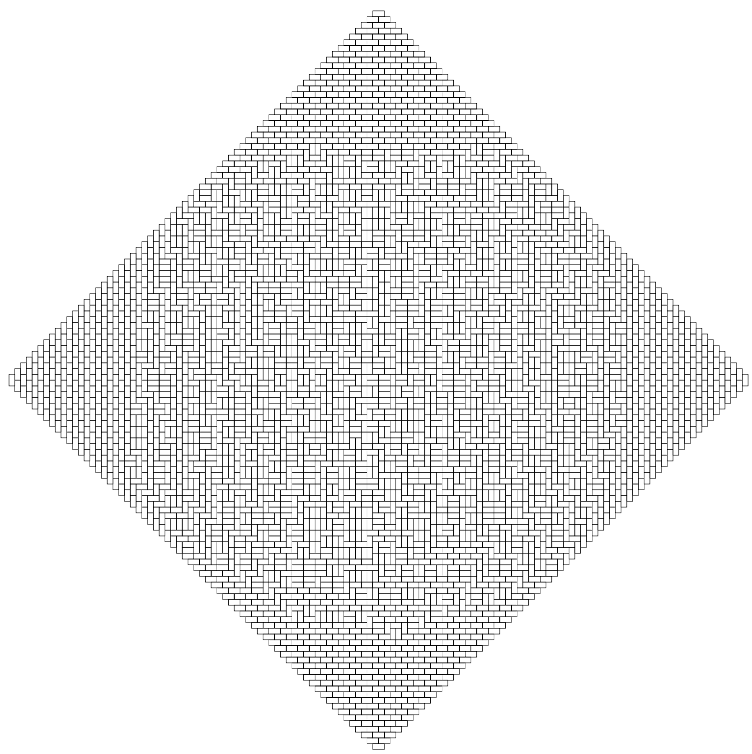

FIGURE 1. A random domino tiling of an Aztec diamond of order 64

In this paper we shall see that groves on standard initial conditions exhibit a very similar behavior. A grove, however, is not a type of tiling. In fact, as the name may suggest, a grove is a collection of trees. From our point of view, groves are forests on a triangular lattice satisfying certain connectivity conditions on the boundary. We will show that outside of the circle inscribed in the triangle, the trees of a large random grove line up uniformly.

[1] Herein we will invoke some of the basic properties of groves without proof. For such arguments, as well as a general treatment of groves and the cube recurrence, the reader is referred to [**CS**].

[2] By random we mean selected from the uniform distribution on all tilings of an Aztec diamond of order n, though other probability distributions may be considered as well. See [**CEP**].

Despite their superficial differences, groves and random domino tilings of Aztec diamonds are linked by more than their asymptotic behavior. In fact it seems that their asymptotic behavior is similar *because* they share a deeper link. The paper of Carroll and Speyer [**CS**] establishes that groves are encoded in terms of a Laurent polynomial given by the cube recurrence. There is a more general form of the cube recurrence:

$$f_{i,j,k}f_{i-1,j-1,k-1} = \alpha f_{i-1,j,k}f_{i,j-1,k-1} + \beta f_{i,j-1,k}f_{i-1,j,k-1} + \gamma f_{i,j,k-1}f_{i-1,j-1,k},$$

where α, β, γ are constants. If $\alpha = \beta = \gamma = 1$ we have the original form of the cube recurrence from whence come groves. If $\alpha = \beta = 1$ and $\gamma = 0$, we have (after re-indexing), the *octahedron recurrence*:

$$g_{i,j,n+1}g_{i,j,n-1} = g_{i-1,j,n}g_{i+1,j,n} + g_{i,j-1,n}g_{i,j+1,n},$$

with which we may encode tilings of Aztec diamonds. In Section 3, we will describe the role that this recurrence plays in the large scale behavior of such tilings.

While the octahedron recurrence is important to us, it has not played a significant role in the study of tilings of Aztec diamonds in the past. Rather, a local move called *domino shuffling* has been used. Domino shuffling was introduced in [**EKLP**] and is generalized in [**P**]. It provides a method for generating tilings of successively larger Aztec diamonds uniformly at random, and has been at least implicit in all probabilistic analysis done to date. Section 1.3 will introduce an analogous local move for groves that we call *grove shuffling*. Like domino shuffling, it will be key to our analysis.

For each of the two models discussed we have a global perspective and a local perspective. Laurent polynomials tell the global story: all groves are encapsulated in $f_{0,0,0}$ (from the cube recurrence), all tilings in $g_{0,0,n}$ (from the octahedron recurrence). A specified shuffling algorithm tells the local story. In this paper we combine these two points of view to build generating functions (for tilings of Aztec diamonds as well as for groves), with which we can study asymptotic behavior.

1.1. Groves on standard initial conditions. The standard initial conditions of order n specify a vertex set $\mathcal{I}(n) = \mathcal{C}(n) \cup \mathcal{B}(n)$ where $\mathcal{C}(n) = \{\, (i,j,k) \in \mathbb{Z}^3 \mid -n-1 \leq i+j+k \leq -n+1, i,j,k \leq 0 \,\}$ and $\mathcal{B}(n) = \{\, (i,j,k) \in \mathbb{Z}^3 \mid i+j+k < -n-1;\ i,j,k \leq 0;$ and $i,j,$ or $k = 0 \,\}$. We draw its projection onto the plane $\mathbb{R}^3/(1,1,1)$ as shown in Figure 2 for the case $n = 2$, and in Figure 4 for the case $n = 5$. One way to generate all groves of order n is to set $f_{i,j,k} := x_{i,j,k}$ for all $(i,j,k) \in I(n)$, and compute $f_{0,0,0}$. Each term in the resulting Laurent polynomial defines a grove as follows. Let $\mathcal{G}(n)$ be the graph on the vertex set $\mathcal{I}(n)$ where vertex (i,j,k) has as its neighbors the vertices $\mathcal{I}(n) \cap \{\, (i \pm 1, j \pm 1, k), (i \pm 1, j, k \pm 1), (i, j \pm 1, k \pm 1) \,\}$. Pictorially, edges of $\mathcal{G}(n)$ connect vertices that lie diagonally across a rhombus.

The terms in $f_{0,0,0}$ are Laurent monomials of the form

$$m(g) = \prod_{(i,j,k) \in \mathcal{I}(n)} x_{i,j,k}^{\deg(i,j,k)-2}.$$

We have the following

Definition 1.1. The *grove* g defined by $m(g)$ is the unique subgraph of $\mathcal{G}(n)$ containing no crossing edges such that vertex (i,j,k) in $\mathcal{I}(n)$ has exactly $\deg(i,j,k)$ incident edges.

The uniqueness of the grove is a consequence of Theorem 3 in [**CS**]. For example, $f_{0,0,0}$ on $\mathcal{I}(2)$ is

$$\frac{x_{-1,-1,0}x_{0,0,-1}}{x_{-1,-1,-1}} + \frac{x_{-1,0,-1}x_{0,-1,0}}{x_{-1,-1,-1}} + \frac{x_{0,-1,-1}x_{-1,0,0}}{x_{-1,-1,-1}},$$

and the corresponding groves are shown in Figure 3.

For a more interesting example, one term of $f_{0,0,0}$ on $\mathcal{I}(5)$ is

$$\frac{x_{-3,0,-2}x_{-2,-1,-1}x_{-1,-3,0}x_{0,-2,-2}}{x_{-3,-1,-2}x_{-2,-3,-1}x_{-1,-2,-2}}.$$

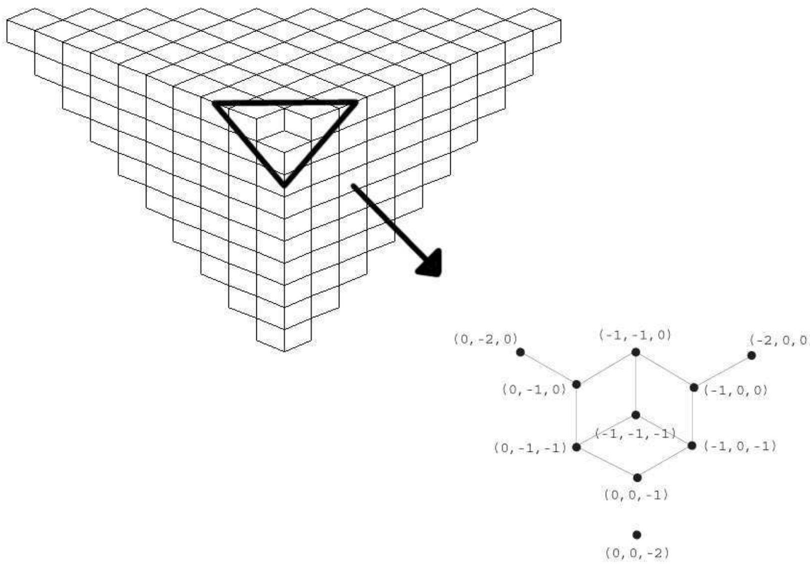

FIGURE 2. Part of the standard initial conditions of order 2

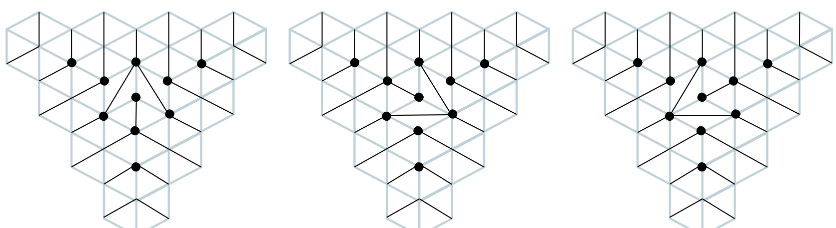

FIGURE 3. The three groves of order 2.

Its corresponding grove, g, is shown in Figure 4. This grove has interesting connectivity properties; in fact these properties are what distinguish groves from arbitrary subgraphs of $\mathcal{G}(n)$. Every vertex on the boundary of $\mathcal{C}(n)$ (where cubes have been pushed down) is connected to another vertex on the boundary of $\mathcal{C}(n)$ if and only if those vertices are equidistant to the nearest corner (i.e. where two coordinates are zero) of the grove. Groves are acyclic — every connected component of a grove is a tree.

Notice that there are two types of edges: *long* edges and *short* edges, depending on whether the long or short diagonal of a rhombus is used. It is shown in [**CS**] that every vertex in $\mathcal{B}(n)$ has degree 2 and only uses its short edges. As a result, there are only finitely many long edges, and these determine the grove. This observation leads to a more convenient way of looking at groves.

1.2. Simplified groves. We begin by constructing a modified form of the cube recurrence. Let $a_{i,j}$, $b_{k,j}$, $c_{i,k}$ be *long edge variables*. The variable $a_{i,j}$ is the label for the edge between vertices $(i, j-1, k+1)$ and $(i-1, j, k+1)$, $b_{k,j}$ is the label for the edge between $(i-1, j, k+1)$ and (i, j, k), and $c_{i,k}$ is the label for the edge between (i, j, k) and $(i, j-1, k+1)$. We write a modified form of the cube recurrence as follows:

$$f_{i,j,k}f_{i-1,j-1,k-1} = b_{i,k}c_{i,j}f_{i-1,j,k}f_{i,j-1,k-1} + c_{i,j}a_{j,k}f_{i,j-1,k}f_{i-1,j,k-1} + a_{j,k}b_{i,k}f_{i,j,k-1}f_{i-1,j-1,k}.$$

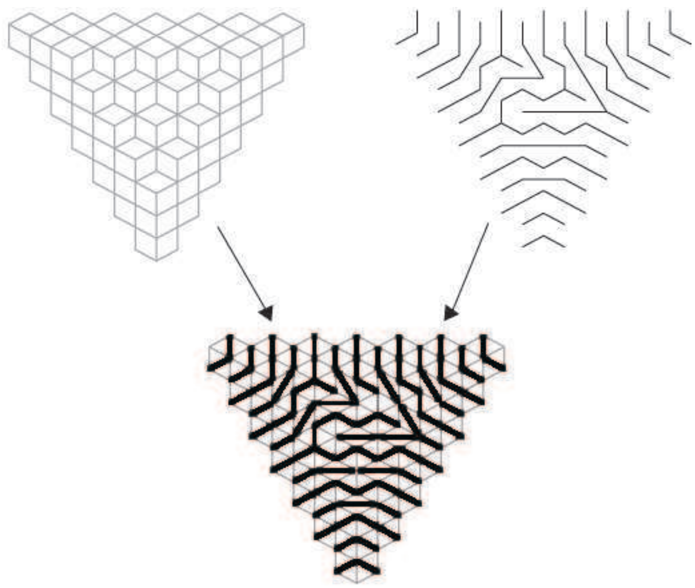

FIGURE 4. A grove g of order 4, superimposed on $\mathcal{I}(4)$

As we said, the long edges determine the grove, so rather than setting $f_{i,j,k} := x_{i,j,k}$ for $(i,j,k) \in \mathcal{I}(n)$, we set $f_{i,j,k} := 1$ for $(i,j,k) \in \mathcal{I}(n)$. Then $f_{0,0,0}$ is simply a polynomial in the edge variables $a_{i,j}, b_{i,j}, c_{i,j}$. Each term describes a unique grove, and we still produce every grove. This form of the cube recurrence is called the *edge variables version*. We can draw a simpler picture of our groves by ignoring all short edges and all of the vertices incident with them. In other words, specify a subset of the standard initial conditions of order n, called the *simplified initial conditions*: $\mathcal{I}'(n) = \{ (i,j,k) \in \mathbf{Z}^3 \mid i+j+k = -n, i,j,k \leq 0 \} \subset \mathcal{I}(n)$. We now represent our groves as graphs on this vertex set – a triangular lattice shown in Figure 5. Also in Figure 5 we see the same grove as in Figure 4, but with only the long edges included. In terms of edge variables, this grove is given by

$$a_{0,0}a_{0,1}a_{0,2}a_{1,0}a_{1,1}a_{2,1}b_{0,0}b_{0,1}c_{0,0}c_{0,1}c_{1,0}c_{2,0}.$$

Another modification of the cube recurrence that we shall like to use is the *edge-and-face variables version*. In the original version of the cube recurrence, the variables $x_{i,j,k}$ such that $i+j+k = -n+1$ were vertex variables. In the simplified picture, we call them the *face variables* of order n, for reasons which will become clear. Rather than setting $f_{i,j,k} := 1$ for all (i,j,k) in $\mathcal{I}(n)$, we give the face variables their formal weights. That is, we set $f_{i,j,k} := 1$ for $(i,j,k) \in \{ (i,j,k) \in \mathbf{Z}^3 \mid -n-1 \leq i+j+k \leq n, i,j,k \leq 0 \}$ and $f_{i,j,k} := x_{i,j,k}$ for $(i,j,k) \in \{ (i,j,k) \in \mathbf{Z}^3 \mid i+j+k = -n+1, i,j,k \leq 0 \}$. Generating $f_{0,0,0}$ using these initial conditions, we get a Laurent polynomial in the edge and face variables. The vertices of the simplified initial conditions can be seen as forming $n(n+1)/2$ downward-pointing equilateral triangles, each with top-left vertex $(i,j-1,k+1)$, top-right vertex $(i-1,j,k+1)$, and bottom vertex (i,j,k). The face variables then correspond to each of these downward-pointing triangles. The triangle with (i,j,k) as its bottom vertex has face variable $x_{i,j,k+1}$. The exponent of the face variable is $-1, 0$, or 1, corresponding to whether the downward-pointing triangle has, respectively, two, one, or zero edges present. Although the face variables don't tell us anything new about a particular grove, they will be useful later in deriving probabilities of edges being present in random groves.

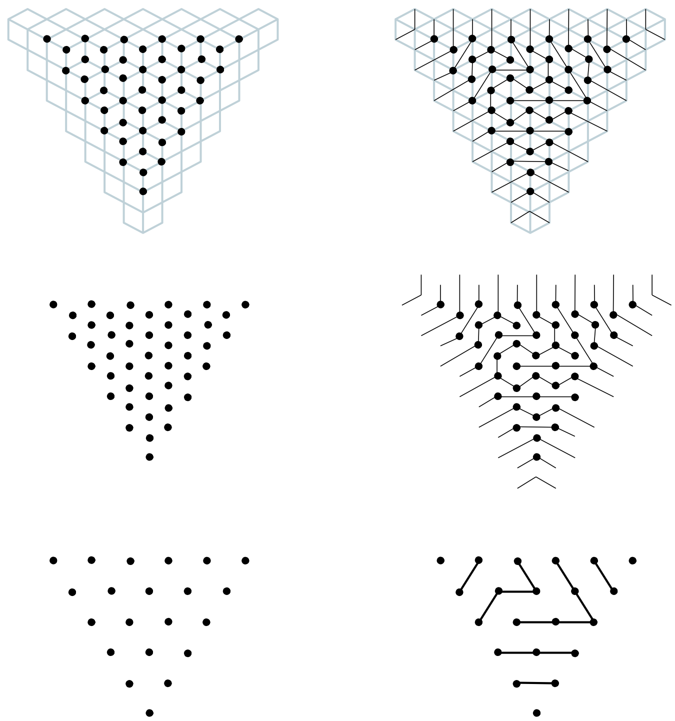

FIGURE 5. On the left: $\mathcal{I}'(4)$ drawn from $\mathcal{I}(4)$. On the right: a simplified grove drawn from a standard grove.

1.3. Grove shuffling. We have given one definition for what groves are, and how they may be generated. The methods and notation introduced in the previous section will be very helpful for later proofs. However, there is another tool we will like to use; an algorithm called *grove shuffling* (or *cube-popping* – see [**CS**]). Grove shuffling not only gives a purely combinatorial definition of groves, but also a method for generating groves of order n uniformly at random. Its inspiration comes from *domino shuffling*, due to Elkies, Kuperberg, Larsen, and Propp [**EKLP**]. The use to which we put grove shuffling is directly motivated by Jim Propp and his paper [**P**]. For proof that grove shuffling does indeed give rise to the same objects as the terms of the Laurent polynomials given by the cube recurrence, see Carroll and Speyer [**CS**]. Here we will only include a description of the algorithm.

Grove shuffling can be thought of as a local move on the downward-pointing triangles of a simplified grove according to whether a triangle has zero, one, or two edges present. See Figure 6. Let x be a generic downward-pointing triangle with possible edges a, b, c as shown, and let x' be an upward-pointing triangle, concentric with x, with possible edges a', b', c' as shown. There are three configurations of x with two edges: ab, ac, bc. Grove shuffling takes each of these triangles and replaces them with an upward-pointing triangle x' having none of its possible edges present. There are three configurations of x with exactly one edge: a, b, c. Each of these is replaced by the upward-pointing triangle x' with only the parallel edge: a', b', c', respectively present. Lastly, there is one configuration of x with none of its possible edges present. This triangle is replaced with the upward-pointing triangle x' containing any two of its three possible edges: $a'b', a'c', b'c'$, chosen randomly with probability 1/3. After we have turned every downward-pointing triangle

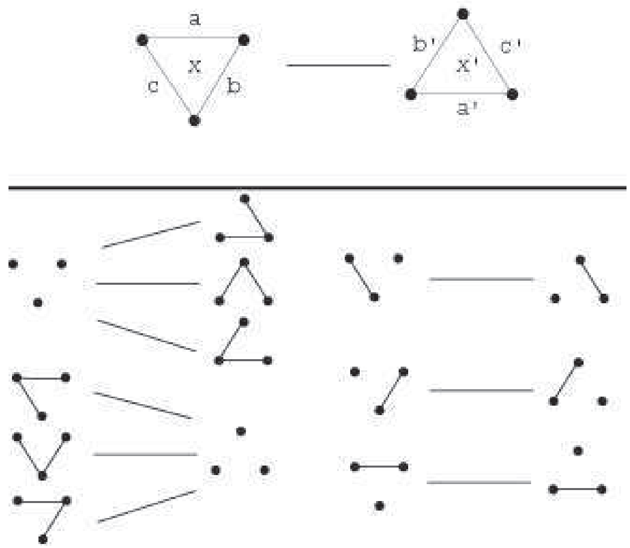

FIGURE 6. Grove shuffling.

into an upward-pointing triangle, we add three new vertices to the corners of the grove so that we may shuffle again.[3]

There is a unique grove of order 1. It has one downward-pointing triangle with zero edges. We now give a purely combinatorial description of simplified groves on standard initial conditions of order n: they are all the possible results of $n-1$ iterations of grove shuffling, beginning with the grove of order 1. It is not hard to show that there are $3^{\lfloor n^2/4 \rfloor}$ groves of order n. We can now make the following claim about grove shuffling.

Theorem 1.2. *Beginning with the unique grove of order one, any grove of order n will be generated after $n-1$ iterations of grove shuffling with probability $1/3^{\lfloor n^2/4 \rfloor}$. In other words, grove shuffling can be used to generate groves uniformly at random.*

The proof follows from some basic observations about grove shuffling.

1.4. Frozen regions. We now describe the phenomenon that we analyze in Section 2. First we observe that edges are indexed relative to the corners perpendicular to them, so in fact the edges a and a' in the previous example have the same name: $a = a' = a_{i,j}$. Horizontal edges are indexed relative to the bottom corner, and the diagonal edges are indexed relative to the top-right and top-left corners. In this way we can think of grove-shuffling as more akin to domino shuffling [**P**]. Rather than replacing edges with parallel edges, we "slide" edges toward the corners along perpendicular lines. When a downward-pointing triangle has two edges, we remove both of those edges because they "annihilate" each other. When a downward-pointing triangle has no edges, we create two new ones randomly.

With this viewpoint, we define an edge to be *frozen* if it cannot be annihilated under any further iterations of grove shuffling. Clearly the bottom corner edge, $a_{0,0}$, is frozen when present. Then the edge $a_{i,j}$ is frozen exactly when the edges $a_{i',j'}$ are frozen, $i \leq i' \leq 0$, $j \leq j' \leq 0$. Diagonal edges behave similarly. In Figure 7 all the highlighted edges are frozen.

[3]To see grove shuffling in action, visit
http://ups.physics.wisc.edu/~hal/SSL/groveshuffler/

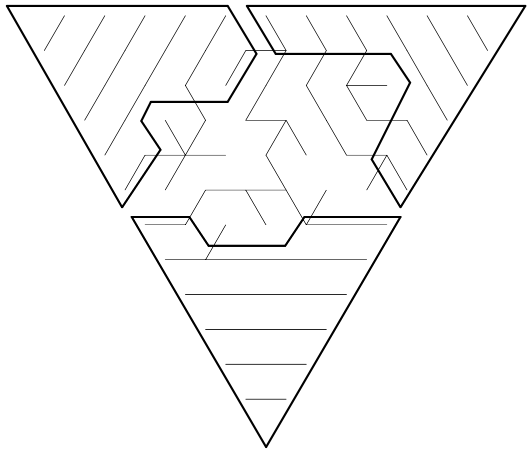

Figure 7. Frozen regions of a random grove of order 12

We conclude this section by examining a picture of a large random grove generated by grove shuffling. In Figure 8, we see that outside of a certain region, all of the edges are parallel. Moreover, the boundary between the less uniform interior and the frozen regions in the corners seems to approximate a circle. Proving that this boundary approaches a circle in the limit is the main goal of this paper.

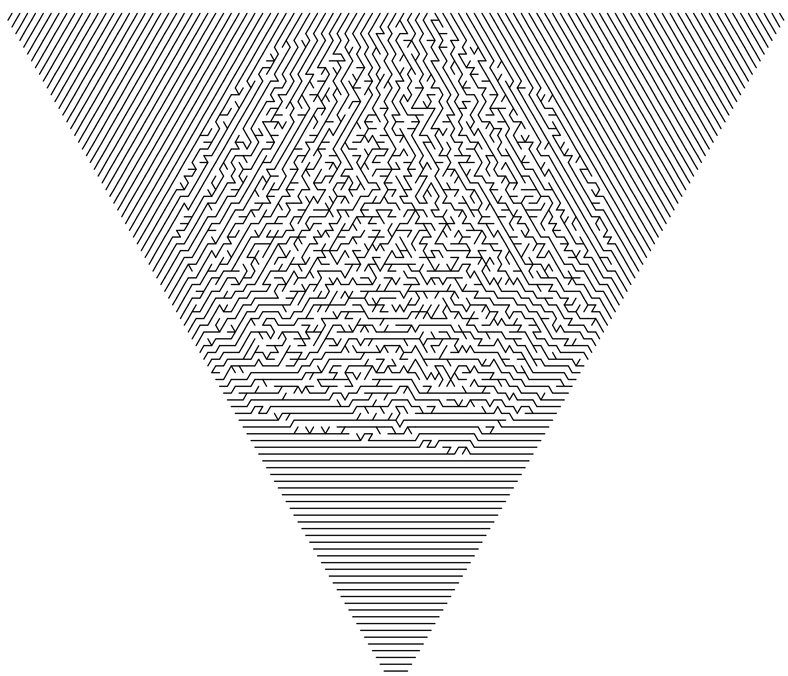

Figure 8. A grove on standard initial conditions of order 100

2. The arctic circle theorem

For any n, we can scale the initial conditions so that they resemble an equilateral triangle with sides of length $\sqrt{2}$. We will show that outside of the circle inscribed in this triangle, there is homogeneity of the edges in an appropriately scaled random grove of order n, with probability approaching 1 as $n \to \infty$. Specifically, we will examine the limiting probability of finding a particular type of edge in a given location outside of the inscribed circle.

2.1. Edge probabilities. Let $p_n(i,j) = p(i,j,k)$, $k = -n-i-j-1$, be the probability that $a_n(i,j)$, the horizontal edge on triangle $x_{i,j,k+1}$, is present in a random grove of order n. Similarly define probabilities $q_n(k,i), r_n(k,j)$ for the diagonal edges of the same triangle. Define $E_n(i,j) = E(i,j,k+1) = 1 - p_n(i,j) - q_n(k,i) - r_n(k,j)$. The numbers $E_n(i,j)$ are analogous to the *creation rates* discussed in [**JPS**], [**CEP**], and [**P**]. We will also refer to them as creation rates. Interestingly, we can also realize the number $E_n(i,j)$ as the expected value of the exponent of the face variable $x_{i,j,k+1}$. We prove the following formula for finding the edge probability $p_n(i,j)$ in terms of creation rates.

Theorem 2.1. *The horizontal edge probabilities are given recursively by* $p_n(i,j) = p_{n-1}(i,j) + \frac{2}{3}E_{n-1}(i,j)$. *Thus,* $p_n(i,j) = \dfrac{2}{3}\sum_{l=1}^{n-1} E_l(i,j)$.

The proof relies only on observations made directly from grove shuffling. We also point out the similarity between this statement and equation 1.5 of [**CEP**].

2.2. A generating function. We now know that to compute the probability of a particular edge being present in a random grove, it will be enough to compute the creation rates $E_l(i,j)$. In this section we derive a generating function for computing these numbers as well as the related generating function for the horizontal edge probabilities.

Let $F(x,y,z) = \sum_{i,j,k\geq 0} E(-i,-j,-k)x^i y^j z^k$ be the generating function for the creation rates. First consider the uniformly weighted version of the cube recurrence:

$$f_{i,j,k}f_{i-1,j-1,k-1} = \frac{1}{3}\left(f_{i-1,j,k}f_{i,j-1,k-1} + f_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j,k-1}f_{i-1,j-1,k}\right).$$

Using this recurrence to calculate $f_{0,0,0}$ we will get each monomial weighted uniformly, so that if we set all the initial conditions equal to 1, $f_{0,0,0} = 1$. If we want the expectation of the exponent of the face variable $x = x_{i_0,j_0,k_0}$, we need only calculate the derivative of $f_{0,0,0}$ with respect to this variable, then set all variables equal to one. In other words,

$$E(i_0,j_0,k_0) = \frac{\partial}{\partial x}\Big(f_{0,0,0}\Big)\Big|_{x_{i,j,k}=1}$$

Furthermore, we can calculate the intermediate creation rates for $(i',j',k') \in \mathcal{I}(n')$ with $n' < n$ by

$$E(i',j',k') = \frac{\partial}{\partial x}\Big(f_{i',j',k'}\Big)\Big|_{x_{i,j,k}=1}$$

(the proof only requires a re-labeling of vertices). With this in mind, let us differentiate the weighted cube recurrence with respect to x:

$$
\begin{aligned}
f'_{i,j,k}f_{i-1,j-1,k-1} + f_{i,j,k}f'_{i-1,j-1,k-1} \\
&= \frac{1}{3}\left(f'_{i-1,j,k}f_{i,j-1,k-1} + f_{i-1,j,k}f'_{i,j-1,k-1}\right) + \\
&\quad \frac{1}{3}\left(f'_{i,j-1,k}f_{i-1,j,k-1} + f_{i,j-1,k}f'_{i-1,j,k-1}\right) + \\
&\quad \frac{1}{3}\left(f'_{i,j,k-1}f_{i-1,j-1,k} + f_{i,j,k-1}f'_{i-1,j-1,k}\right).
\end{aligned}
$$

Now by setting $x_{i,j,k} = 1$ for all (i, j, k), we get a linear recurrence for the expectations in question:

$$
\begin{aligned}
E(i,j,k) + E(i-1,j-1,k-1) &= \frac{1}{3}\left(E(i-1,j,k) + E(i,j-1,k-1)\right) + \\
&\quad \frac{1}{3}\left(E(i,j-1,k) + E(i-1,j,k-1)\right) + \\
&\quad \frac{1}{3}\left(E(i,j,k-1) + E(i-1,j-1,k)\right).
\end{aligned}
$$

We can form the rational generating function in the variables x, y, z:

$$
\begin{aligned}
F(x,y,z) &= \sum_{i,j,k\geq 0} E(-i,-j,-k)x^i y^j z^k \\
&= \frac{1}{1 + xyz - \frac{1}{3}(x+y+z+xy+xz+yz)}.
\end{aligned}
$$

Now using the fact that $p(i, j, k) = p(i, j, k + 1) + (2/3)E(i, j, k)$, we can derive the formula for the probability generating function:

$$
\begin{aligned}
G(x,y,z) &= \sum_{i,j,k\geq 0} p(-i,-j,-k)x^i y^j z^k \\
&= \frac{2F(x,y,z)}{3(1-z)}.
\end{aligned}
$$

2.3. Asymptotic analysis. With our generating function in hand, we can prove our main theorem. First let us embed a triangle in three-space by $T := \{\,(x, y, z) \in \mathbf{R}^3 \mid x, y, z \leq 0, x + y + z = -1\,\}$. This is the triangle that we will scale $\mathcal{I}(n)$ to fit. A point $(x, y, z) \in T$ is outside of the inscribed circle (what will show is the arctic circle) if and only if the angle between the vector (x, y, z) and vector $(-1, -1, -1)$ is greater than $\cos^{-1}(\sqrt{2/3})$.

Notice that for any point (x, y, z) outside of the inscribed circle, we have either $x \leq y + z$, $y \leq x + z$, or $z \leq x + y$, depending on the region in which (x, y, z) lies. We call the coordinates on the right hand side *small* coordinates.

Theorem 2.2 (Weak Arctic Circle)**.** *Let (x_0, y_0, z_0) be a point in T outside of the inscribed circle for which z_0 is a small coordinate. Let (i_n, j_n, k_n), $i_n + j_n + k_n = -n - 1$, be a sequence of nonpositive integer triples such that*

$$\lim_{n\tilde{\Omega}\infty} \frac{1}{n+1}(i_n, j_n, k_n) = (x_0, y_0, z_0).$$

Then $\lim_{n\tilde{\Omega}\infty} p(i_n, j_n, k_n) = 0$.

In other words, the theorem states that in the upper two regions of T outside of the arctic circle, the probability of finding a horizontal edge goes to zero as the order of a (scaled) random grove goes to infinity. By symmetry, there can be no diagonal edges in the lower region, and in order to satisfy the connectivity properties of groves, all the edges in the lower region must be horizontal. The following lemma is the heart of the proof.

Lemma 2.3. *Fix a point (x_0, y_0, z_0) in T outside of the inscribed circle. Then there are real constants A, B, C such that*

$$p(-i,-j,-k) = O(e^{-(Ai+Bj+Ck)})$$

for all $i, j, k \geq 0$ and $Ax_0 + By_0 + Cz_0 < 0$.

The proof of the lemma is the most subtle part of the argument. It relies on the Cauchy integral formula and an examination of the singular variety of the generating function. Asymptotics of multivariate generating functions is described in general in the sequence of papers [**PW1**], [**PW2**], [**PW3**], by Robin Pemantle and Mark Wilson. Perhaps their techniques will lead to a stronger version of Theorem 2.2. In particular, we hope for a theorem that describes the statistics throughout the grove, similar to Theorem 1 of [**CEP**].

3. Domino tilings of Aztec diamonds

We now draw parallels between the examination of the behavior of large groves on standard initial conditions, and the behavior of tilings of large Aztec diamonds. This approach yields no new results for Aztec diamonds, but presents an alternative approach to their study. In this section we derive a generating function for the probabilities $p_n(i,j)$ that position (i,j) in a tiling of an Aztec diamond of order n is covered by a particular type of horizontal domino. The asymptotics for the function we will derive are discussed as an example in [**PW1**]. The first derivation of the function is due to Jim Propp and Dan Ionescu, though their (different) derivation has never been published. Some recursive formulas for $p_n(i,j)$ are given in [**P**], and are the inspiration for our derivation of the edge probabilities for groves. We list the analogous results.

Theorem 3.1 ([**P**]). *The horizontal edge probabilities are given recursively by $p_n(i,j) = p_{n-1}(i,j) + \frac{1}{2}E_{n-1}(i,j)$. Thus, $p_n(i,j) = \frac{1}{2}\sum_{l=1}^{n-1} E_l(i,j)$.*

The theorem follows more or less directly from the definition of domino shuffling, where $E_n(i,j)$ is the *net creation rate* (see [**EKLP**], [**P**]).

By differentiating the uniformly weighted version of the octahedron recurrence

$$g_{i,j,n+1}g_{i,j,n-1} = \frac{1}{2}\left(g_{i-1,j,n}g_{i+1,j,n} + g_{i,j-1,n}g_{i,j+1,n}\right),$$

and because

$$E_n(i_0,j_0) = \frac{\partial}{\partial x}\Big(g_{0,0,n}\Big)\Big|_{x_{i,j}=1}$$

we obtain

$$\begin{aligned} E_{n+1}(i,j) + E_{n-1}(i,j) = &\frac{1}{2}\left(E_n(i-1,j) + E_n(i+1,j)\right) + \\ &\frac{1}{2}\left(E_n(i,j-1) + E_n(i,j+1)\right). \end{aligned}$$

From this recurrence and Theorem 4 we get the generating function:

$$\begin{aligned} G(x,y,z) = &\sum_{n\geq 0}\sum_{|i|+|j|\leq n} p_n(i,j)x^i y^j z^n \\ = &\frac{z/2}{(1-yz)(1+z^2-\frac{z}{2}(x+x^{-1}+y+y^{-1}))}. \end{aligned}$$

This is the form of the generating function used as an example in [**PW1**]. A weak arctic circle theorem like ours for groves follows directly from that example. Probabilities throughout the diamond could be extracted from this function in principle, though the analysis is more difficult.

4. Further speculation on statistics of groves

As mentioned, we hope to apply the methods of Pemantle and Wilson to determine asymptotic probabilities throughout a random grove. Based on computer experiments and the similarity of groves and Aztec diamond tilings seen so far, we believe a formula for such probabilities exists.

Another future aim is to apply the methods of growth models and statistical mechanics to groves, in the style of Johansson [**J1**], [**J2**]. One clever way for determining the boundary of the frozen region for Aztec diamond tilings is to look at a frozen corner as a randomly growing Young diagram. See [**JPS**] for the first description of this interpretation. A nearly identical projection of the frozen region of a grove yields some sort of randomly growing Young diagram, but it seems to follow more intricate rules of growth than those of Aztec diamond tilings.

In [**CEP**], the authors considered non-uniform distributions on the set of all tilings of the Aztec diamond. In the shuffling algorithm, rather than having horizontal or vertical tiles chosen with equal probability, the choice is biased towards one type of tile or the other. In this situation, there still appear frozen regions and a temperate zone, but the boundary is no longer a circle, but an ellipse. By analogy, we have also considered biased groves. Rather than making the random choice in grove shuffling be uniform, we make one choice with probability α, another with probability β and the third with probability $\gamma = 1 - \alpha - \beta$. This bias emerges in the generating function for creation rates as:

$$F(x, y, z) = \frac{1}{1 + xyz - \alpha(x + yz) - \beta(y + xz) - \gamma(z + xy)}.$$

The boundary from temperate zone to frozen regions generalizes from a circle to an ellipse just as in the Aztec diamond case, here given by the intersection of the plane $x + y + z = -1$ with the surface

$$rs + rt + st = \frac{r^2 + s^2 + t^2}{2},$$

where $r = (1 - \alpha)x$, $s = (1 - \beta)y$, and $t = (\alpha + \beta)z$.

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Department of Mathematics, Brandeis University, Waltham, MA, USA, 02454
E-mail address: tkpeters@brandeis.edu
URL: http://people.brandeis.edu/~tkpeters

Department of Mathematics, University of California Berkeley, Berkeley, CA, USA 94720
E-mail address: speyer@math.berkeley.edu