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The weak and Kazhdan-Lusztig orders on standard Young tableaux

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Abstract. *Let SYT_n be the set of all standard Young tableaux with n cells. After recalling the definition of a partial order on SYT_n first defined by Melnikov, which we call the weak order, we prove two main results:*

- *Intervals in the weak order essentially describe the product in a Hopf algebra of tableaux defined by Poirier and Reutenauer.*
- *The map sending a tableau to its descent set induces a homotopy equivalence of the proper parts of either weak order or Kazhdan-Lusztig order on tableaux with the Boolean algebra $2^{[n-1]}$. In particular, the Möbius function for either of these orders on tableaux is $(-1)^{n-1}$.*

The methods use in an essential way the Kazhdan-Lusztig order on SYT_n, and in some cases apply to other orders between the weak order and KL-order.

1. Introduction

The weak order on standard Young tableaux was introduced by Melnikov [**15**] (who called it the *induced Duflo order*), in connection with the Robinson-Schensted (RSK) correspondence and the weak Bruhat order on permutations. Roughly speaking, this order is the weakest partial ordering on SYT_n, such that the map from the weak Bruhat order on the symmetric group S_n which takes a permutation w to its RSK insertion tableau $P(w)$ is order preserving; see Figure 1 for $n = 2, 3, 4, 5$.

This order is closely related to the Kazhdan-Lusztig preorder on the symmetric group, and the partial order on SYT_n that it induces, which we will call the KL *order*. In general, the weak order on SYT_n is weaker than the KL order, although they are equivalent up to $n = 5$. The goal of this paper is to prove two main results, Theorems 1.1 and 1.2, about the weak and KL orders on SYT_n.

The first result relates to algebra structures defined by Malvenuto and Reutenauer, Poirier and Reutenauer, and is motivated by results of Loday and Ronco [**13**]; the same result was also asserted without proof in [**8**, middle of p. 579]. Malvenuto and Reutenauer [**14**] defined a (Hopf) algebra structure on $\mathbb{Z}\mathfrak{S} = \oplus_{n\geq 0}\mathbb{Z}\mathfrak{S}_n$, whose product sends a pair of permutations u, v to the sum of all shuffles $\mathrm{sh}(u, v)$ of u and v (after raising the values of all letters in v by the length of u). Poirier and Reutenauer [**17**] observed that this product restricts to a product on the subalgebra spanned by sums over Knuth/plactic classes in $\mathfrak{S}_n$ (or right Kazhdan-Lusztig cells), which are indexed by Young tableaux T. This defines the product $T * S$ in the Poirier-Reutenauer Hopf algebra $\mathbb{Z}SYT = \oplus_{n\geq 0}\mathbb{Z}SYT_n$. The following is proven in Section 3, where T/S and $T\backslash S$ are defined more precisely.

Theorem 1.1.

$$T * S = \sum_{\substack{R\in SYT_n: \\ T/S\leq_{weak} R\leq_{weak} T\backslash S}} R$$

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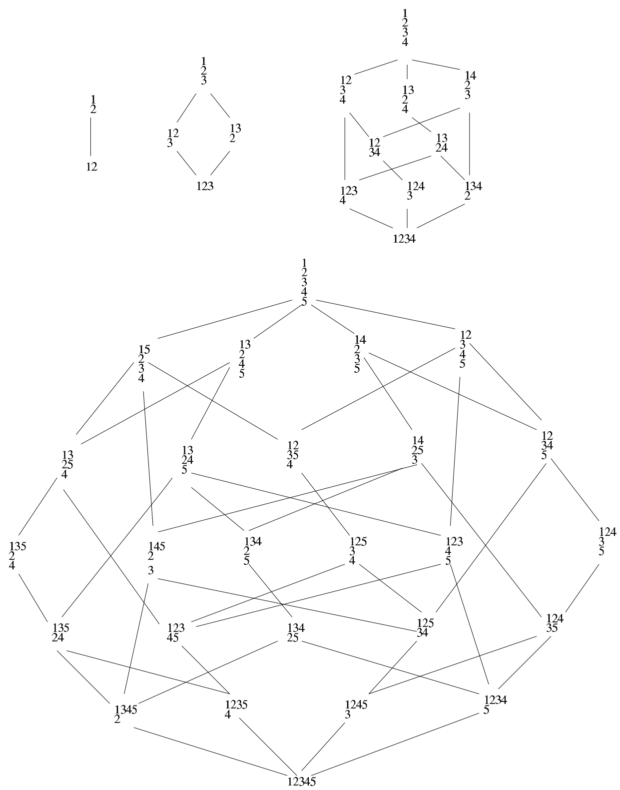

FIGURE 1. The weak order and the KL-order on SYT_n, which coincide for for $n = 2, 3, 4, 5$ (but not in general).

where T/S and $T\backslash S$ are obtained by sliding S over T from the left and from the bottom respectively.

The second main result is about the Möbius function and homotopy type for the weak order and KL-order on SYT_n. The weak Bruhat order on $\mathfrak{S}_n$ is well-known to have each interval homotopy equivalent to either a sphere or a point, and hence have Möbius function values all in $\{\pm 1, 0\}$. This is false for intervals in general in $(SYT_n, \leq weak)$; see Figure 2 below. However, it is true for the interval from bottom to top.

Theorem 1.2. *Let $\leq$ be any partial order on SYT_n that lies between $\leq_{weak}$ and $\leq_{KL}$ (e.g. $\leq_{weak}$ or $\leq_{KL}$ itself).*

Then the map of sets $SYT_n \mapsto 2^{[n-1]}$ sending a tableau to its descent set is order-preserving, and induces a homotopy equivalence of proper parts. In particular, $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}$ for any such order.

To clarify the context and motivation for Theorems 1.1 and 1.2, we recall two commutative diagrams appearing in the work of Loday and Ronco [**13**]

$$(1.1)\qquad \begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & Y_n \\ & \searrow & \downarrow \\ & & 2^{[n-1]} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}\mathfrak{S} & \longleftarrow & \mathbb{Z}Y \\ & \nwarrow & \uparrow \\ & & \Sigma \end{array} .$$

In the left diagram, Y_n denotes the set of plane binary trees with n vertices. The horizontal map sends a permutation w to a certain tree $T(w)$, and has been considered in many contexts (see e.g. [**22**, §1.3], [**5**, §9]). The southeast map $\mathfrak{S}_n \to 2^{[n-1]}$ sends a permutation w to its descent set $\mathrm{Des}_L(w)$. These maps of sets become order-preserving if one orders $\mathfrak{S}_n$ by weak order, Y_n by the *Tamari order* (see [**5**, §9]), and $2^{[n-1]}$ by inclusion. In [**5**, Remark 9.12], Björner and Wachs (essentially) show that the triangle on the left induces a diagram of homotopy equivalences on the proper parts of the posets involved. Theorem 1.2 and the stronger assertion in Corollary 4.3 below give the analogue of this statement in which one replaces $(Y_n, \leq_{Tamari})$ by $(SYT_n, \leq_{weak})$. We were further motivated in proving Theorem 1.2 by the results of Aguiar and Sottile [**1**], where the Möbius function of the weak order on $\mathfrak{S}_n$ plays a role in understanding the structure of the Malvenuto-Reutenauer algebra.

The second diagram in (1.1) consists of induced inclusions of Hopf algebras, in which $\mathbb{Z}\mathfrak{S}$ is the Malvenuto-Reutenauer algebra, $\mathbb{Z}Y$ is a subalgebra isomorphic to Loday and Ronco's free *dendriform algebra* on one generator [**12**], and Σ is a subalgebra known as the algebra of *noncommutative symmetric functions*. In [**13**], Loday and Ronco proved a description of the product structure for each of these three algebras very much analogous to Theorem 1.1, which should be viewed as the analogue replacing $\mathbb{Z}Y$ by $\mathbb{Z}SYT$.

The analogy between the standard Young tableaux SYT_n and the plane binary trees Y_n is tightened further by recent work of Hivert, Novelli and Thibon [**8**]. They show that the planar binary trees Y_n can be interpreted as the plactic monoid structure given by a Knuth-like relation similar to the interperation of the set of standard Young tableaux as Knuth/plactic classes.

2. Definition and properties of the weak order on SYT_n

Before giving the definition of the weak order, it is necessary to recall the Robinson-Schensted (RSK) correspondence; see [**18**, §3] for more details and references on RSK. The RSK correspondence is a bijection between $\mathfrak{S}_n$ and $\{(P,Q) : P, Q \in SYT_n \text{ of same shape}\}$. Here P and Q are called the *insertion* and *recording tableau* respectively. Knuth [**11**] defined an equivalence relation $\underset{K}{\sim}$ on $\mathfrak{S}_n$ with the property that $\sigma \underset{K}{\sim} \tau$ if and only if they have the same insertion tableaux $P(\sigma) = P(\tau)$.

It turns out that RSK is closely related to the *Kazhdan-Lusztig* preorders on $\mathfrak{S}_n$. Recall that a *preorder* on a set X is a binary relation $\leq$ which is reflexive ($x \leq x$) and transitive ($x \leq y, y \leq z$ implies $x \leq z$). It need not be antisymmetric, that is, the equivalence relation $x \sim y$ defined by $x \leq y, y \leq x$ need not have singleton equivalence classes. Note that a preorder induces a *partial order* on the set $X/\sim$ of equivalence classes. Kazhdan and Lusztig [**9**] introduced two preorders (the left and right KL preorders) on Coxeter groups. In this paper we will denote by $\leq_{KL}^{op}$ the *opposite* of the usual KL right preorder on $\mathfrak{S}_n$. For example, with our convention, the identity element 1 and the longest element w_0 satisfy $1 \leq_{KL}^{op} w_0$. It turns out [**9**] (and explicitly in [**6**, p. 54]) that the associated equivalence relation for this KL preorder is the Knuth equivalence $\underset{K}{\sim}$. Hence an equivalence class (usually called either a *Knuth class* or *plactic class* or a *Kazhdan-Lusztig right cell* in $\mathfrak{S}_n$) corresponds to a tableau T in SYT_n. Denote this equivalence class C_T. We denote by $(SYT_n, \leq_{KL}^{op})$ the partial order induced by the KL preorder.

Proposition 2.1. *Let $\leq$ be any preorder on $\mathfrak{S}_n$ which is weaker than $\leq_{KL}^{op}$. Then $\leq$ induces an order on SYT_n, by taking the transitive closure of the relation which has $S \leq T$ whenever $\sigma \leq \tau$ for some σ, τ in $\mathfrak{S}_n$ with $P(\sigma) = S, P(\tau) = T$.*

Furthermore, the map $(\mathfrak{S}, \leq) \to (SYT_n, \leq)$ sending $\sigma \mapsto P(\sigma)$ is order-preserving.

PROOF. Straightforward, but omitted in this extended abstract. □

We now recall the *(right) weak (Bruhat) order* $\leq_{weak}$ $\mathfrak{S}_n$. It is the transitive closure of the relation $\sigma \leq_{weak} \tau$ if $\tau = \sigma \cdot s_i$ for some i with $\sigma_i < \sigma_{i+1}$, and where s_i is the adjacent transposition $(i\ i+1)$. The weak order has an alternative characterization [**3**, Prop. 3.1] in terms of *(left) inversion sets*

$$\mathrm{Inv}_L(\sigma) := \{(i,j) : 1 \leq i < j \leq n \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\},$$

namely $\sigma \leq_{weak} \tau$ if and only if $\mathrm{Inv}_L(\sigma) \subset \mathrm{Inv}_L(\tau)$.

It is known [**9**, page 171] that the (right) weak order $\leq_{weak}$ on $\mathfrak{S}_n$ is weaker than the (right) KL preorder $\leq_{KL}^{op}$ on $\mathfrak{S}_n$, leading to the following definition.

Definition 2.2. The *weak order* $(SYT_n, \leq_{weak})$, first introduced by Melnikov [**15**] under the name *induced Duflo order*, is the partial order induced by $(\mathfrak{S}_n, \leq_{weak})$ via Proposition 2.1.

Implicitly the definition of $(SYT_n, \leq_{weak})$ involves taking transitive closure; the necessity of this is illustrated by the following example (cf. Melnikov [**15**, Example 4.3.1]).

Example 2.3. Let $R = \begin{smallmatrix} 1\,2\,5 \\ 3\,4 \end{smallmatrix}$, $S = \begin{smallmatrix} 1\,4\,5 \\ 2 \\ 3 \end{smallmatrix}$, $T = \begin{smallmatrix} 1\,4 \\ 2\,5 \\ 3 \end{smallmatrix}$ with

$$\begin{aligned} C_R &= \{31425, 34125, 31452, 34152, 34512\}, \\ C_S &= \{32145, 32415, 32451, 34215, 34251, 34521\}, \\ C_T &= \{32154, 32514, 35214, 32541, 35241\}. \end{aligned}$$

Here $R <_{weak} S$ since $34125 <_{weak} 34215 = 34125 \cdot s_3$, and $S <_{weak} T$ since $32145 <_{weak} 32154 = 32145 \cdot s_4$. Therefore $R < T$.

On the other hand, for every $\rho \in C_R$ one has $(2,4) \in \mathrm{Inv}_L(\rho)$, whereas for every $\tau \in C_T$ one has $(2,4) \notin \mathrm{Inv}_L(\tau)$. This shows that there is no $\rho \in C_R$ and $\tau \in C_T$ such that $\rho <_R \tau$.

It happens that $(SYT_n, \leq_{weak})$ and $(SYT_n, \leq^{op}_{KL})$ coincide for $n \leq 5$, but the following examples show that they differ for $n = 6$.

Example 2.4. Let

$$S = \begin{smallmatrix} 1\,2\,3 \\ 4\,5\,6 \end{smallmatrix}, \qquad T_1 = \begin{smallmatrix} 1\,2\,5 \\ 3\,6 \\ 4 \end{smallmatrix}, \qquad T_2 = \begin{smallmatrix} 1\,3\,6 \\ 2\,4 \\ 5 \end{smallmatrix}$$

Computer calculations show that $S \leq^{op}_{KL} T_1, T_2$, but $S \not\leq_{weak} T_1, T_2$. By using the anti-automorphism of $\leq^{op}_{KL}, \leq_{weak}$ that transposes a standard Young tableau (see Proposition 2.6) one obtains two more examples of pairs of tableaux which are comparable in $\leq^{op}_{KL}$, but not in $\leq_{weak}$. These are the *only* such examples in SYT_6.

An important property of both $\leq_{weak}$ and $\leq^{op}_{KL}$ are their interactions with descent sets. The *(left) descent set* of a permutation σ is defined by

$$\begin{aligned} \mathrm{Des}_L(\sigma) &:= \{(i, i+1) : 1 \leq i \leq n-1 \text{ and } \sigma^{-1}(i) > \sigma^{-1}(i+1)\} \\ &= \mathrm{Inv}_L(\sigma) \cap S \end{aligned}$$

where $S = \{(i, i+1) : 1 \leq i \leq n-1\}$. In what follows, we will often identify the set S of adjacent transposition with the numbers $[n-1] := \{1, 2, \ldots, n-1\}$ via the obvious map $(i, i+1) \mapsto i$.

Property (i) in the next proposition is well-known [**9**, Prop. 2.4], and property (ii) follows from the characterization of $\leq_{weak}$ by inclusion of left inversion sets.

Proposition 2.5. *For σ, τ in $\mathfrak{S}_n$,*

(i) $\sigma \leq^{op}_{KL} \tau$ *implies* $\mathrm{Des}_L(\sigma) \subset \mathrm{Des}_L(\tau)$.
(ii) $\sigma \leq_{weak} \tau$ *implies* $\mathrm{Des}_L(\sigma) \subset \mathrm{Des}_L(\tau)$.

As a consequence of this proposition (or well-known properties of RSK), the left descent set $\mathrm{Des}_L(-)$ is constant on Knuth classes C_T; the *descent set* of the standard Young tableau T is described intrinsically by

$$\begin{aligned} \mathrm{Des}(T) := \{(i, i+1) : 1 \leq i \leq n-1 \text{ and} \\ i+1 \text{ appears in a row below } i \text{ in } T\}. \end{aligned}$$

For the record, we note here some well-known symmetries of $\leq_{weak}$ and $\leq^{op}_{KL}$ on SYT_n, and some obvious order-preserving maps to other posets. Let $(2^{[n-1]}, \subseteq)$ be the Boolean algebra of all subsets of $[n-1]$ ordered by inclusion. Let $(\mathrm{Par}_n, \leq_{dom})$ denote the set of all partitions of the number n ordered by *dominance*, that is, $\lambda \leq_{dom} \mu$ if

$$\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \text{ for all } k.$$

Proposition 2.6. *The following maps are order-preserving:*

(i) *The map*

$$(SYT_n, \leq_{weak}) \to (2^{[n-1]}, \subseteq)$$

sending a tableau T *to its descent set* $\mathrm{Des}(T)$.

(ii) *The same map*

$$(SYT_n, \leq_{KL}^{op}) \to (2^{[n-1]}, \subseteq).$$

(iii) *The map*

$$(SYT_n, \leq_{weak}) \to (\mathrm{Par}_n, \leq_{dom})^{opp}$$

sending T *to its shape* $\lambda(T)$, *where here* $(-)^{opp}$ *denotes the opposite or dual poset.*

Also, Schützenberger's evacuation map [**20**] *on* SYT_n *gives a poset automorphism of both* $\leq_{weak}$ *and* $\leq_{KL}^{op}$, *and the transpose map on* SYT_n *gives a poset anti-automorphism of both.*

PROOF. The first two assertions are immediate from Proposition 2.5 (i) and (ii). For (iii), one can apply Greene's Theorem [**7**].

The assertions about transposing and evacuation follow from the fact that the involutive maps

$$w \mapsto w_0 w \text{ and } w \mapsto w w_0$$

are antiautomorphisms of both $(\mathfrak{S}_n, \leq_{KL}^{op})$ [**6**] and $(\mathfrak{S}_n, \leq_{weak})$. Hence $w \mapsto w_0 w w_0$ is an automorphism of both. On the other hand $P(ww_0)$ is just the transpose tableau of $P(w)$ [**19**] and $P(w_0 w w_0)$ is nothing but the evacuation of $P(w)$ [**20**]. □

3. The Hopf Algebra of SYT_n

Malvenuto and Reutenauer, in [**14**] construct two graded Hopf algebra structure on the $\mathbb{Z}$ module of all permutations $\mathbb{Z}S = \oplus_{n\geq 0}\mathbb{Z}S_n$ which are dual to each other, and shown to be free as associative algebras by Poirier and Reutenauer in [**17**]. The product structure of the one that concerns us here is given by, $\alpha * \beta = \mathrm{sh}(\alpha, \overline{\beta})$ where $\overline{\beta}$ is obtained by increasing the indices of β by the length of α and sh denotes the shuffle product.

Poirier and Reutenauer also show that $\mathbb{Z}$ module of all plactic classes $\{PC_T\}_{T\in SYT}$, where $PC_T = \sum_{P(\alpha)=T} \alpha$ becomes a Hopf subalgebra of permutations and the product is given by the formula

$$PC_T * PC_{T'} = \sum_{\substack{P(\alpha)=T \\ P(\beta)=T'}} sh(\alpha, \overline{\beta}) \tag{3.1}$$

Then the bijection sending each plactic class to its defining tableau gives us a Hopf algebra structure on the $\mathbb{Z}$ module of all standard Young tableaux, $\mathbb{Z}SYT = \oplus_{n\geq 0}\mathbb{Z}SYT_n$.

For example,

$$PC_{\substack{1\\2}} * PC_{12} = \mathrm{sh}(21,34) = PC_{\substack{134\\2}} + PC_{\substack{14\\2\\3}}$$

since $\mathrm{sh}(21,34) = 2134 + 2314 + 2341 + 3241 + 3421$. In other words,

$$\begin{matrix}1\\2\end{matrix} * 12 = \begin{matrix}134\\2\end{matrix} + \begin{matrix}14\\2\\3\end{matrix}.$$

Another approach to calculate the product of two tableaux is given in [**17**] where Poirier and Reutenauer explain this product using jeu de taquin slides. Our goal is to show that it can also be described by a formula using partial orders, analogous to a result of Loday and Ronco [**13**, Thm. 4.1]. To state their result, given $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$, with $n := k + \ell$, let $\overline{\tau}$ be obtained from τ by adding k to each letter. Then let σ/τ and $\sigma\backslash\tau$ denote the concatenations of $\sigma, \overline{\tau}$ and of $\overline{\tau}, \sigma$, respectively.

Theorem 3.1. *For $\tau \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_\ell$, with $n := k+\ell$, one has in the Malvenuto-Reutenauer Hopf algebra*

$$\tau * \sigma = \sum_{\substack{\rho \in \mathfrak{S}_n: \\ \sigma/\tau \leq \rho \leq \sigma\backslash\tau}} \rho.$$

Equivalently, the shuffles $\mathrm{sh}(\sigma,\tau)$ *are the interval* $[\sigma/\tau, \sigma\backslash\tau]_{\leq_{weak}}$.

The next definition identifies a crucial property for transporting the Loday and Ronco result to SYT_n.

Definition 3.2. Given σ in $\mathfrak{S}_n$, and $k \in [n]$, let I and I^c be the initial and final segments $I = [k]$ and $I^c = [n] - [k] = [k+1, n]$ of the alphabet $[n]$. Let σ_I and σ_{I^c} be the subwords of σ obtained by restricting to the alphabets I and I^c. Let $\mathrm{std}(\sigma_{I^c})$ in $\mathfrak{S}_{n-k}$ be the word obtained from σ_{I^c} by subtracting k from each letter.

Say that a family of preorders $\leq$ on $\mathfrak{S}_n$ for all n *restricts to initial and final segments* if $\sigma \leq \tau$ implies $\sigma_I \leq \tau_I$ and $\mathrm{std}(\sigma_{I^c}) \leq \mathrm{std}(\tau_{I^c})$.

We need analogous definitions for tableaux. Given a tableaux T and $k \in [n]$ with initial and final segments $I = [k], I^c$ as before, let T_I denote subtableau of T obtained by restricting to the values in I. Let $\mathrm{std}(T_{I^c})$ denote the tableau obtained by first restricting T to its skew subtableau on the values in I^c, then lowering all these entries by k, and then sliding into normal shape by jeu-de-taquin [**21**].

The following are two basic facts about RSK, Knuth equivalence, and jeu-de-taquin are essentially due to Knuth and Schützenberger; see Knuth [**10**, Section 5.1.4] for detailed explanations.

Lemma 3.3. *Given $\rho \in \mathfrak{S}_n$ and $k \in [n]$, let $I = [k], I^c$ be initial and final segments as before. Then*

(i) $P(w_I) = P(w)_I$, *and*
(ii) $\mathrm{std}(P(w)_{I^c}) = P(\mathrm{std}(w_{I^c}))$.

Let $\sigma \in \mathfrak{S}_k, \tau \in \mathfrak{S}_\ell$. When $P(\sigma) = S$ and $P(\tau) = T$, let $\overline{T}$ denote the result of adding k to every entry of T. It is easily seen that $P(\sigma/\tau) = S/T$ and $P(\sigma\backslash\tau) = S\backslash T$, where S/T (respectively, $S\backslash T$) is the tableaux whose columns (resp. rows) are obtained by concatenating the columns (resp. rows) of S and $\overline{T}$. Note also that Lemma 3.3 shows

$$\begin{aligned} (S/T)_I = S \quad st((S/T)_{I^c}) = T \\ (S\backslash T)_I = S \quad st((S\backslash T)_{I^c}) = T. \end{aligned}$$

The following theorem is a consequence of Lemma 3.3, Proposition 2.1 and Theorem 3.1. For the sake of space we omit the detailed proof.

Theorem 3.4. *Let $\leq$ be a family of preorders on $\mathfrak{S}_n$ for all n that*

(a) *lies between $\leq_{weak}$ and $\leq_{KL}^{op}$, and*
(b) *restricts to initial and final segments.*

Let $(SYT_n, \leq)$ denote the partial order on tableaux which it induces as in Proposition 2.1.

Then in the Poirier-Reutenauer Hopf algebra,

$$S * T = \sum_{\substack{R \in SYT_n: \\ S/T \leq R \leq S\backslash T}} R.$$

Proof of Theorem 1.1. The poset $(\mathfrak{S}_n, \leq_{weak})$ satisfies both hypotheses of Theorem 3.4: it lies between itself and $\leq_{KL}^{op}$, and its characterization via inclusion of left inversion sets shows immediately that it restricts to initial and final segments. $\square$

Example 3.5. Let $T = \begin{matrix}1\,2\\3\end{matrix}$ and $S = P(\beta) = \begin{matrix}1\\2\end{matrix}$. Then the product on the corresponding the plactic classes gives

$$T * S = \begin{matrix}1\,2\\3\end{matrix} * \begin{matrix}1\\2\end{matrix} = \begin{matrix}1\,2\,4\\3\,5\end{matrix} + \begin{matrix}1\,2\,4\\3\\5\end{matrix} + \begin{matrix}1\,2\\3\,4\\5\end{matrix} + \begin{matrix}1\,2\\3\\4\\5\end{matrix}.$$

On the other hand, $T/S = \begin{matrix}1\,2\,4\\3\,5\end{matrix}$ and $T\backslash S = \begin{matrix}1\,2\\3\\4\\5\end{matrix}$. The Hasse diagram of SYT_5 in Figure 1 shows that the product above is equal to the sum of all tableaux in the interval $[T/S, T\backslash S]_{\leq_{weak}}$.

4. Möbius function and homotopy equivalences

In this section, we prove Theorem 1.2, but in greater generality. We will view the the commutative diagram

$$\begin{array}{ccc} S_n & \longrightarrow & SYT_n \\ & \searrow & \downarrow \\ & & 2^{[n-1]} \end{array} \tag{4.1}$$

as an instance of the following set-up, involving closure relations, equivalence relations, order-preserving maps, and the topology of posets. For background on poset topology, see [**2**].

Let P be a partial order and $p \mapsto \bar{p}$ a closure relation on P, that is,

$$\bar{\bar{p}} = \bar{p}, \quad p \leq_P \bar{p} \quad \text{and } p \leq_P q \text{ implies } \bar{p} \leq_P \bar{q}.$$

It is well-known that in this instance, the order-preserving closure map $P \to \overline{P}$ has the property that its associated simplicial map of order complexes $\Delta(P) \to \Delta(\overline{P})$ is a strong deformation retraction.

Now assume $\sim$ be an equivalence relation on P such that, as maps of sets, the closure map $P \to \overline{P}$ factors through the quotient map $P \to P/_{\sim}$. Equivalently, the vertical map below is well-defined, and makes the diagram commute:

$$\begin{array}{ccc} P & \longrightarrow & P/_{\sim} \\ & \searrow & \downarrow \\ & & \overline{P} \end{array} \tag{4.2}$$

Proposition 4.1. *In the above situation, partially order $\overline{P}$ by the restriction of $\leq_P$, and assume that $P/_{\sim}$ has been given a partial order $\leq$ in such a way that the horizontal and vertical maps in the* (4.2) *are also order-preserving.*

Then the commutative diagram of associated simplicial maps of order complexes are all homotopy equivalences.

PROOF. The proof is omitted for the sake of space. ☐

Lemma 4.2. *Given any subset $D \subset [n-1]$, there exists a maximum element $\tau(D)$ in $(\mathfrak{S}_n, \leq_{weak})$ for the descent class*

$$\mathrm{Des}_L^{-1}(D) := \{\sigma \in \mathfrak{S}_n : \mathrm{Des}_L(\sigma) = D\}.$$

Consequently, the map $\mathfrak{S}_n \to \mathfrak{S}_n$ defined by $\sigma \mapsto \tau(\mathrm{Des}_L(\sigma))$ is a closure relation, with image isomorphic to $(2^{[n-1]}, \subseteq)$.

PROOF. It is known that [**3**, page 98-100] $\mathrm{Des}_L^{-1}(D) := \{\sigma \in \mathfrak{S}_n : \mathrm{Des}_L(\sigma) = D\}$ is actually an interval of the weak Bruhat order on S_n. The rest follows from this fact easily. ☐

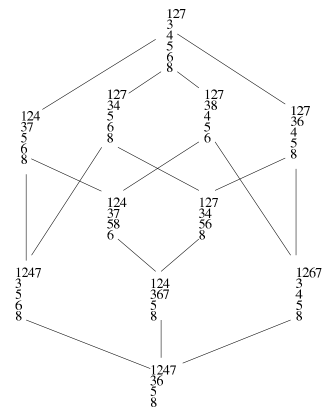

FIGURE 2. An interval in $(SYT_8, \leq_{weak})$, having Mobius function -2.

Corollary 4.3. *Order $\mathfrak{S}_n$ by $\leq_{weak}$ and $2^{[n-1]}$ by $\subseteq$, and let $\leq$ be any order on SYT_n such that the commuting diagram* (4.1) *has all the maps order-preserving.*

Then these restrict to a commuting diagram of order-preserving maps on the proper parts, each of which induces a homotopy equivalence of order complexes. Consequently, $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}$ for each of the three orders.

PROOF. Straightforward from Proposition 4.1 and Lemma 4.2, but omitted in this extended abstract. □

Proof of Theorem 1.2. Any partial order $\leq$ on SYT_n between $\leq_{weak}$ and $\leq_{KL}^{op}$ satisfies the hypotheses of Corollary 4.3. □

The example shown in Figure 2 illustrates that the Möbius function values need not all lie in $\{\pm 1, 0\}$ for $\leq_{weak}$ on SYT_n.

Remark 4.4. In light of Theorems 1.2 and 3.4 one might ask if there are other natural orders on SYT_n which lie between $\leq_{weak}$ and $\leq_{KL}^{op}$? And if so, do any of them restrict to initial and final segments?

Conjecture 4.5. *The Kazhdan-Lusztig order $\leq_{KL}$ on SYT_n restricts to initial and final segments. Equivalently, the Kazhdan-Lusztig right pre-order on $\mathfrak{S}_n$ restricts to initial and final segments.*

By the evacuation symmetry on $\leq_{KL}$ (see Proposition 2.6), one need only check that it restricts to initial segments. Computer calculations have verified this for SYT_n with $n \leq 7$.

Remark 4.6. One might ask to what extent the definitions and results in this paper apply to other Coxeter systems (W, S). The weak order on W is well-defined, as are the KL-cells (replacing SYT_n) and the KL-order, so Proposition 2.1, Definition 2.2 make sense and remain valid. Proposition 2.5 is also well-known ([**9**]; see [**6**, Fact 7]), and hence Proposition 2.6(i),(ii) remain valid.

For the analysis of Möbius function and homotopy types, the crucial Lemma 4.2 was proven by Bjorner and Wachs [**4**, Theorem 6.1] for all *finite* Coxeter groups W. Hence Corollary 4.3 and Theorem 1.2 are valid also in this generality, with the same proof.

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