Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

Deformed Universal Characters for Classical and Affine Algebras and the $X = M = K$ Conjecture

Mark Shimozono and Mike Zabrocki

Abstract. *Creation operators are given for three distinguished bases of the type BCD universal character ring of Koike and Terada. Deformed versions of these operators create symmetric functions whose expansion in the universal character basis, has coefficient polynomials* $K \in \mathbb{Z}_{\geq 0}[q]$. *We conjecture that for every nonexceptional affine root system, these polynomials coincide with the graded tensor product multiplicities for affine characters that occur in the* $X = M$ *conjecture of Hatayama, Kuniba, Okado, Takagi, Tsuboi, and Yamada, which asserts the equality of an affine crystal theoretic formula* X *with a rigged configuration fermionic formula* M.

Résumé. *Nous donnons les opérateurs qui créent trois bases spéciales du type BCD de l'anneau des caractères de Koike et Terada. Les versions difformes de ces opérateurs créent les fonctions symétriques avec les coefficients* $K \in \mathbb{Z}_{\geq 0}[q]$. *Nous conjecturons que pour tous les systèmes des racines affines et non-exceptionnels, ces polynômes coïncident avec les multiplicités des produit tensoriels des charactres affines qui apparaissent dans le conjecture* $X = M$ *de Hatayama, Kuniba, Okado, Takagi, Tsuboi, et Yamada. Cette conjecture affirme qu'une formule pour* X *liée aux crystaux affines, est égale à une formule fermionique des configurations 'gréées' pour* M.

1. Introduction

It is well-known that the ring Λ of symmetric functions is the universal character ring of type A, with universal characters given by the Schur functions. That is, for every $n \in \mathbb{Z}_{>0}$ there is a ring epimorphism $\Lambda \to R(GL(n))$ from Λ onto the ring of polynomial representations of $GL(n)$, which sends the Schur function s_λ to the isomorphism class of the irreducible $GL(n)$-module of highest weight λ.

Using identities of Littlewood [**13**], Koike and Terada [**12**] showed that that the common universal character ring for types B, C, and D, is isomorphic to Λ, constructing two distinguished bases which correspond to the irreducible characters of the symplectic and orthogonal groups. These bases have the same structure constants under a suitable labeling of dominant weights by partitions. This ring captures the behavior (as the rank goes to infinity) of the representation ring of the simple Lie group, or more precisely, the subring generated by the vector representation.

There is a third basis of Λ with the same structure constants as the above two bases. This basis is implicitly defined by Kleber [**7**], who showed that up to a constraint involving Schur function expansions, these are the only three bases of Λ with the given set of structure constants. This basis also appears with

1991 *Mathematics Subject Classification.* Primary 05E10, 22E46.

Key words and phrases. symmetric functions, algebraic combinatorics, Kostka polynomials.

The first author is partially supported by NSF grant DMS-0100918.

a slight deformation in [**11**, def 6.4, eq (7.2.6)]. It is noteworthy that [**7**] was motivated by identities for characters of finite dimensional modules over affine algebras, and that one only sees the third basis upon considering the twisted affine root system $D_{n+1}^{(2)}$.

Bernstein's creation operator B_r is a degree r linear endomorphism of Λ. The operators B_r create the Schur basis by adding a row at a time to a Schur function, in the sense that $B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_k} 1 = s_\lambda$ where $\lambda = (\lambda_1, \dots, \lambda_k)$. Jing [**3**] defined a q-analogue of Bernstein's operator and showed that they create the Hall-Littlewood symmetric functions. In [**22**] the authors defined parabolic analogues of Jing's Hall-Littlewood creation operators and showed that they create symmetric functions, which, when expanded in the Schur basis, have coefficients given by the generalized Kostka polynomials of [**21**].

We consider the analogous constructions for the three bases of the BCD universal character ring using the general q-analogue of a symmetric function operator given in [**23**]. Such operators create q-analogues of products of universal characters. In the row-adding case one obtains polynomials with nonnegative integer coefficients, but in the parabolic case the nonnegativity fails. Corresponding to the three bases of the BCD universal character ring, we define three analogues of the type A deformed parabolic creation operators, and observe that the coefficients are polynomials with nonnegative coefficients which we call K.

To identify the polynomials K we turn to affine algebras. Kirillov and Reshetikhin [**8**] defined a family of finite-dimensional modules over Yangians and conjectured that tensor products multiplicities of such modules, are given by a fermionic formula. This inspired Hatayama, Kuniba, Okado, Takagi, Tsuboi, and Y. Yamada [**2**] [**1**] to formulate the $X = M$ conjecture. First, they conjecture the existence of a family of irreducible finite-dimensional modules over quantum affine algebras called Kirillov-Reshetikhin (KR) modules. Using the theory of affine crystal graphs, they define a formula X, which is a q-analogue of the multiplicities of the restriction to the canonical simple Lie subalgebra, of the tensor product of KR modules. They also define the fermionic formula M by generalizing to any affine root system, the q-analogue of the fermionic formula in [**8**]. They then assert that $X = M$.

We observe that for each infinite family of affine root systems, the formula M has a stable limit as the rank goes to infinity. Using the stable M polynomials we define a symmetric function called a universal affine character, which corresponds to the character of a tensor product of KR modules for large rank. We conjecture that $X = M = K$. There are eight infinite families of affine root systems if one distinguishes the two ways to achieve $A_{2n}^{(2)}$ based on whether the 0 root is short (denoted $A_{2n}^{(2)}$) or extra long (written $A_{2n}^{(2)\dagger}$). In this stable limit we observe that there are only four distinct families of universal affine characters, which are in natural correspondence with the four bases of symmetric functions given by the Schur functions and the three other aforementioned bases. For any of the four families, the corresponding K polynomials are related to those of type A in a simple way. Moreover the K polynomials satisfy a Macdonald-type level-rank duality. Via the $X = M = K$ conjecture these observations have remarkable implications for the affine characters.

2. Plethystic formulae

Let Λ be the ring of symmetric functions, to which we apply the 'plethystic notation'. Instead of defining this notation precisely, we list most of the necessary identities in this section; see also subsection 3.3. Assume that the letters X, Y, Z and W represent sums of monomials with coefficient 1 and expressions like $x \in X$ indicate that x is a single monomial in the multiset X. Let $\widehat{\Lambda}$ be the completion of Λ given by formal sums $f_0 + f_1 + f_2 + \dots$ where $f_i \in \Lambda$ has degree i.

2.1. Cauchy kernel.

There is an element $\Omega \in \widehat{\Lambda}$ defined by

$$\Omega[X - Y] = \frac{\prod_{y \in Y}(1-y)}{\prod_{x \in X}(1-x)} = \left(\sum_{r \geq 0} (-1)^r s_{(1^r)}[Y] \right) \left(\sum_{r \geq 0} s_r[X] \right). \tag{2.1}$$

It satisfies $\Omega[X+Y]=\Omega[X]\Omega[Y]$. The reproducing kernel for $\langle\cdot\,,\,\cdot\rangle$ is

$$\Omega[XY]=\sum_\lambda s_\lambda[X]s_\lambda[Y]. \tag{2.2}$$

2.2. Skewing operators. Given $P[X]\in\Lambda$, the skewing operator $P[X]^\perp\in\mathrm{End}(\Lambda)$ is the linear operator that is adjoint to multiplication by $P[X]$:

$$P[X]^\perp(\Omega[XY])=\Omega[XY]P[Y]. \tag{2.3}$$

By linearity it follows that for all $P[X]\in\Lambda$,

$$\Omega[XZ]^\perp(P[X])=P[X+Z] \tag{2.4}$$

For all $P[X]\in\Lambda$ one obtains the operator identity

$$\Omega[XW]^\perp\circ P[X]=P[X+W]\circ\Omega[XW]^\perp \tag{2.5}$$

where $P[X]$ denotes multiplication by $P[X]\in\Lambda$.

2.3. Coproduct. The coproduct $\Delta:\Lambda\to\Lambda\otimes\Lambda$ may be computed as follows. For $P\in\Lambda$, expand $P[X+Y]$ as a sum of products of the form $P_1[X]P_2[Y]$: $P[X+Y]=\sum_{(P)}P_1[X]P_2[Y]$. Then $\Delta(P)=\sum_{(P)}P_1\otimes P_2$.

The skewing operators $P^\perp$ act on products via the coproduct:

$$P^\perp(QR)=\sum_{(P)}P_1^\perp(Q)P_2^\perp(R). \tag{2.6}$$

2.4. Deforming an operator on Λ. Given any operator $V\in\mathrm{End}(\Lambda)$, one of the authors [**23**] defined its t-analogue $\widetilde{V}\in\mathrm{End}(\Lambda)$ by

$$\widetilde{V}(P[X])=V^Y(P[tX+(1-t)Y])|_{Y\to X} \tag{2.7}$$

where V^Y acts on the Y variables and $Y\to X$ is the substitution map. Applying this construction to $V=\Omega[XZ]\circ\Omega[XW]^\perp$, we have that for $P[X]\in\Lambda$,

$$\widetilde{V}(P[X])=\Omega[XZ]\Omega[XW(1-t)]^\perp P[X].$$

By linearity, for all $P[X],Q[X]\in\Lambda$, if $V=P[X]\circ Q[X]^\perp$, then

$$\widetilde{V}=P[X]\circ Q[X(1-t)]^\perp. \tag{2.8}$$

At $t=0$ the operator V is recovered:

$$\widetilde{V}|_{t=0}=P[X]\circ Q[X]^\perp=V. \tag{2.9}$$

At $t=1$, the operator

$$\widetilde{V}|_{t=1}=P[X]Q[0] \tag{2.10}$$

is multiplication by $P[X]Q[0]$.

Let $e_r[X]=s_{(1^r)}[X]$ be the elementary symmetric function. The following result is used in later proofs.

Proposition 2.1.

$$\Omega[We_2[X]]^\perp\circ\Omega[ZX]=\Omega[ZX+We_2[Z]]\Omega[W(e_2[X]+ZX)]^\perp. \tag{2.11}$$

3. Four bases of symmetric functions

3.1. Littlewood's formulae. Let

$$\begin{aligned} f_{\varnothing}[X] &= 0 \\ f_{\square}[X] &= s_1[X] + e_2[X] \\ f_{\boxminus}[X] &= e_2[X] \\ f_{\boxbar}[X] &= s_2[X]. \end{aligned} \tag{3.1}$$

To explain the notation, for $\diamondsuit \in \{\varnothing, \square, \boxminus, \boxbar\}$, let $\mathcal{P}^{\diamondsuit}$ be the set of partitions that can be tiled using the shape $\diamondsuit$. That is, $\mathcal{P}^{\varnothing} = \{\varnothing\}$ is the singleton set containing the empty partition, $\mathcal{P} = \mathcal{P}^{\square}$ is the set of all partitions, $\mathcal{P}^{\boxbar}$ is the set of partitions with even rows, and $\mathcal{P}^{\boxminus}$ is the set of partitions with even columns. Littlewood proved that

$$\Omega[f_{\diamondsuit}] = \sum_{\lambda \in P^{\diamondsuit}} s_\lambda[X]. \tag{3.2}$$

3.2. Definition of the four bases. For $\lambda \in \mathcal{P}$ define

$$s^{\diamondsuit}_\lambda[X] = \Omega[-f_{\diamondsuit}]^{\perp} s_\lambda[X]. \tag{3.3}$$

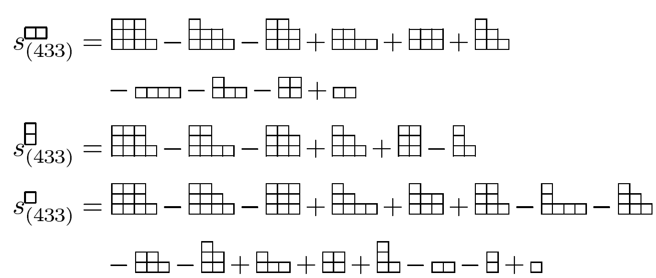
All of the four families $\{s^{\diamondsuit}_\lambda \mid \lambda \in \mathcal{P}\}$ are bases of Λ, due to the inverse formula

$$s_\lambda[X] = \Omega[f_{\diamondsuit}]^{\perp} s^{\diamondsuit}_\lambda[X]. \tag{3.4}$$

Of course $s^{\varnothing}_\lambda = s_\lambda$ is the basis of Schur functions, which are the universal characters for the special/general linear groups. The bases $\{s^{\boxminus}_\lambda\}$ and $\{s^{\boxbar}_\lambda\}$ appear in [**12**] as the universal characters for the symplectic and orthogonal groups respectively. The basis $\{s^{\square}_\lambda\}$ is not mentioned in [**12**] but appears implicitly in [**7**].

Example 3.1. The following elements may computed by the Littlewood-Richardson rule, (3.3), and Littlewood's inverse relations to (3.3) [**15**]. Each Schur function s_μ will be represented by the Ferrers diagram of the partition μ.

(3.5)



3.3. Change of basis. In plethystic formulae let ε represent a variable that has been specialized to the scalar -1. We will consider ε to be a special element with the property $\varepsilon^2 = 1$ and

$$\Omega[\varepsilon X - \varepsilon Y] = \frac{\prod_{y \in Y}(1+y)}{\prod_{x \in X}(1+x)}. \tag{3.6}$$

For $\diamondsuit, \heartsuit \in \{\varnothing, \square, \boxminus, \boxbar\}$ define the linear isomorphism $i^{\heartsuit}_{\diamondsuit} : \Lambda \to \Lambda$ by

$$i^{\heartsuit}_{\diamondsuit}(s^{\diamondsuit}_\lambda[X]) = s^{\heartsuit}_\lambda[X] \tag{3.7}$$

for all λ. It is given by

$$i^{\heartsuit}_{\diamondsuit} = \Omega[f_{\diamondsuit} - f_{\heartsuit}]^{\perp}. \tag{3.8}$$

Proposition 3.2. *For all $P \in \Lambda$,*

(3.9) $$i^{\Box}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} P[X] = P[X-1] \qquad i^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\Box} P[X] = P[X+1]$$

(3.10) $$i^{\Box\Box}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} P[X] = P[X-1-\varepsilon] \qquad i^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\Box\Box} P[X] = P[X+1+\varepsilon]$$

(3.11) $$i^{\Box\Box}_{\Box} P[X] = P[X-\varepsilon] \qquad i^{\Box}_{\Box\Box} P[X] = P[X+\varepsilon]$$

Since substitution maps are algebra homomorphisms, one has the following result, which was obtained in [**12**] for $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and $\Box\Box$. The full result is proved in [**7**], although the basis $s^{\Box}_{\lambda}[X]$ is not explicitly mentioned.

Corollary 3.3. $i^{\heartsuit}_{\diamondsuit}$ *is an algebra isomorphism for* $\diamondsuit, \heartsuit \in \{\Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \Box\Box\}$.

3.4. BCD structure constants and uniqueness of bases. Define the structure constants ${}^{\diamondsuit}c^{\lambda}_{\mu\nu}$ by

(3.12) $$s^{\diamondsuit}_{\mu}[X] s^{\diamondsuit}_{\nu}[X] = \sum_{\lambda} {}^{\diamondsuit}c^{\lambda}_{\mu\nu} s^{\diamondsuit}_{\lambda}[X].$$

The coefficient ${}^{\varnothing}c^{\lambda}_{\mu\nu}$ is the ordinary Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$. By Corollary 3.3, the other three sets of structure constants coincide; call this common structure constant $d_{\lambda\mu\nu}$.

Theorem 3.4. [**7**] *Suppose $\{v_\lambda\}$ is a basis of Λ such that*

(3.13) $$v_\mu v_\nu = \sum_{\lambda} d_{\lambda\mu\nu} v_\lambda$$

for all μ, ν and that

(3.14) $$s_\lambda \in v_\lambda + \sum_{\mu < \lambda} \mathbb{Z}_{\geq 0}\, v_\mu$$

where $\mu \leq \lambda$ means that $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all i (but μ and λ need not have the same number of cells). Then $\{v_\lambda\}$ must be one of the bases $\{s^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\lambda}\}$, $\{s^{\Box\Box}_{\lambda}\}$, or $\{s^{\Box}_{\lambda}\}$.

The structure constants $d_{\lambda\mu\nu}$ can be expressed in terms of the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ using the Newell-Littlewood formula.

Proposition 3.5. [**14**] [**16**]

(3.15) $$d_{\lambda\mu\nu} = \sum_{\rho,\sigma,\tau} c^{\mu}_{\rho\tau} c^{\nu}_{\sigma\tau} c^{\lambda}_{\rho\sigma}.$$

Example 3.6. For $\diamondsuit \in \{\Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \Box\Box\}$,

$$s^{\diamondsuit}_{(21)} s^{\diamondsuit}_{(3)} = s^{\diamondsuit}_{(2)} + s^{\diamondsuit}_{(11)} + s^{\diamondsuit}_{(4)} + 2\, s^{\diamondsuit}_{(31)} + s^{\diamondsuit}_{(22)} + s^{\diamondsuit}_{(211)} + s^{\diamondsuit}_{(51)} + s^{\diamondsuit}_{(42)} + s^{\diamondsuit}_{(411)} + s^{\diamondsuit}_{(321)}$$

The well-known transpose symmetry of Littlewood-Richardson coefficients $c^{\lambda^t}_{\mu^t\nu^t} = c^{\lambda}_{\mu\nu}$ immediately implies the following result.

Corollary 3.7. [**12**] $d_{\lambda^t\mu^t\nu^t} = d_{\lambda\mu\nu}$.

4. Bernstein operators for the bases $s^{\diamondsuit}_{\lambda}$ and determinants

4.1. The Schur basis. The Schur functions $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ are the unique family of symmetric functions, which for $\lambda = (r)$ are given by

(4.1) $$\sum_{r \in \mathbb{Z}} s_r[X] z^r = \Omega[zX]$$

and for $\lambda \in \mathcal{P}$ are given by the Jacobi-Trudi determinant

(4.2) $$s_\lambda[X] = \det |s_{\lambda_i - i + j}[X]|_{1 \leq i,j \leq \ell(\lambda)}$$

where $\ell(\lambda)$ is the number of parts of λ. One may define $s_\nu[X]$ for $\nu \in \mathbb{Z}^n$ using (4.1) and (4.2).

Bernstein's operators $\{B_r \mid r \in \mathbb{Z}\} \subset \mathrm{End}(\Lambda)$ are defined by

$$B(z) = \sum_{r\in\mathbb{Z}} B_r z^r = \Omega[zX]\Omega[-z^*X]^\perp \tag{4.3}$$

where $z^* = 1/z$. For $\nu \in \mathbb{Z}^n$, define

$$B_\nu = B_{\nu_1} \circ B_{\nu_2} \circ \cdots \circ B_{\nu_n} \in \mathrm{End}(\Lambda).$$

It is well-known that

$$B_\nu 1 = s_\nu[X]. \tag{4.4}$$

4.2. Creating the bases $s^\Diamond$. For $\nu \in \mathbb{Z}^n$ and $Z = (z_1, z_2, \ldots, z_n)$, define $B^\Diamond_\nu \in \mathrm{End}(\Lambda)$ by

$$B^\Diamond(Z) = \sum_{\nu\in\mathbb{Z}^n} z^\nu B^\Diamond_\nu = i^\Diamond_\varnothing \circ B(Z) \circ i^\varnothing_\Diamond. \tag{4.5}$$

For $\nu \in \mathbb{Z}^n$ it follows from (4.4) and (3.7) that

$$B^\Diamond_{\nu_1} \cdots B^\Diamond_{\nu_n} 1 = B^\Diamond_\nu 1 = s^\Diamond_\nu[X]. \tag{4.6}$$

The operator $B^\Diamond(Z)$ has the following plethystic formula.

Proposition 4.1. *For $\Diamond \in \{\square, \boxminus, \boxbar\}$,*

$$B^\Diamond(Z) = R(Z)\Omega[-f_\Diamond[Z]]\Omega[ZX]\Omega[-(Z+Z^*)X]^\perp, \tag{4.7}$$

where $Z^* = z_1^* + z_2^* + \cdots + z_n^*$ and

$$R(Z) = \prod_{1\le i<j\le n} (1 - z_j/z_i).$$

It follows that

$$B^\square(Z) = \Omega[-Z]B^\boxminus(Z) \tag{4.8}$$

$$B^\boxbar(Z) = \Omega[-(1+\varepsilon)Z]B^\boxminus(Z) \tag{4.9}$$

4.3. Determinantal formulae. Recall that the Schur functions satisfy the Jacobi-Trudi identity (4.2). The other three bases satisfy a common determinantal formula due to Weyl for $s^\boxminus$ and $s^\boxbar$. See [**12**, Thm. 2.3.3].

Proposition 4.2. *For $\Diamond \in \{\square, \boxminus, \boxbar\}$ the basis $\{s^\Diamond_\lambda \mid \lambda \in \mathcal{P}\}$ of Λ is characterized by*

$$\begin{aligned} s^\boxminus_r &= s_r \\ s^\square_r &= s_r - s_{r-1} \\ s^\boxbar_r &= s_r - s_{r-2} \end{aligned} \tag{4.10}$$

for $r \in \mathbb{Z}$ and

$$s^\Diamond_\lambda = \frac{1}{2} \det \left| s^\Diamond_{\lambda_i - i + j} + s^\Diamond_{\lambda_i - i - j + 2} \right|_{1\le i,j\le \ell(\lambda)} \tag{4.11}$$

5. Hall-Littlewood symmetric functions and analogues

5.1. Deformed Schur basis. Define

$$\widetilde{B}(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu \widetilde{B}_\nu \tag{5.1}$$

where $\widetilde{B}_\nu$ is the t-analogue of B_ν given by equation (2.7). This is the "parabolic modified" analogue of Jing's Hall-Littlewood creation operator. It was studied in [**22**], where it is denoted by H^t_ν. It is given by

$$\widetilde{B}(Z) = R(Z)\Omega[ZX]\Omega[(t-1)Z^*X]^\perp. \tag{5.2}$$

Let $Z^{(1)}, \dots, Z^{(L)}$ be a family of finite ordered alphabets and R_1 through R_L partitions such that the number of parts of R_j is equal to the number of letters in $Z^{(j)}$ for all j. Define the symmetric functions $\mathbb{B}_R[X;t]$ and polynomials $c_{\lambda;R}(t)$ by

$$\widetilde{B}_{R_1} \cdots \widetilde{B}_{R_L} 1 = \mathbb{B}_R[X;t] = \sum_\lambda s_\lambda[X] c_{\lambda;R}(t). \tag{5.3}$$

The $c_{\lambda;R}(t)$ are the generalized Kostka polynomials of [**21**], as proved in [**22**].

By (2.9) and (4.4) we have

$$\mathbb{B}_R[X;0] = B_{R_1} \cdots B_{R_L} 1 = s_{(R_1,\dots,R_L)}[X] \tag{5.4}$$

where $(R_1, \dots, R_L)$ denotes the sequence of integers obtained by juxtaposing the parts of the partitions R_j. By (2.10) and (4.4) we have

$$\mathbb{B}_R[X;1] = s_{R_1}[X] \cdots s_{R_L}[X]. \tag{5.5}$$

5.2. Deformed $s^\diamondsuit_\lambda$ basis. Let $\widetilde{B}^\diamondsuit_\nu$ be the t-analogue of $B^\diamondsuit_\nu$. For $\diamondsuit \in \{\square, \boxminus, \square\!\square\}$ define

$$\widetilde{B}^\diamondsuit(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu \widetilde{B}^\diamondsuit_\nu. \tag{5.6}$$

By (2.8), Proposition 4.1, (4.8) and (4.9),

$$\begin{aligned} \widetilde{B}^{\boxminus}(Z) &= R(Z)\Omega[-e_2[Z]]\Omega[ZX]\Omega[(Z+Z^*)(t-1)X]^\perp \\ \widetilde{B}^{\square}(Z) &= \widetilde{B}^{\boxminus}(Z)\Omega[-Z] \\ \widetilde{B}^{\square\!\square}(Z) &= \widetilde{B}^{\boxminus}(Z)\Omega[-(1+\varepsilon)Z]. \end{aligned} \tag{5.7}$$

For a sequence of partitions $R = (R_1, R_2, \dots, R_L)$, define the symmetric function $\mathbb{B}^\diamondsuit_R[X;t]$ and the polynomials $d^\diamondsuit_{\lambda R}(t)$ by

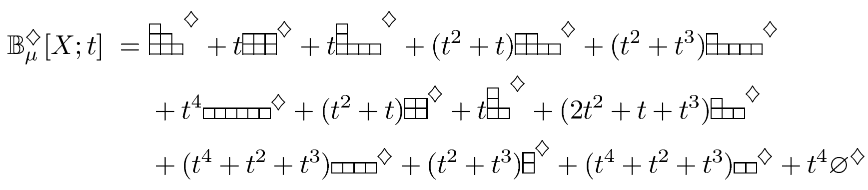
$$\mathbb{B}^\diamondsuit_R[X;t] = \widetilde{B}^\diamondsuit_{R_1} \widetilde{B}^\diamondsuit_{R_2} \cdots \widetilde{B}^\diamondsuit_{R_L} 1 = \sum_\lambda d^\diamondsuit_{\lambda R}(t) s^\diamondsuit_\lambda. \tag{5.8}$$

Theorem 5.1. *$d^\diamondsuit_{\lambda R}(t)$ is constant over $\diamondsuit \in \{\square, \boxminus, \square\!\square\}$.*

Let us call these polynomials $d_{\lambda R}(t)$. When R consists of single-rowed rectangles of sizes given by the partition μ, write $d_{\lambda\mu}(t)$ instead of $d_{\lambda R}(t)$.

Theorem 5.2. *$d_{\lambda\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

Example 5.3. Let $\mu = (3, 2, 1)$. For $\diamondsuit \in \{\square, \boxminus, \square\!\square\}$, we will represent the function $s^{\diamondsuit}_{\lambda}$ by the diagram for the partition λ superscripted by $\diamondsuit$. By Theorem 5.1 the expansion is independent of $\diamondsuit$.



6. Parabolic Hall-Littlewood operators and $\diamondsuit$-analogues

For each $\diamondsuit \in \{\varnothing, \square, \boxminus, \square\!\square\}$ we define a variant of the type A parabolic Hall-Littlewood creation operator $\widetilde{B}_{\nu}$. These will be the creation operators for the universal affine characters.

6.1. $\diamondsuit$-analogues of $\widetilde{B}_{\nu}$. Write $\widetilde{B}^{\diamondsuit}_{t^2}(Z)$ for $\widetilde{B}^{\diamondsuit}(Z)$ with t replaced by t^2. Let

$$H^{\diamondsuit}(Z) = \sum_{\nu \in \mathbb{Z}^k} z^{\nu} H^{\diamondsuit}_{\nu} = \Omega[f_{\diamondsuit}[tX] - f_{\diamondsuit}[X]]^{\perp} \widetilde{B}_{t^2}(Z) \Omega[f_{\diamondsuit}[X] - f_{\diamondsuit}[tX]]^{\perp}. \tag{6.1}$$

Proposition 6.1. *For* $\diamondsuit \in \{\varnothing, \square, \boxminus, \square\!\square\}$,

$$H^{\diamondsuit}(Z) = \Omega[f_{\diamondsuit}[tZ]] \widetilde{B}^{\diamondsuit}_{t^2}(Z). \tag{6.2}$$

6.2. The K polynomials. Let $R = (R_1, R_2, \ldots, R_L)$ be a sequence of partitions. For $\diamondsuit \in \{\varnothing, \square, \boxminus, \square\!\square\}$ define $\mathbb{H}^{\diamondsuit}_{R}[X; t]$ and $K^{\diamondsuit}_{\lambda;R}(t)$ by

$$\mathbb{H}^{\diamondsuit}_{R}[X; t] = \sum_{\lambda} K^{\diamondsuit}_{\lambda;R}(t)\, s^{\diamondsuit}_{\lambda}[X] = H^{\diamondsuit}_{R_1} H^{\diamondsuit}_{R_2} \cdots H^{\diamondsuit}_{R_L} 1. \tag{6.3}$$

Using (5.4) and (5.5) one obtains the specializations at $t = 0$ and $t = 1$, for all $\diamondsuit$.

$$\mathbb{H}^{\diamondsuit}_{R}[X; 0] = s^{\diamondsuit}_{(R_1, R_2, \ldots, R_L)}[X] \tag{6.4}$$

$$\mathbb{H}^{\diamondsuit}_{R}[X; 1] = s_{R_1}[X] s_{R_2}[X] \cdots s_{R_L}[X]. \tag{6.5}$$

Remark 6.2. For any $\diamondsuit$, $\mathbb{H}^{\diamondsuit}_{R}[X; t]$ is a t-deformation of the product of Schur functions, rather than $s^{\diamondsuit}_{R_i}$. Note also that $K^{\varnothing}_{\lambda;R}(t) = c_{\lambda;R}(t^2)$; see (5.3).

6.3. $K^{\diamondsuit}$ in terms of $K^{\varnothing}$. Let $|R| = \sum_i |R_i|$. Observe that

$$\mathbb{H}^{\diamondsuit}_{R}[X; t] = \Omega[f_{\diamondsuit}[tX] - f_{\diamondsuit}[X]]^{\perp} \mathbb{H}^{\varnothing}_{R}[X; t].$$

It follows that for $\diamondsuit \in \{\varnothing, \square, \boxminus, \square\!\square\}$,

$$K^{\diamondsuit}_{\lambda;R}(t) = t^{|R| - |\lambda|} \sum_{\substack{\tau \in \mathcal{P} \\ |\tau| = |R|}} K^{\varnothing}_{\tau;R}(t) \sum_{\substack{\mu \in \mathcal{P}^{\diamondsuit} \\ |\mu| = |R| - |\lambda|}} c^{\tau}_{\lambda\mu}. \tag{6.6}$$

Example 6.3.

$$
\begin{aligned}
H^{\square\square}_{(32)}[X;t] &= s^{\square\square}_{(32)} + t s^{\square\square}_{(41)} + t^2 s^{\square\square}_{(5)} + t^2\left(1+t+t^2\right) s^{\square\square}_{(3)} \\
&\quad + t^2(1+t)\, s^{\square\square}_{(21)} + t^4\left(1+t+t^2\right) s^{\square\square}_{(1)} \\
H^{\substack{\square\\\square}}_{(32)}[X;t] &= s^{\substack{\square\\\square}}_{(32)} + t s^{\substack{\square\\\square}}_{(41)} + t^2 s^{\substack{\square\\\square}}_{(5)} + t^3 s^{\substack{\square\\\square}}_{(3)} + t^2 s^{\substack{\square\\\square}}_{(21)} + t^4 s^{\substack{\square\\\square}}_{(1)} \\
H^{\square}_{(32)}[X;t] &= s^{\square}_{(32)} + t s^{\square}_{(41)} + t^2 s^{\square}_{(5)} + t s^{\square}_{(22)} + \left(t+t^2\right) s^{\square}_{(31)} \\
&\quad + \left(t^2+t^3\right) s^{\square}_{(4)} + \left(2\,t^2+t^3\right) s^{\square}_{(21)} + \left(t^2+2\,t^3+t^4\right) s^{\square}_{(3)} \\
&\quad + t^3 s^{\square}_{(11)} + \left(t^3+t^4\right) s^{\square}_{(2)} + t^4 s^{\square}_{(1)}
\end{aligned}
$$

6.4. Level-rank (transpose) duality. Let $||R|| = \sum_{i<j} |R_i \cap R_j|$, $\varnothing^t = \varnothing$, $\square^t = \square$, $\substack{\square\\\square}^{\,t} = \square\square$, and $\square\square^{\,t} = \substack{\square\\\square}$.

Proposition 6.4. *Let R be a dominant sequence of rectangles (that is, one whose widths weakly decrease) and R' a dominant rearrangement of R^t. Then for all partitions λ,*

$$K^{\diamond^t}_{\lambda^t;R'}(t) = t^{2(||R||+|R|-|\lambda|)} K^{\diamond}_{\lambda;R}(t^{-1}). \tag{6.7}$$

6.5. Connection between $\mathbb{B}^\diamond$ and $\mathbb{H}^\diamond$.

Proposition 6.5. *Let R be the sequence of single-rowed partitions of sizes given by the partition μ. Then*

$$\mathbb{H}^{\substack{\square\\\square}}_R[X;t] = \mathbb{B}^{\substack{\square\\\square}}_R[X;t^2] \tag{6.8}$$

$$K^{\substack{\square\\\square}}_{\lambda;R}(t) = d_{\lambda R}(t^2). \tag{6.9}$$

7. Universal affine characters and $X = M = K$

Let $\mathfrak{g}$ be any affine Lie algebra, say, of type $X_N^{(r)}$ [**6**], with canonical simple Lie subalgebra $\overline{\mathfrak{g}}$ of rank n, and let $U_q'(\mathfrak{g})$ and $U_q(\overline{\mathfrak{g}})$ the corresponding quantized enveloping algebras. Motivated by the work of [**8**] on finite-dimensional modules over Yangians, the papers [**2**] and [**1**] conjecture the existence of finite-dimensional $U_q'(\mathfrak{g})$-modules called Kirillov-Reshetikhin (KR) modules. In type A the restriction of a KR module to $U_q(\overline{\mathfrak{g}})$ has character given by a Schur function indexed by a rectangle. In general one can think of the KR-modules as being indexed by rectangles, but the restriction of a KR module to $U_q(\overline{\mathfrak{g}})$ is generally reducible. The KR modules are conjectured to have a natural grading that is constant on $U_q(\overline{\mathfrak{g}})$-components.

The above two papers propose the $X = M$ conjecture, which give two ways to compute the graded multiplicity of a $U_q(\overline{\mathfrak{g}})$-irreducible in a tensor product of KR modules over $U_q'(\mathfrak{g})$. The symbols X and M represent two families of polynomials that are indexed by a pair (R, λ) where R is a sequence of rectangles which corresponds to a tensor product of KR modules, and λ is a partition which corresponds to a dominant weight of $\overline{\mathfrak{g}}$. The X formula can be stated entirely in terms of the affine crystal graph of a tensor product of KR modules; its definition depends on the existence of KR modules and some of their conjectured properties. The M formula is a q-analogue of the fermionic formula in [**8**], but extended to all affine root systems. It is well-defined and independent of the existence of KR modules. See [**2**] and [**1**] for details on this remarkable conjecture.

The $X = M$ conjecture is only completely proved for type A [**10**] and in this case the polynomials are essentially the generalized Kostka coefficients $c_{\lambda;R}(t)$ of equation (5.3). In general the KR modules have not even been constructed, although strong additional hints on their structure have been provided by Kashiwara [**4**] [**5**].

Proposition 7.1. *Consider a nonexceptional family* $\{X_N^{(r)}\}$ *of affine root systems. There is a well-defined limiting polynomial*

$$\lim_{n\to\infty} \overline{M}_{R,\lambda}(t) \tag{7.1}$$

as the rank n goes to infinity. It depends only on R, λ, *and the affine family of* $X_N^{(r)}$. *Moreover, there are only four distinct families of such polynomials, which shall be named as follows.*

(1) *For* $A_n^{(1)}$: $\overline{M}^{\varnothing}_{R,\lambda}(t)$.

(2) *For* $B_n^{(1)}$, $D_n^{(1)}$, *and* $A_{2n-1}^{(2)}$: $\overline{M}^{\genfrac{}{}{0pt}{}{\square}{\square}}_{R,\lambda}(t)$.

(3) *For* $C_n^{(1)}$ *and* $A_{2n}^{(2)\dagger}$: $\overline{M}^{\square\!\square}_{R,\lambda}(t)$.

(4) *For* $D_{n+1}^{(2)}$ *and* $A_{2n}^{(2)}$: $\overline{M}^{\square}_{R,\lambda}(t)$.

The families are grouped according to the decomposition of a KR module upon restriction to $U_q(\overline{\mathfrak{g}})$; see the appendices of [**2**] [**1**]. We define the **universal affine character** associated to $\diamondsuit$ and R to be the symmetric function $\sum_\lambda \overline{M}^{\diamondsuit}_{R,\lambda}(t) s^{\diamondsuit}_\lambda$; it corresponds to the graded character of the tensor product of KR modules indexed by R in the large rank limit.

Conjecture 7.2. *For* R *a dominant sequence of rectangles and all* $\diamondsuit \in \{\varnothing, \square, \genfrac{}{}{0pt}{}{\square}{\square}, \square\!\square\}$,

$$K^{\diamondsuit}_{\lambda;R}(t) = \overline{M}^{\diamondsuit^t}_{R^t,\lambda^t}(t^{2/\epsilon}) \tag{7.2}$$

where $\epsilon = 1$ *except for* $\diamondsuit = \square$, *in which case* $\epsilon = 2$.

At $t = 1$ this was essentially known [**7**]. However the formulae for the powers of t occurring in the affine characters given either by X or the M formulae, do not at all suggest such a simple relationship. Perhaps the virtual crystal methods of [**17**] can be used to prove Conjecture 7.2.

Equation (7.2) holds for $\diamondsuit = \varnothing$ by combining [**10**] [**18**] [**19**] [**20**] [**22**]. It also holds for a single rectangle in all nonexceptional affine types; see [**1**, Appendix A] and [**2**, Appendix A].

Observe that by combining Conjecture 7.2 and Proposition 6.4 one obtains the following conjecture.

Conjecture 7.3.

$$\overline{M}^{\diamondsuit}_{R;\lambda}(t) = t^{\epsilon(||R||+|R|-|\lambda|)} \overline{M}^{\diamondsuit^t}_{R^t,\lambda^t}(t^{-1}). \tag{7.3}$$

This was proved in [**9**] via a direct bijection for $\diamondsuit = \varnothing$. This is a striking conjecture as it relates the fermionic formulae of different types. This kind of relation is not apparent from the structure of the fermionic formulae.

References

[1] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Z. Tsuboi, Paths, Crystals and Fermionic Formulae, MathPhys odyssey, 2001, 205–272, Prog. Math. Phys., **23**, Birkhäuser Boston, Boston, MA, 2002.

[2] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, Remarks on fermionic formula, in Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 243–291, Contemp. Math., **248**, Amer. Math. Soc., Providence, RI, 1999.

[3] N. Jing, Vertex operators and Hall-Littlewood symmetric functions. Adv. Math. **87** (1991) 226–248.

[4] M. Kashiwara, On level-zero representation of quantized affine algebras, Duke Math. J. **112** (2002), 117–195.

[5] M. Kashiwara, Level zero fundamental representations over quantized affine algebras and Demazure modules, preprint math.QA/0309142.

[6] V. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, 1990.

[7] M. Kleber, Embeddings of Schur functions into types $B/C/D$, J. Algebra **247** (2002) 452–466.

[8] A. N. Kirillov and N. Yu. Reshetikhin, Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras, J. Soviet Math. **52** (1990) 3156–3164.

[9] A. N. Kirillov and M. Shimozono, A generalization of the Kostka-Foulkes polynomials, J. Algebraic Combin. **15** (2002) 27–69.

[10] A. N. Kirillov, A. Schilling, and M. Shimozono, A bijection between Littlewood-Richardson tableaux and rigged configurations, Selecta Math. (N.S.) **8** (2002) 67–135.

[11] K. Koike, Representations of spinor groups and the difference characters of $SO(2n)$, Advances in Mathematics, Vol. 128, 1997, pp.58–62

[12] K. Koike and I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n, J. Algebra **107** (1987) 466–511.

[13] D. E. Littlewood, The theory of group characters and matrix representation of groups, second edition, Clarendon Press, Oxford, 1950.

[14] D. E. Littlewood, Canad. J. Math. **10** (1958) 17–32.

[15] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[16] M. J. Newell, Proc. Roy. Irish Acad. Sect. A **54** (1951) 153–163.

[17] M. Okado, A. Schilling, and M. Shimozono, Virtual crystals and fermionic formulas for types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$, Represent. Theory **7** (2003), 101–163.

[18] A. Schilling and S. O. Warnaar, Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials, Comm. Math. Phys. **202** (1999) 359–401.

[19] M. Shimozono, A cyclage poset structure for Littlewood-Richardson tableaux, European J. Combin. **22** (2001) 365–393.

[20] M. Shimozono, Affine Type A Crystal Structure on Tensor Products of Rectangles, Demazure characters, and Nilpotent Varieties, J. Algebraic Combin. **15** (2002) 151–187.

[21] M. Shimozono and J. Weyman, Graded characters of modules supported in the closure of a nilpotent conjugacy class, European J. Combin. **21** (2000) 257–288.

[22] M. Shimozono and M. Zabrocki, Hall-Littlewood vertex operators and generalized Kostka polynomials. Adv. Math. **158** (2001) 66–85.

[23] M. Zabrocki, q-Analogs of symmetric function operators, Disc. Math. **256** (2002) 831–853.

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123 USA
E-mail address: `mshimo@math.vt.edu`

Dept. of Math and Stats, York University, Toronto, Canada, M3J 1P3
E-mail address: `zabrocki@mathstat.yorku.ca`