Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

The Phase Transition for Random Subgraphs of the n-cube

Gordon Slade

Abstract. *We describe recent results, obtained in collaborations with C. Borgs, J.T. Chayes, R. van der Hofstad and J. Spencer, which provide a detailed description of the phase transition for random subgraphs of the n-cube.*

Résumé. *Nous présentons des résultats récents qui donnent une description détaillée de la transition de phase des sous-graphes aléatoires du n-cube. Ces résultats sont obtenus en collaboration avec C. Borgs, J.T. Chayes, R. van der Hofstad et J. Spencer.*

Extended Abstract

The phase transition for random subgraphs of the complete graph, or the *random graph* for short, was first studied by Erdős and Rényi [**7**], and has been analyzed in considerable detail since then [**2, 11**]. Let K_V denote the complete graph on V vertices, so that there is an edge joining each of the $\binom{V}{2}$ pairs of vertices. In the random graph, edges of the complete graph are independently occupied with probability p and vacant with probability $1-p$, as in the bond percolation model. The occupied edges naturally determine connected components, called *clusters*. There is a phase transition as p is varied, in the sense that there is an abrupt change in the number of vertices $|\mathcal{C}_{\max}|$ in a cluster $\mathcal{C}_{\max}$ of maximal size, as p is varied through the critical value $p_c = \frac{1}{V}$.

We will say that a sequence of events E_V occurs *with high probability*, denoted w.h.p., if $\mathbb{P}(E_V) \to 1$ as $V \to \infty$. The basic fact of the phase transition is that when p is scaled as $(1+\epsilon)V^{-1}$, there is a phase transition at $\epsilon = 0$ in the sense that w.h.p.

$$|\mathcal{C}_{\max}| = \begin{cases} \Theta(\log V) & \text{for } \epsilon < 0, \\ \Theta(V^{2/3}) & \text{for } \epsilon = 0, \\ \Theta(V) & \text{for } \epsilon > 0. \end{cases} \tag{1}$$

The asymptotic results of (1) are valid for fixed ϵ, independent of V. These results have been substantially strengthened to show that there is a scaling window of width $V^{-1/3}$, in the sense that if $p = (1+\Lambda_V V^{-1/3})V^{-1}$, then w.h.p.

$$|\mathcal{C}_{\max}| \begin{cases} \mathbf{l} V^{2/3} & \text{for } \Lambda_V \to -\infty, \\ = \Theta(V^{2/3}) & \text{for } \Lambda_V \text{ uniformly bounded in } V, \\ \gg V^{2/3} & \text{for } \Lambda_V \to -\infty. \end{cases} \tag{2}$$

Here, we are using the notation $f(V)\mathbf{l}g(V)$ to mean that $f(V)/g(V) \to 0$ as $V \to \infty$, while $f(V) \gg g(v)$ means that $f(V)/g(V) \to \infty$ as $V \to \infty$. A great deal more is known, and can be found in [**2, 11**].

1991 *Mathematics Subject Classification.* Primary 05C80, 60K35; Secondary 82B43.
Key words and phrases. random graph, phase transition, n-cube, percolation.
Research supported by an NSERC Discovery Grant.

Our goal is to understand how these results can be extended to apply to the n-cube $\mathbb{Q}_n$. This graph has vertex set $\{0,1\}^n$, with an edge joining pairs of vertices which differ in exactly one coordinate. It has $V = 2^n$ vertices, each of degree n. Edges are again independently occupied with probability p. If $p = (1+\epsilon)n^{-1}$ with $\epsilon < 0$ independent of n, then $|\mathcal{C}_{\max}|$ turns out to be $\Theta(\log V)$. On the other hand, for $\epsilon > 0$ independent of n, it was shown in [**1**] that $|\mathcal{C}_{\max}| = \Theta(V)$. Thus, a transition takes place at the value $\frac{1}{n}$ of p.

In [**3**], the results of [**1**] were extended to show that w.h.p.

$$|\mathcal{C}_{\max}| = \begin{cases} 2\epsilon^{-2}\log V\big(1+o(1)\big) & \text{for } \epsilon \le -(\log n)^2(\log\log n)^{-1}n^{-1/2}, \\ 2\epsilon V & \text{for } \epsilon \ge 60(\log n)^3 n^{-1}. \end{cases} \tag{3}$$

Thus, ϵ as in the first line of (3) gives a subcritical p, whereas in the second line p is supercritical. The gap between these ranges of p is much bigger than the $V^{-1/3}$ (here $2^{-n/3}$) seen above as the size of the scaling window for the complete graph.

The following result from [**9, 10**], which builds on results of [**4, 5, 6**], gives bounds for ϵ on an arbitrary scale that is polynomial in n^{-1}.

Theorem 1.1. *For the n-cube, there exists a sequence of rational numbers $a_1, a_2, a_3, \ldots$, with $a_1 = a_2 = 1$ and $a_3 = \frac{7}{2}$, such that for any $M \ge 1$, for $p_c^{(M)} = \sum_{i=1}^M a_i n^{-i}$, and for $p = p_c^{(M)} + \delta n^{-M}$ with δ independent of n, the following bounds hold w.h.p.:*

$$|\mathcal{C}_{\max}| \begin{cases} \le 2(\log 2)\delta^{-2}n^{2M-1}[1+o(1)] & \text{for } \delta < 0, \\ \ge \text{const}\,\delta n^{1-M}2^n & \text{for } \delta > 0. \end{cases} \tag{4}$$

More is proved in [**9, 10**], but (4) is highlighted here because it shows subcritical behaviour for negative δ and supercritical behaviour for positive δ. Theorem 1.1 suggests that the critical value for the n-cube should be $\sum_{i=1}^\infty a_i n^{-i}$, but circumstantial evidence leads us to conjecture that this infinite series is divergent (see [**8**] for a general discussion of such issues). If the conjecture is correct, the critical value cannot be defined in this way. This difficulty was bypassed in [**4**], where the critical value for the phase transition on a “high-dimensional” finite graph $\mathbb{G}$ was defined to be the value $p_c = p_c(\mathbb{G}, \lambda)$ for which

$$\chi(p_c) = \lambda V^{1/3}, \tag{5}$$

where $\chi(p)$ is by definition the expected number of vertices in the component of an arbitrary fixed vertex (e.g., the origin of the n-cube), V is the number of vertices in the graph $\mathbb{G}$, and λ is a fixed positive number. This definition is by analogy with the random graph, where it is known that $\chi(1/V) = \Theta(V^{1/3})$. The parameter λ allows for some flexibility, associated with the fact that criticality corresponds to a scaling window of finite width and not to a single point. The following theorem is proved in [**10**], building on results in [**4, 5, 6**].

Theorem 1.1. *For the n-cube, let $M \ge 1$, fix constants c, c' (independent of n but possibly depending on M), and choose p such that $\chi(p) \in [cn^M, c'n^{-2M}2^n]$. Then for a_i given by Theorem 1.1,*

$$p = \sum_{i=1}^M a_i n^{-i} + O(n^{-M-1}) \qquad \text{as } n \to \infty. \tag{6}$$

The constant in the error term depends on M, c, c', but does not depend otherwise on p.

Fix $\lambda > 0$ independent of n. Then $\chi(p_c(\mathbb{Q}_n, \lambda)) = \lambda 2^{n/3}$ is in an interval $[cn^M, c'n^{-2M}2^n]$ for every M, with c, c' dependent on M and λ. By Theorem 1.1, (6) holds for $p = p_c(\mathbb{Q}_n, \lambda)$, for *every* fixed choice of λ and for *every* M. Thus,

$$p_c(\mathbb{Q}_n, \lambda) \sim \sum_{i=1}^\infty a_i n^{-i} \tag{7}$$

is an asymptotic expansion for $p_c(\mathbb{Q}_n, \lambda)$, for every positive λ.

By analogy with the complete graph, we would like to prove that the critical scaling window for the n-cube has size $V^{-1/3} = 2^{-n/3}$. This exponential scale is not accessible using the asymptotic expansion of Theorems 1.1–1.1. The following result from [**6**], which builds on the results of [**4, 5**], does not quite prove that the scaling window has size $2^{-n/3}$, but does show that it is smaller than any inverse power of n.

Theorem 1.2. *For the n-cube, let $V = 2^n$, let λ_0 be a fixed sufficiently small constant, and let $p = p_c(\mathbb{Q}_n, \lambda_0) + \epsilon n^{-1}$. If $\epsilon < 0$ and $\epsilon V^{1/3} \to -\infty$ as $V \to \infty$, then w.h.p.*

$$|\mathcal{C}_{\max}| \leq 2\epsilon^{-2} \log V\big(1 + o(1)\big). \tag{8}$$

If $|\epsilon| V^{1/3} \leq B$ for some constant B, then there is a constant b (depending on B and λ_0) such that, for any $\omega \geq 1$,

$$\mathbb{P}\Big(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega}. \tag{9}$$

Finally, there are positive constants c, c_1 such that if $e^{-cn^{1/3}} \leq \epsilon \leq 1$ then w.h.p.

$$|\mathcal{C}_{\max}| \geq c_1 \epsilon V. \tag{10}$$

Additional estimates can be found in [**6**], but those in Theorem 1.2 show that the critical window in ϵ is of size $V^{-1/3} = 2^{-n/3}$ on the subcritical side of $p_c(n)$, and has at most size $e^{-cn^{1/3}}$ on the supercritical side. We expect that the window actually has size $V^{-1/3} = 2^{-n/3}$ on both sides of $p_c(n)$, and that, more generally, the scaling window in high-dimensional graphs has size $V^{-1/3}$.

An interesting consequence of the above theorems is that the approximate critical values $p_c^{(M)} = \sum_{i=1}^{M} a_i n^{-i}$ will lie outside the critical window around $p_c(\mathbb{Q}_n, \lambda)$, for *every* M, unless the sequence a_i is eventually zero and the asymptotic series is actually a polynomial in n^{-1}. We expect the series to be divergent, and not a polynomial. We regard the definition (5) as superior to any definition based on the asymptotic expansion. In particular, the coefficients a_i are obtained from an asymptotic expansion for $p_c(\mathbb{Q}_n, \lambda)$, so the latter contains all information contained in the former.

There are several ingredients in the proof of these theorems, most of which are more familiar in mathematical physics than in combinatorics. These include differential inequalities, the triangle condition, finite-size scaling ideas, and the lace expansion. The method of [**1**], which we call sprinkling, is used in conjunction with estimates obtained via these other methods to prove the lower bound (10). In [**4, 5**], other graphs besides the n-cube are also treated, including finite periodic approximations to $\mathbb{Z}^n$ for n large, with less complete results.

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