Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Cambrian Lattices

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Abstract. *For an arbitrary finite Coxeter group W, we define the family of Cambrian lattices for W as quotients of the weak order on W with respect to certain lattice congruences. We associate to each Cambrian lattice a complete fan, which we conjecture is the normal fan of a polytope combinatorially isomorphic to the generalized associahedron for W. In types A and B, we obtain, by means of a fiber-polytope construction, combinatorial realizations of the Cambrian lattices in terms of triangulations and in terms of permutations. Using this combinatorial information, we prove that in types A and B the Cambrian fans are combinatorially isomorphic to the normal fans of the generalized associahedra, and that one of the Cambrian fans is linearly isomorphic to Fomin and Zelevinsky's construction of the normal fan as a "cluster fan." Our construction does not require a crystallographic Coxeter group and therefore suggests a definition, at least on the level of cellular spheres, of a generalized associahedron for any finite Coxeter group. The Tamari lattice is one of the Cambrian lattices of type A, and two "Tamari" lattices in type B are identified, and characterized in terms of signed pattern avoidance. We also show that intervals in Cambrian lattices are either contractible or homotopy equivalent to spheres.*

Résumé. *Pour un groupe fini arbitraire de Coxeter W, nous définissons la famille des treillis cambriens pour W comme des quotients de l'ordre faible sur W par certaines congruences de treillis. Nous associons à chaque treillis cambrien un éventail complet et nous conjecturons que cet éventail est l'éventail normal d'un polytope isomorphe, au sens combinatoire, à un associèdre généralisé. Dans le cas des types A et B, nous obtenons, par une construction de fibre-polytope, des réalisations combinatoires des treillis cambriens en termes de triangulations et en termes de permutations. En utilisant cette information combinatoire, nous montrons que, dans le cas des types A et B, les éventails cambriens sont isomorphes, au sens combinatoire, aux éventails normaux des associaèdre généralisés, et qu'un des éventails cambriens est linéairement isomorphe à l'éventail normal construit par Fomin et Zelevinsky sous forme de l'éventail des amas. Notre construction n'exige pas que le groupe de Coxeter soit cristallographique et suggère une définition, du moins au niveau des sphères cellulaires, d'un associaèdre généralisé pour tout groupe fini de Coxeter. Le treillis de Tamari est un des treillis cambriens du type A, et deux "treillis de Tamari" dans le type B sont identifiés, et caractérisés en termes des permutations signées à motifs exclus. Nous prouvons également que les intervalles dans les treillis cambriens sont soit contractibles, soit équivalent aux sphères, par homotopie.*

2000 *Mathematics Subject Classification.* Primary 20F55, 06B10; Secondary 52C07.
The author was partially supported by NSF grant DMS-0202430.

1. Overview

The Catalan numbers $C_n = (1/(n+1))\binom{2n}{n}$ count a variety of combinatorial objects [**18**, Excercise 6.19] including triangulations of a convex $(n+2)$-gon. A *finite Coxeter group* W is a finite group that can be presented as a group generated by Euclidean reflections, and a *root system* associated to W is a collection of *roots*, that is, normal vectors to the reflecting hyperplanes satisfying certain conditions. The W-permutohedron is the zonotope which is the Minkowsi sum of the roots. There is a W-Catalan number associated to any W, and the ordinary Catalan numbers are associated to irreducible Coxeter groups of type A (the *symmetric groups*). The *associahedron* or *Stasheff polytope* is a polytope whose vertices are counted by the Catalan numbers. Chapoton, Fomin and Zelevinsky [**6, 7**] have recently generalized the associahedron to all finite Coxeter groups which are *crystallographic*. Earlier, the type-B associahedron, or cyclohedron, was defined by Bott and Taubes [**5**] and Simion [**16**].

A related Catalan structure is the Tamari lattice, whose Hasse diagram is the 1-skeleton of the associahedron. Simion [**16**, §5] asked if the vertices of the type-B associahedron could be partially ordered so as to obtain a lattice whose Hasse diagram is the 1-skeleton of the type-B associahedron. Reiner [**15**, Remark 5.4] used an equivariant fiber polytope construction to identify a family of maps from the type-B permutohedron to the type-B associahedron, in analogy to well-known maps in type A. He asked whether one of these maps can be used to define a partial order on the vertices of the type-B associahedron with similar properties to the Tamari lattice, such that the map from the type-B permutohedron to the type-B associahedron shared the pleasant properties of the corresponding map in type A (see [**4**, §9]).

The (right) weak order is a partial order on a Coxeter group W, and is a lattice when W is finite [**3**]. On the symmetric group of permutations, one moves up in the weak order by switching two adjacent entries so as to put them out of order. The starting point of the present research is the observation that the Tamari lattice is a lattice-homomorphic image of the weak order on the symmetric group. This fact has, to our knowledge, never appeared in the literature, although essentially all the ingredients of a proof were assembled by Björner and Wachs in [**4**]. This lattice-theoretic point of view suggests a search among Reiner's maps to determine which induces a lattice homomorphism on the weak order. Surprisingly, for each of these maps, one can choose a vertex of the type-B permutohedron to label as the identity element so that the map induces a lattice homomorphism. Thus each of Reiner's maps defines a lattice structure on the type-B associahedron. Furthermore, the analogous family of maps in type A yields a family of lattices on vertices of the type-A associahedron. A close look at the lattice homomorphisms in types A and B leads to a type-free generalization of these families of lattices which we call *Cambrian lattices*. The name "Cambrian" can be justified by a geological analogy: The Cambrian layer of rocks marks a dramatic increase in the diversity of the fossil record and thus the sudden profusion of Catalan-related lattices arising from the single (Pre-Cambrian) example of the Tamari lattice might fittingly be called *Cambrian*.

A *congruence* on a lattice L is an equivalence relation on L which respects the operations of meet and join in the same way that, for example, a congruence on the integers respects addition and multiplication. The congruences on a finite lattice L are in particular partitions, so we can partially order the set of congruences of L by refinement. This partial order is known to be a distributive lattice [**8**]. In particular, it is a lattice, so one can specify a set of equivalences, and ask for the smallest congruence containing those equivalences.

A finite Coxeter group has a diagram G, a graph whose vertices are a certain set of generating reflections, with edges labeled by pairwise orders $m(s,t)$ of the generators. The pairwise order $m(s,t)$ is the smallest integer such that $(st)^{m(s,t)} = 1$. If $m(s,t)$ is 2 then there is no edge in G connecting s and t. An *orientation* $\vec{G}$ of G is a directed graph with the same vertex set as G, with one directed edge for each edge of G. Thus if G has e edges, there are 2^e orientations of G. For each orientation $\vec{G}$ of G, there is a Cambrian lattice, defined as follows. For a directed edge $s\tilde{\Omega}t$ in $\vec{G}$, require that t be equivalent to the element of W represented by the word $tsts\cdots$ with $m(s,t)-1$ letters. The *Cambrian congruence* associated to $\vec{G}$ is the smallest congruence of the (right) weak order on W satisfying this requirement for each directed edge in $\vec{G}$. The Cambrian lattice

$\mathcal{C}(\vec{G})$ is defined to be the (right) weak order on W modulo the Cambrian congruence associated to $\vec{G}$. Two Cambrian lattices are isomorphic (respectively anti-isomorphic) exactly when the associated diagrams are isomorphic (respectively anti-isomorphic), taking edge labelings into account.

For any finite Coxeter group W, let $\mathcal{F}$ be the fan defined by the reflecting hyperplanes of W. One can identify the maximal cones of $\mathcal{F}$ with the elements of W. In [**12**] a fan $\mathcal{F}_\Theta$ is defined for any lattice congruence Θ of the weak order on W, such that the maximal cones of $\mathcal{F}_\Theta$ are the unions over congruence classes of the maximal cones of $\mathcal{F}$. The lattice quotient W/Θ is a lattice whose elements are the maximal cones of $\mathcal{F}_\Theta$. Let $\mathcal{F}(\vec{G})$ be the *Cambrian fan* constructed in this way from the Cambrian congruence associated to $\vec{G}$.

Conjecture 1.1. *For any finite Coxeter group W and any orientation $\vec{G}$ of the associated Coxeter diagram, the fan $\mathcal{F}(\vec{G})$ associated to the Cambrian lattice $\mathcal{C}(\vec{G})$ is the normal fan of a convex polytope which is combinatorially isomorphic to the generalized associahedron for W.*

Each statement in the following conjecture is would be implied by Conjecture 1.1, and proofs of any of these weaker conjectures would be interesting. Statements b.–e. are weakenings of a.

Conjecture 1.2. *For any Coxeter group W with digram G:*

a. *Given any orientation $\vec{G}$ of G, the fan $\mathcal{F}(\vec{G})$ is combinatorially isomorphic to the normal fan of the generalized associahedron for W.*
b. *Given any orientation $\vec{G}$ of G, the fan $\mathcal{F}(\vec{G})$ is combinatorially isomorphic to the normal fan of some polytope.*
c. *All Cambrian fans arising from different orientations of G are combinatorially isomorphic.*
d. *All Cambrian fans arising from different orientations of G have the same number of maximal cones.*
e. *$\mathcal{F}(\vec{G})$ is simplicial for any orientation $\vec{G}$ of G.*

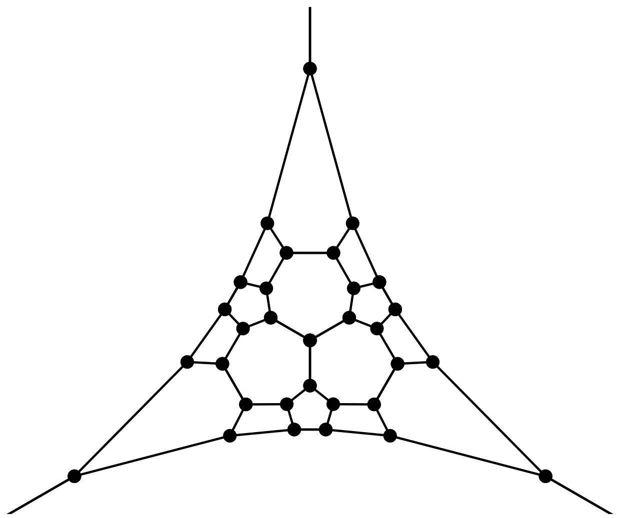
The fan $\mathcal{F}_\Theta$ is PL for any Θ, so that in particular $\mathcal{F}(\vec{G})$ has a dual cellular sphere $\Gamma(\vec{G})$. If Conjecture 1.1 holds, then in particular, $\Gamma(\vec{G})$ is a polytope, and the Cambrian fans offer an alternate definition of the generalized associahedra. In the absence of a proof of Conjecture 1.1, we will nonetheless refer to the dual spheres $\Gamma(\vec{G})$ as generalized associahedra. This construction lifts the restriction to crystallographic Coxeter groups imposed by the definition in [**7**], giving the first definition of associahedra of types H and I. The associahedra for H_3 and $I_2(m)$ constructed from Cambrian lattices have the numbers of faces of each dimension one would expect from generalized associahedra, and the facets of the H_3 associahedron are the correct generalized associahedra of lower dimension. The $I_2(m)$-associahedron is an $(m+2)$-gon and the 1-skeleton of the H_3-associahedron is pictured in Figure 1.

The Cambrian lattices and fans have the following properties which follow from the results of [**12**]. First, $\mathcal{C}(\vec{G})$ is a partial order induced on the maximal cones of $\mathcal{F}(\vec{G})$ by a linear functional, and the Hasse diagram of $\mathcal{C}(\vec{G})$ is isomorphic to the 1-skeleton of the dual sphere $\Gamma(\vec{G})$. The set of cones containing a given face F of $\mathcal{F}(\vec{G})$ is an interval in $\mathcal{C}(\vec{G})$ called a *facial interval*. Non-facial intervals in $\mathcal{C}(\vec{G})$ are contractible and facial intervals are homotopy-equivalent to spheres with the dimension of the sphere depending on the dimension of the corresponding face of $\mathcal{F}(\vec{G})$. If $\mathcal{F}(\vec{G})$ is indeed simplicial, the corresponding simplicial sphere is flag and any linear extension of $\mathcal{C}(\vec{G})$ is a shelling of the corresponding simplicial sphere. If $\mathcal{F}(\vec{G})$ is indeed polytopal, then since it refines the normal fan of $\mathcal{F}$, the associated polytope in a Minkowski summand of the W-permutohedron.

Because disconnected Coxeter diagrams lead to direct product decompositions of all of the relevant objects, it is enough to prove Conjecture 1.1 in the case of connected Coxeter diagrams, or equivalently irreducible Coxeter groups. We use (equivariant) fiber polytope constructions to work out the combinatorics of Cambrian lattices of types A and B in detail, both in terms of triangulations and in terms of permutations. In particular, we prove that:

Theorem 1.3. *Conjecture 1.2.a holds when W is of type A or B.*

FIGURE 1. The 1-skeleton of the H_3-associahedron. The vertex at infinity completes the three unbounded regions to heptagons.



Fomin's and Zelevinsky's definition [**7**] of generalized associahedra is in terms of *clusters* of roots in the root system Φ associated to W. For the purposes of this extended abstract, it suffices to say the following about clusters. One begins with a bipartition $G = I_+ \cup I_-$ of the Coxeter diagram for W, and uses the bipartition to construct piecewise linear maps τ_+ and τ_- acting on the root space of Φ, and generating a dihedral group of piecewise linear maps. These two maps are used to define the *clusters*, certain subsets of Φ whose cardinality is the rank of W. The *cluster fan* is the fan whose maximal cones are the cones generated by the clusters, and the W-associahedron is defined as the polytope whose normal fan is the cluster fan. In particular, the clusters index the vertices of the generalized associahedron for W, and the edges are pairs of clusters which differ by exchanging one root. Using τ_+ and τ_- to compare the roots that are exchanged along an edge, we define a partial order on the clusters called the *cluster poset*.

Naturally associated to the bipartition $G = I_+ \cup I_-$ is an orientation of G which we denote $I_+ \longrightarrow I_-$, and call a *bipartite orientation*. Specifically, any edge in G connects an element s of I_+ to an element t of I_-, and we direct the edge $s\tilde{\Omega}t$.

Conjecture 1.4. *The Cambrian fan for the orientation $I_+ \longrightarrow I_-$ is linearly isomorphic to the cluster fan, and the Cambrian lattice for the same orientation is the cluster poset.*

This conjecture would in particular imply that the cluster poset is a lattice, that it is induced on the vertices of the generalized associahedron by a linear functional, that its Hasse diagram is isomorphic to the 1-skeleton of the generalized associahedron, and that it has the pleasant homotopy and shelling properties described above. General proofs of any of these weaker statements would also be interesting.

Conjecture 1.4 can be proven in types A and B. This provides a proof of Conjecture 1.1 in the special case where $\vec{G}$ is a bipartite orientation of the diagram of a Coxeter group of type A or B. As further support for Conjecture 1.4, we prove the following fact which would be a consequence of Conjecture 1.4.

Theorem 1.5. *The cluster fan refines a fan that is linearly isomorphic to the normal fan of the W-permutohedron.*

In light of the combinatorial description of Cambrian lattices of type A which will be given in Section 2, the Tamari lattice is the type-A Cambrian lattice associated to a path directed linearly, that is, with the arrows all pointing the same direction. Call this the *Tamari orientation* of the diagram. By the symmetry of the path and the fact that directed diagram anti-automorphisms induce lattice anti-automorphisms, we recover the fact that the Tamari lattice is self-dual. The Coxeter diagram for type B is a path as well, but has an asymmetric edge-labeling. There are two Tamari orientations, linear orientations of the type B diagram, yielding two "Type-B Tamari lattices," which are not isomorphic but dual to each other. In support of their claim to the title of "Tamari" is the fact that they can be constructed as the restriction of the weak order to signed permutations avoiding certain signed patterns (Proposition 2.6). The type-A Tamari lattice has a well-known realization in terms of pattern-avoidance. Because the type-B Tamari elements are counted by the type-B Catalan numbers, this result has some bearing on a question posed by Simion in the introduction to [**17**], which asked for signed permutation analogues of counting formulas for restricted permutations. Thomas [**20**], working independently and roughly simultaneously, used one of Reiner's maps to construct the type-B Tamari lattice and proposed a type-D Tamari lattice.

Stasheff and Schnider [**19**] gave a realization of the type-A associahedron by specifying facet hyperplanes, and Loday [**11**] determined the vertices of this realization. The Cambrian fan for the type-A Tamari orientation is the normal fan of this realization of the associahedron, thus proving Conjecture 1.1 for the Tamari orientation in type A. Thus in type A, the Cambrian fans interpolate between the cluster fan and the normal fan of Stasheff's and Shnider's realization of the associahedron.

In general, the quotient of a lattice L with respect to some congruence is isomorphic to an induced subposet of L, but need not be a sublattice. However, in types A and B, the Cambrian lattices are sublattices of the weak order. This fact was proven for the Tamari lattices in [**4**].

Conjecture 1.6. *For any finite Coxeter group W and any orientation $\vec{G}$ of the associated Coxeter diagram, the Cambrian lattice $\mathcal{C}(\vec{G})$ is a sublattice of the weak order on W.*

The Cambrian lattices also inherit any lattice property from the weak order which is preserved by homomorphisms. Notably, the Cambrian lattices are *congruence uniform*, generalizing a theorem of Geyer [**9**] on the Tamari lattice.

For a finite Coxeter group W, let the *(left) descent map* $\mathrm{des} : W\tilde{\Omega}2^S$ be the map which associates to each $w \in W$ its (left) descent set. This map is a lattice homomorphism from the (right) weak order on W onto a boolean algebra [**10**] (see also [**13**]). The homomorphism η from the weak order to a Cambrian lattice factors through the map des in the sense that there is a lattice homomorphism also called des from the Cambrian lattice to 2^S such that $\mathrm{des} \circ \eta = \mathrm{des} : W\tilde{\Omega}2^S$. In types A and B we identify this map on triangulations.

2. Combinatorics of Cambrian lattices of type A

Space does not permit us to elaborate on every assertion made in the overview. We will conclude this extended abstract by describing combinatorial realizations of the Cambrian lattices of type A which arise naturally from a fiber polytope construction, and adding a few words about type B. For background information on these fiber polytope constructions, see [**1, 2, 15**].

Consider a tower of surjective linear maps of polytopes

$$\Delta^{n+1} \xrightarrow{\sigma} Q_{n+2} \xrightarrow{\rho} I,$$

where I is a 1-dimensional polytope, Q_{n+2} is a polygon with $n+2$ vertices, and Δ^{n+1} is the $(n+1)$-dimensional simplex whose vertices are the coordinate vectors $e_0, e_1, \ldots, e_{n+1}$ in $\mathbb{R}^{n+2}$. When n has already been specified, we will sometimes refer to these polytopes simply as Δ and Q. Let $a_i := \rho(\sigma(e_i))$ and $v_i := \sigma(e_i)$, and suppose that $a_0 < a_1 < \cdots < a_{n+1}$. Let f be a non-trivial linear functional on $\ker \rho$. We may as well take ρ to be an orthogonal projection of Q onto the line segment whose endpoints are v_0 and v_{n+1} and think of f as giving the positive or negative "height" of each vertex of Q_{n+2} above that line segment. We abbreviate $f_i := f(\sigma(e_i))$ and use the shorthand $\overline{i}$ to denote an $i \in [n]$ with $f_i \geq 0$. In this case we will call v_i an *up vertex* and i an *up index.* Similarly, $\underline{i}$ will denote an $i \in [n]$ with $f_i \leq 0$, called a *down index.* Thus for example the phrase "Let $\underline{i} \in [n]$" means "Let $i \in [n]$ have $f_i \leq 0$." For any $H \subseteq [n]$, let $\underline{H} = \{\underline{i} \in H\}$ and let $\overline{H} = \{\overline{i} \in H\}$.

The fiber polytope $\Sigma(\Delta \xrightarrow{\rho\circ\sigma} I)$ is a cube whose vertices correspond to triangulations of the point configuration $\{a_0, a_1, \ldots, a_{n+1}\}$. Such a triangulation can be thought of as a subset H of $[n]$ where H is the set of points (other than the endpoints of I) appearing as vertices of the triangulation. We will write $H = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$, with $0 = i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1} = n + 1$.

The iterated fiber polytope $\Sigma(\Delta\tilde{\Omega}Q\tilde{\Omega}I)$ is the f-monotone path polytope of $\Sigma(\Delta \xrightarrow{\rho\circ\sigma} I)$. (In this case it is known [**2**] that every f-monotone path is coherent.) The f-monotone paths are permutations, and $\Sigma(\Delta\tilde{\Omega}Q\tilde{\Omega}I)$ is combinatorially isomorphic to the (A_{n-1})-permutohedron [**2**]. Specifically, an f-monotone path $\underline{[n]} = H_0\tilde{\Omega}H_1\tilde{\Omega}\cdots\tilde{\Omega}H_n = \overline{[n]}$ is associated to the permutation $x_1x_2\cdots x_n$ where a_{x_i} is the unique element in the symmetric difference of H_i and H_{i-1}. Two such monotone paths are connected by an edge in $\Sigma(\Delta\tilde{\Omega}Q\tilde{\Omega}I)$ if they differ in only one vertex. Thus edges correspond to cover relations in the (right) weak order.

The fiber polytope $\Sigma(\Delta^{n+1} \xrightarrow{\sigma} Q_{n+2})$ is combinatorially isomorphic [**1**] to the (A_{n-1})-associahedron, whose vertices are the triangulations of Q. By a general theorem in [**2**], the normal fan of $\Sigma(\Delta\tilde{\Omega}Q\tilde{\Omega}I)$ refines that of $\Sigma(\Delta\tilde{\Omega}Q)$. In other words there is a map $\eta : \Sigma(\Delta\tilde{\Omega}Q\tilde{\Omega}I) \longrightarrow \Sigma(\Delta\tilde{\Omega}Q)$ respecting the facial structure.

We are most interested in the restriction of η to the vertices of the permutohedron (i.e. to permutations), so from now on η will refer to that restriction. The map η takes a permutation $x = x_1x_2\cdots x_n$ to a triangulation of Q, and has a characterization in terms of polygonal paths [**2**]. The edges of the triangulation arise as a union of polygonal paths $\gamma_0, \gamma_1, \ldots, \gamma_n$ in Q such that each vertex of each path is a vertex of Q, and such that each path visits vertices in the order given by their subscripts. Specifically, if x is the permutation associated to the monotone path $H_0\tilde{\Omega}H_1\tilde{\Omega}\cdots\tilde{\Omega}H_n$, then $\gamma_i(x)$ visits the vertices $\{v_j : j \in H_i\}$ in the order given by their subscripts. Alternately, let $\gamma_0(x)$ be the path from v_0 to v_{n+1} passing through the points v_i for $\underline{i} \in [n]$ and define γ_i recursively: If x_i is $\underline{x_i}$, define γ_i by deleting v_{x_i} from the list of vertices visited by γ_{i-1}. If x_i is $\overline{x_i}$, define γ_i by adding v_{x_i} to the list of vertices visited by γ_{i-1}. The union of the paths $\gamma_0, \gamma_1, \ldots, \gamma_n$ is the union of the edges in the triangulation $\eta(x)$.

The combinatorics of the map η from permutations to triangulations of Q derive from the sign of f on each vertex of Q. Thus to be more exact, we should name the map η_f. Usually, however, the choice of f will be fixed, so we will drop the subscript f and pick it up again when we want to emphasize the fact that f can vary.

Theorem 2.1. *The fibers of η are the congruence classes of a lattice congruence Θ on the weak order on permutations.*

Congruence classes of a congruence on a finite lattice L are all intervals, and the quotient of L mod the congruence is isomorphic to the subposet induced on the set of bottom elements of congruence classes. Thus, by identifying the set of triangulations of Q with the set of permutations which are the bottom of their congruence class, we induce a partial order on the triangulations. The content of Theorem 2.1 is that η is a lattice homomorphism from the weak order onto this partial order. We call the associated congruence Θ_f.

Orientations $\vec{G}$ of the Coxeter diagram for A_{n-1} correspond to choices of the linear functional f as follows. Given f, define the orientation $\vec{G}_f$ to be $s_b\tilde{\Omega}s_{b-1}$ for every $\overline{b} \in [2, n-1]$ and $s_{b-1}\tilde{\Omega}s_b$ for every $\underline{b} \in [2, n-1]$. By the reverse process, an orientation $\vec{G}$ specifies which indices in $[2, n-1]$ are up or down, and the indices 1 and n can be arbitrarily chosen as up or down indices. Denote any polygon corresponding to such a choice of up and down vertices as $Q(\vec{G})$.

In [**13**], the author determined the congruence lattice of the weak order on S_n. Knowing the congruence lattice allows us to prove the following:

Theorem 2.2. *The quotient lattice S_n/Θ_f is the Cambrian lattice $\mathcal{C}(\vec{G}_f)$.*

Say that x contains the pattern $\overline{2}31$ if there exist $1 \leq i < j < k \leq n$ with $x_k < \overline{x_i} < x_j$. Recall that this means $x_k < x_i < x_j$ and $f_{x_i} > 0$. No conditions are placed on f_{x_j} or f_{x_k}. Similarly, x contains the pattern $31\underline{2}$ if there exist $1 \leq i < j < k \leq n$ with $x_j < \underline{x_k} < x_i$. If a permutation does not contain a given pattern, we say it avoids that pattern.

Theorem 2.3. *For $\vec{G}$ an orientation of the Coxeter diagram for A_{n-1}, the Cambrian lattice $\mathcal{C}(\vec{G})$ is isomorphic to the subposet of the (right) weak order on S_n consisting of permutations avoiding both $\overline{2}31$ and $31\underline{2}$, where up and down indices are determined by the vertices of $Q(\vec{G})$.*

The edges of the (A_{n-1})-associahedron correspond to diagonal flips. The slope of a diagonal will refer to the usual slope, relative to the convention that the positive horizontal direction is the direction of a ray from v_0 through v_{n+1} and the positive vertical direction is the positive direction of the functional f.

Theorem 2.4. *For $\vec{G}$ an orientation of the Coxeter diagram for A_{n-1}, the Cambrian lattice $\mathcal{C}(\vec{G})$ is isomorphic to the partial order on triangulations of an $(n+2)$-gon $Q(\vec{G})$ whose cover relations are diagonal flips, where going up in the cover relation corresponds to increasing the slope of the diagonal.*

Also as a consequence of Theorem 2.2, we can prove the type-A case of Theorem 1.3. Finally, we have the following theorem.

Theorem 2.5. *For any orientation $\vec{G}$ of the Coxeter diagram associated to A_{n-1}, the Cambrian lattice $\mathcal{C}(\vec{G})$ is a sublattice of the weak order on A_{n-1}.*

An equivariant fiber-polytope construction produces combinatorial realizations which are related to the type-A case by the standard "folding" construction. These folded lattices can be realized combinatorially by centrally symmetric triangulations of a centrally symmetric polygon, or by restricted signed permutations. However, the fact that these are combinatorial realizations of the Cambrian lattices does not follow from the type-A proof by folding, but must be argued separately, using the characterization of the congruence lattice of the weak order on B_n from [**13**].

When $\vec{G}$ is the diagram for a Coxeter group of type B, directed linearly from one endpoint to the other, we call $\mathcal{C}(\vec{G})$ a type-B Tamari lattice. The justification for the name comes from the fact that, in analogy to type A, these are the unique Cambrian lattices of type B which can be defined via signed pattern avoidance, without reference to up indices and down indices.

To any sequence $(a_1, a_2, \ldots, a_p)$ of distinct nonzero integers, we associate a *standard signed permutation* $\mathrm{st}(a_1, a_2, \ldots, a_p)$. This is the signed permutation $\pi \in B_p$ such that $\pi_i < 0$ if and only if $a_i < 0$ and $|\pi_i| < |\pi_j|$ if and only if $|a_i| < |a_j|$. So for example st(7-3-51) = 4-2-31. Rephrasing [**17**], we say that a signed permutation π *contains* a signed permutation τ if there is a subsequence of the entries of π whose standard signed permutation is τ. Otherwise, say that π *avoids* τ.

Proposition 2.6. *One of the type-B Tamari lattices is the sublattice of the weak order on signed permutations consisting of signed permutations avoiding the signed patterns -2-1, 2-1, -231, -12-3, 12-3 and 231. The other is the sublattice consisting of signed permutations avoiding -21, 1-2, -2-1-3, -13-2, 3-12 and 312.*

3. Acknowledgements

The author wishes to thank Sergey Fomin, Vic Reiner, John Stembridge and Hugh Thomas for helpful conversations.

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