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# Tamari Lattices and Non-crossing Partitions in Types $B$ and $D$

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**Abstract.** *The usual, or type $A_n$, Tamari lattice is a partial order on $T_n^A$, the triangulations of an $(n+3)$-gon. We define a partial order on $T_n^B$, the set of centrally symmetric triangulations of a $(2n+2)$-gon. We show that it is a lattice, and that it shares certain other nice properties of the $A_n$ Tamari lattice; it can therefore be considered the $B_n$ Tamari lattice.*

*We define a bijection between $T_n^B$ and the non-crossing partitions of type $B_n$ defined by Reiner. Reiner has also defined the noncrossing partitions of type $D_n$ as a subset of those of type $B_n$. We show that the elements of $T_n^B$ which correspond to the noncrossing partitions of type $D_n$ form a lattice under the order induced from their inclusion in $T_n^B$, which therefore can be considered the $D_n$ Tamari lattice.*

*This is a somewhat abridged version of a longer paper with the same title, available at* `www.arxiv.org/math.CO/0311334`.

**Résumé.** *Le treillis de Tamari standard (de type $A_n$) est un ordre partiel sur $T_n^A$, les triangulations d'un $(n+3)$-gone. Nous définissons un ordre partiel sur $T_n^B$, l'ensemble des triangulations centralement symétriques d'un $(2n+2)$-gone. Nous montrons que c'est un treillis et qu'il possède aussi d'autres propriétés intéressantes similaires au treillis de Tamari de type $A_n$. Ce treillis peut donc être considéré comme le treillis de Tamari de type $B_n$.*

*Nous définissons une bijection entre $T_n^B$ et les partages non-croisés de type $B_n$ définis par Reiner. Reiner a aussi défini les partages non-croisés de type $D_n$ comme un sous-ensemble de ceux de type $B_n$. Nous montrons que les éléments de $T_n^B$ qui correspondent aux partages non-croisés de type $D_n$ forment un treillis sous l'ordre induit par leur inclusion dans $T_n^B$, qui peut donc être considéré comme le treillis de Tamari de type $D_n$.*

*Cet exposé est une version plus courte d'un exposé du même titre qui est disponible sur* `www.arxiv.org/math.CO/0311334`.

## 1. Introduction

Let $T_n^A$ denote the set of triangulations of an $(n+3)$-gon. By a triangulation of a polygon, we mean a division of it into triangles by connecting pairs of its vertices with straight lines which do not cross in the interior of the polygon. Conventionally, we will number the vertices of our $(n+3)$-gon clockwise from 0 to $n+2$, with a long top edge connecting vertices 0 and $n+2$. An example triangulation is shown in Figure 1 below.

Let $S \in T_n^A$. As in [**Lee**], we colour the chords of $S$ red and green, as follows. A chord $C$ of $S$ is the diagonal of a quadrilateral $Q(C)$ in $S$. If $C$ is the diagonal of $Q(C)$ which is connected to the vertex with

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the largest label, we colour it green; otherwise we colour it red. In Figure 1, the red chords are indicated by thick lines.

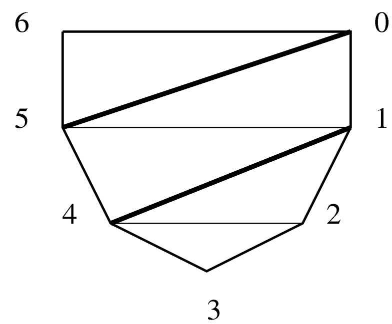


FIGURE 1

We partially order $T_n^A$ by giving covering relations: $T$ covers $S$ if they coincide except that some green chord in $S$ has been replaced by the other diagonal of $Q(C)$ (which is red). This is one way to construct the Tamari lattice, which was introduced in [**Tam**] and which has since been studied by several authors (see [**HT, Pal, BW2**]).

Although this is not clear from the elementary description given here, the Tamari lattice should be thought of as being associated to type $A$. One indication of why can be found in [**BW2**], where it is shown that $T_n^A$ is a quotient of the weak order on the symmetric group $S_{n+1}$ (the type $A_n$ reflection group). Another reason is that the elements of $T_n^A$ index clusters in the $A_n$ root system (see [**FZ**]). Once one has the idea that the Tamari lattice is type $A$, it is natural to ask whether there exist Tamari lattices in other types.

For reasons which we shall go into further below, the $B_n$ triangulations, denoted $T_n^B$, are the triangulations of a $(2n+2)$-gon which are fixed under a half-turn rotation. These triangulations have already appeared in the work of Simion [**Sim**], and in that of Fomin and Zelevinsky [**FZ**] where they index the clusters in the $B_n$ root system. One goal of our paper is to define a partial order on $T_n^B$ and to prove that it is a lattice. The definition is analogous to that already given for the $A_n$ Tamari lattice: it is given in terms of covering relations, and $S$ covers $T$ in $T_n^B$ if $S$ is obtained from $T$ by replacing a symmetric pair of chords $C, \bar{C}$ by the other diagonals of $Q(C), Q(\bar{C})$. The details of the definition are a trifle complicated, so we defer them for the main body of the paper. This definition was arrived at independently and more or less simultaneously by Reading [**Rea**]. He has also proved that $T_n^B$ is a lattice, using a rather different approach. Two alternative partial orders on $T_n^B$ with similar (but somewhat easier to describe) covering relations were suggested by Simion [**Sim**]; one is studied further in [**San**]. But since neither of these is a lattice, neither is completely satisfying as a type $B$ analogue of the usual Tamari lattice.

What objects should be considered the $D_n$ triangulations is not as settled as in type $B_n$, although certain information is known, such as the desired cardinality. One candidate is provided in [**FZ**], and used there to index the clusters in the $D_n$ root system. We follow a different approach. First, we find a bijection between $B_n$ triangulations and $B_n$ non-crossing partitions, which were introduced by Reiner [**Rei**]. Motivated by Reiner's definition of non-crossing partitions for type $D_n$ as a subset of those for type $B_n$ (which has the desired cardinality), we take our $D_n$ triangulations $T_n^D$ to be the corresponding subset of $T_n^B$. (It is not clear whether there is any natural bijection between our $T_n^D$ and the $D_n$ triangulations of [**FZ**].) Our second chief result is to show that the order induced on $T_n^D$ from its inclusion in $T_n^B$ gives it a lattice structure also. In fact, our approach to type $D_n$ works for any of the interpolating pseudo-types indexing hyperplane arrangements between $B_n$ and $D_n$. (We shall recall the definition of these pseudo-types below.)

We show that $T_n^B$ and $T_n^D$ have an unrefinable chain of left modular elements, a property also shared by the usual Tamari lattice [**BS**]. One consequence of this, due to Liu [**Liu**], is that these lattices have

EL-labellings. Using these labellings, we show that, as for the usual Tamari lattice (see [**BW2**]), the order complex of any interval is either homotopic to a sphere or contractible.

From the results in this paper one could proceed in two directions. One direction is to consider the existence of Tamari lattices in all Coxeter types. The other direction is to investigate further the lattices defined here, to see how many more of the properties of the usual Tamari lattice carry over.

## 2. Type $B$ Triangulations

Recall that the $B_n$ Weyl group consists of signed permutations of $n$. We can think of these as permutations of $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ fixed under interchanging $i$ and $\bar{i}$ for all $1 \leq i \leq n$. By analogy, $B_n$ triangulations, $T_n^B$, are defined to be type $A$ triangulations of a $(2n+2)$-gon fixed under a half-turn. There is general consensus that this is the correct definition of $B_n$ triangulation: see [**Sim, FZ**].

We number the vertices of our standard $(2n+2)$-gon counterclockwise from $n+1$ to 1 and then from $\overline{n+1}$ to $\bar{1}$. A typical triangulation is shown in Figure 2.

We will frequently distinguish two types of chords: *pure* and *mixed*. A chord is pure if it connects two barred vertices or two unbarred vertices; otherwise it is mixed. For $S \in T_n^B$, consider a chord $C$ of $S$. The chord $C$ is the diagonal of a quadrilateral, which we denote $Q(C)$. If $C$ is pure, then we colour it red if $Q(C)$ contains another vertex of the same type as those of $C$ whose label is higher, and green otherwise. If it is mixed, we colour it red if $Q(C)$ contains an unbarred vertex whose label is higher than the label of the unbarred vertex of $C$, or a barred vertex whose label is higher than the label of the barred vertex of $C$. Otherwise we colour it green. In Figure 2, the red chords are indicated by thick lines.

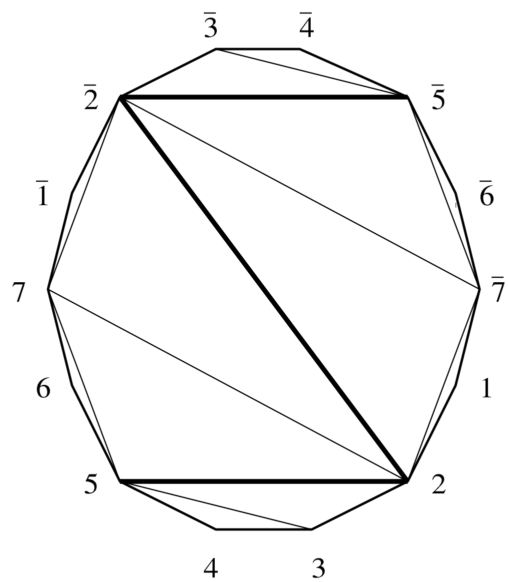


FIGURE 2

For $C$ a chord, we write $\bar{C}$ for its symmetric partner (that is to say, the image of $C$ under a half turn). Observe that $C$ and $\bar{C}$ are assigned the same colour.

**Lemma 2.1.** *Consider a chord $C$ in a triangulation $S$. Let $S'$ be the triangulation obtained by replacing $C$ by $C'$, the other diagonal of $Q(C)$, and also replacing $\bar{C}$ by $\bar{C}'$. Then the colours of $C$ in $S$ and $C'$ in $S'$ are opposite.*

We can now state the first main theorem of this paper (which, as was already remarked, was arrived at and proved independently and more or less simultaneously by Reading [**Rea**]).

**Theorem 2.2.** *There is a lattice structure on $T_n^B$ whose covering relations are given by $S \lessdot T$ iff $S$ and $T$ differ in that green chords $C$, $\bar{C}$ in $S$ are replaced in $T$ by the other diagonals of $Q(C)$ and $Q(\bar{C})$ (which will be red). Note that we allow $C = \bar{C}$ (i.e. $C$ being a diameter). We call this lattice the $B_n$ Tamari lattice.*

The first ingredient in our proof of Theorem 1 is some further analysis of the red and green chords of triangulations.

Fix a triangulation $S$. For $1 \le i \le n$, consider those chords of $S$ which are attached to $i$ and let $C_i(S)$ be the first of these encountered in searching clockwise starting at $\bar{1}$. If none is encountered before reaching $i-1$, then $C_i(S)$ is not defined. Let $R(S)$ be the set of these chords, together with their symmetric partners.

**Lemma 2.3.** *For any triangulation $S$, the chords in $R(S)$ are red, the other chords of $S$ are green, and $S$ is the unique triangulation whose red chords are exactly $R(S)$.*

## 3. Bracket Vectors in types $A$ and $B$

We briefly recall some well-known facts about the type $A$ Tamari lattice, which serve as motivation for our work in type $B$.

Any triangulation $S \in T_n^A$ has a bracket vector $r(S) = (r_1(S), \ldots, r_{n+1}(S))$. Let $c_i(S)$ be the least vertex attached to $i$. Then $r_i(S) = i - 1 - c_i(S)$. For example, the bracket vector of the triangulation shown in Figure 1 is (0,0,0,2,4). This approach to representing elements of the Tamari lattice goes back to [**HT**], though we make some different choices of convention here.

**Proposition 3.1.** *An $(n+1)$-tuple of positive integers is a bracket vector for some triangulation in $T_n^A$ iff it satisfies the following two properties:*

*(i) For $1 \le i < j \le n+1$, $r_i \le r_j - (j-i)$ provided $r_j - (j-i)$ is non-negative.*

*(ii) $0 \le r_i \le i-1$.*

The order relation on triangulations has a simple interpretation in terms of bracket vectors, which we summarize in the following proposition:

**Proposition 3.2.** *The lattice structure on $T_n^A$ can be described as follows:*

*(i) $S \le T$ iff $r_i(S) \le r_i(T)$ for all $i$.*

*(ii) $r(S \wedge T)_i = \min(r_i(S), r_i(T))$.*

*(iii) For $x$ any $n+1$-tuple of numbers satisfying only the second condition of Proposition 1, there is a unique triangulation $\uparrow(x)$ such that that for $S \in T_n^A$,*

$$r_i(S) \ge x_i \text{ for all } i \text{ iff } S \ge \uparrow(x).$$

*(iv) $r(S \vee T) = \uparrow(\max(r(S), r(T)))$, where max is taken coordinatewise.*

We now proceed to describe a similar construction in type $B$. To a triangulation $S \in T_n^B$ we associate a bracket vector $r(S) = (r_1(S), \ldots, r_n(S))$, as follows. For $1 \le i \le n$, let $c_i(S)$ denote the first vertex adjacent to $i$ encountered proceeding clockwise starting at $\bar{1}$. If the counter-clockwise distance from $i-1$ to $c_i(S)$ is less than or equal to $n-1$, set $r_i(S)$ to be that distance. Otherwise, set $r_i(S) = *$. Thus, the triangulation shown in Figure 2 has bracket vector $(0, *, 0, 0, 2, 0)$.

**Conventions regarding $*$.** $*$ is considered to be greater than any integer. $*$ plus an integer (or $*$) equals $*$.

**Lemma 3.3.** *The map from $T_n^B$ to bracket vectors is injective.*

**Proposition 3.4.** *$B_n$ bracket vectors are $n$-tuples of symbols from $[0, n-1] \cup \{*\}$ characterized by the following two properties:*

*i) For $1 \le i < j \le n$, $r_i \le r_j - (j-i)$ if $r_j - (j-i)$ is non-negative.*

*ii) If $* > r_i \geq i$, then $r_{n+i-r_i} = *$.*

We will now define an order on $T_n^B$. For $S, T \in T_n^B$, let $S \leq T$ iff for all $i$, $r_i(S) \leq r_i(T)$.

**Proposition 3.5.** *The covering relations in this order on $T_n^B$ are exactly those described by Theorem 1.*

Our next goal is to show that the $B_n$ Tamari order is really a lattice. Before we can do that, we need some preliminary results.

Let $M_n$ denote the $n$-tuples with entries in $[0, n-1] \cup \{*\}$, with the Cartesian product order. Let $M_n^{(i)}$ denote the elements of $M_n$ which satisfy condition (i) of Proposition 3. Let $M_n^{(ii)}$ denote the elements of $M_n$ which satisfy condition (ii) of Proposition 3.

**Proposition 3.6.** *There exist maps $\uparrow : M_n^{(ii)} \to T_n^B$, $\downarrow : M_n^{(i)} \to T_n^B$, which satisfy the following conditions:*

$$f \leq r(S) \text{ iff } \uparrow(f) \leq S$$
$$r(S) \leq f \text{ iff } S \leq \downarrow(f).$$

Using these maps, we can prove that meet and join exist in $T_n^B$ by giving simple descriptions of them.

**Proposition 3.7.** *The Tamari order on $T_n^B$ is a lattice. The lattice operations are as follows: For $S, T \in T_n^B$, $S \vee T = \uparrow(\max(r(S), r(T)))$ and $S \wedge T = \downarrow(\min(r(S), r(T)))$.*

This completes the proof of Theorem 1. The Hasse diagram of $T_B^3$ is shown in Figure 5, at the end of the paper.

## 4. Non-crossing partitions

The $A_n$ non-crossing partitions, $\mathrm{NC}_n^A$, are partitions of $n+1$ into sets such that if $v_1, \ldots, v_{n+1}$ are $n+1$ points on a circle, labelled in cyclic order, and if $B_1, \ldots, B_r$ are the convex hulls of the sets of vertices corresponding to the blocks of the partition, then the $B_i$ are non-intersecting.

There is a bijection from $T_n^A$ to $\mathrm{NC}_n^A$ as follows. For $S \in T_n^A$, erase all the green chords of $S$ and the vertices 0 and $n+2$. Then move the endpoints of each red chord $ij$ a little bit, the lower-numbered end point a little clockwise, the higher-numbered endpoint a little counterclockwise (so $i$ and $j$ are both on the upper side of the chord). These chords now divide the vertices in $[n+1]$ into subsets, which form a non-crossing partition by construction. Figure 3 shows the triangulation from Figure 1, together with the non-crossing partition which it induces: $\{14, 23, 5\}$

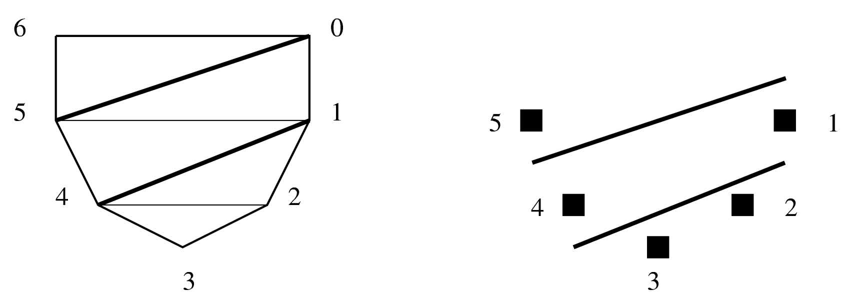


FIGURE 3

Note that the non-crossing partitions are often considered as being ordered by refinement; this order is quite different from the Tamari order.

As defined by Reiner [**Rei**], the $B_n$ non-crossing partitions, $\mathrm{NC}_n^B$, are partitions of the set $1, \ldots, n$, $\bar{1}, \ldots, \bar{n}$, which have the properties that the partition remains fixed under interchanging barred and unbarred elements, and that if $2n$ points are chosen around a circle and labelled cyclically $v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\bar{n}}$, then the convex hulls of the vertices corresponding to the blocks of the partition do not intersect.

We now define a map $\psi$ from $T_n^B$ to $\mathrm{NC}_n^B$, analogous to that in type $A$. Erase all green chords. Move both endpoints of mixed red chords slightly counterclockwise. Move the endpoints of pure red chords slightly together (so that the vertices both lie on the side of the chord which includes the larger part of the polygon). Erase the vertices $n+1$ and $\overline{n+1}$. The remaining vertices are now partitioned by the red chords, in what is clearly a $B_n$ non-crossing partition. Figure 4 shows the triangulation from Figure 2, together with the $B_n$ non-crossing partition which it induces: $\{1\bar{2}\bar{5}\bar{6}, 34, \bar{1}256, \bar{3}\bar{4}\}$.

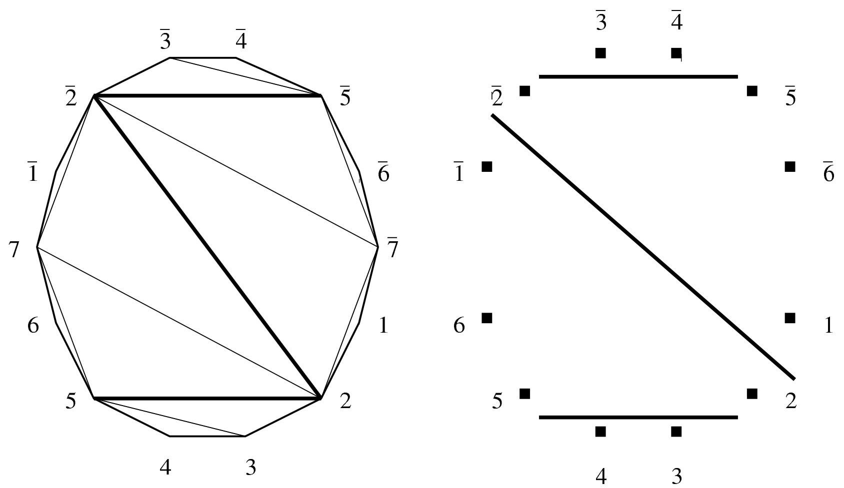


FIGURE 4

**Proposition 4.1.** *The map $\psi$ is a bijection from $T_n^B$ to $\mathrm{NC}_n^B$.*

## 5. EL-Shellability

Recall that an element $x$ of a lattice $L$ is said to be left modular if, for all $y < z \in L$,

$$(y \vee x) \wedge z = y \vee (x \wedge z).$$

For $1 \leq i \leq n$ and $t \in [1, n-1] \cup \{*\}$, let $S_{i,t}$ denote the triangulation with bracket vector as follows:

$$r_j(S_{i,t}) = \begin{cases} * & \text{for } j > i \\ t & \text{for } j = i \\ 0 & \text{for } j < i. \end{cases}$$

**Lemma 5.1.** *$S_{i,t} \in T_n^B$ is left modular.*

Since the $S_{i,t}$ together with $\hat{0}$ form an unrefinable chain, we have the following theorem:

**Theorem 5.2.** *$T_n^B$ has an unrefinable chain of left modular elements.*

The analogous fact that $T_n^A$ posesses a maximal chain of left modular elements was first proved by Blass and Sagan [**BS**].

It was shown in [**Liu**] that a lattice having an unrefinable chain of left-modular elements has an EL-labelling. In particular, this shows that the order complex of any interval in such a lattice is shellable and hence contractible or homotopy equivalent to a wedge of spheres. For more on EL-labelling, and EL-shellability, see [**Bj, BW1, BW2**].

Thus, Theorem 2 implies the following corollary:

**Corollary 5.3.** *$T_n^B$ is EL-shellable.*

## 6. Homotopy types of intervals

As we have already remarked, the fact that $T_n^B$ is EL-shellable implies that the order complex of any interval is either contractible or has the homotopy type of a wedge of spheres. In this section, we shall show that it is in fact either contractible or homotopic to a single sphere. One reason that such a result is of interest is that it implies that the Möbius function of any interval in $T_n^B$ is 0, $-1$, or 1.

**Theorem 6.1.** *The order complex of an interval in $T_n^B$ is either contractible or homotopy equivalent to a single sphere.*

To sketch the proof of Theorem 3, we begin by recalling the EL-labelling of [**Liu**]. Let $L$ be a lattice, and let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = \hat{1}$ be an unrefinable chain of left modular elements. Let $\mathcal{W}_i$ be the set of join irreducibles below $x_i$ but not below $x_{i-1}$. For $y < z$ in $L$, let $\mathcal{W}(y,z)$ be the set of irreducibles below $z$ but not below $y$. For any $S \lessdot T$ in $L$, label the corresponding edge of the Hasse diagram by:

$$\gamma(S,T) = \min\{i \mid \mathcal{W}_i \cap \mathcal{W}(S,T) \neq \emptyset\}.$$

**Proposition 6.2** ([**Liu**])**.** *For $L$ a lattice with an unrefinable left modular chain, the labelling $\gamma$ defined above is an EL-labelling.*

In order to interpret this labelling in our case, we need some results about the join-irreducibles of $T_n^B$.

For $1 \leq t \leq i-1$, let $W_{i,t}$ denote the triangulation whose bracket vector consists of $t$ in the $i$-th place, all the other entries being zero.

For $i \leq t < n$, let $W_{i,t}$ denote the triangulation defined by:

$$r_j(W_{i,t}) = \begin{cases} t & \text{for } j = i \\ * & \text{for } j = n+i-t \\ 0 & \text{otherwise} \end{cases}$$

Let $W_{i,*}$ denote the triangulation whose bracket vector consists of a single $*$ in the $i$-th place, all the other entries being zero.

Write $\mathcal{W}$ for the set of all the $W_{i,t}$.

**Proposition 6.3.** *The join-irreducibles of $T_n^B$ are exactly $\mathcal{W}$. The unique join-irreducible below $S_{i,t}$ and not below any smaller $S_{i',t'}$ is $W_{i,t}$.*

Recall from [**BW2**] that given a poset with an EL-labelling, the order complex of an interval $[y,z]$ is homotopic to a wedge of spheres, one for each unrefinable chain from $y$ to $z$ such that the labels strictly decrease as one reads up the chain. Such chains are called *decreasing chains.*

Thus, Theorem 3 follows from the following lemma:

**Lemma 6.4.** *For $Y < Z \in T_n^B$, there is at most one decreasing chain from $Y$ to $Z$.*

## 7. Generalizing to Type $BD_n^S$

Here we fix $n$ and a subset $S$ of $[n]$. We will be operating in type $BD_n^S$, a concept introduced in [**Rei**] which we now explain. This is not a type in the usual sense. Rather, it refers to a certain hyperplane arrangement between those associated to $B_n$ and $D_n$.

Recall that a root system gives rise to a hyperplane arrangement by taking all the hyperplanes through the origin perpendicular to roots. The $B_n$ arrangement therefore consists of all those hyperplanes defined by $x_i \pm x_j = 0$, together with those defined by $x_i = 0$, for $1 \leq i,j \leq n$, while the $D_n$ arrangement consists only of those hyperplanes defined by $x_i \pm x_j = 0$ for $1 \leq i,j \leq n$. Now, for $S \subset [n]$, the $BD_n^S$ hyperplane arrangement consists of those hyperplanes defined by $x_i \pm x_j = 0$ together with $x_i = 0$ for $i \notin S$. When $S = \emptyset$ we recover $B_n$, while if $S = [n]$ we recover $D_n$.

The $B_n$ partitions, $\Pi_n^B$, are by definition those partitions of the set $\{1,\ldots,n,\,\bar{1},\ldots,\bar{n}\}$ which are fixed under the map interchanging $i$ and $\bar{i}$, and such that there is at most one block which contains any $i$ and $\bar{i}$ simultaneously. This is a suitable definition of $\Pi_n^B$ because its elements are naturally in bijection with the elements of the intersection lattice of the $B_n$ arrangement. $\mathrm{NC}_n^B$ is a subset of $\Pi_n^B$.

The intersection lattice of the $BD_n^S$ hyperplane arrangement is a subset of that of type $B_n$. This allows a natural definition of $BD_n^S$ partitions, $\Pi_n^S$, as a subset of $\Pi_n^B$. By this approach, one obtains that $\Pi_n^S$ consists of those partitions of $\Pi_n^B$ which do not contain any block $\{i, \bar{i}\}$ for $i \in S$. In [**Rei**], Reiner defined $\mathrm{NC}_n^S$, the non-crossing partitions of type $BD_n^S$, by $\mathrm{NC}_n^S = \mathrm{NC}_n^B \cap \Pi_n^S$. (For more details on the material sketched in the preceding paragraphs, see [**Rei**].)

We now define $T_n^S$ to be those triangulations which correspond under $\psi$ to partitions in $\mathrm{NC}_n^S$. We can describe them more directly as follows:

**Lemma 7.1.** *$T_n^S$ consists of those triangulations which do not contain the triangles $i, \bar{i}, i+1$ and $i, \bar{i}, \overline{i+1}$ for any $i \in S$. $T_n^S$ can also be characterized as the set of triangulations $T$ such that $r_i(T) \neq n-1$ for any $i \in S$.*

The remainder of the paper is devoted to sketching the proof of the following theorem, which generalizes Theorems 1, 2, and 3 to the broader context of type $BD_n^S$.

**Theorem 7.2.** *$T_n^S$ admits a lattice structure which is a quotient of that on $T_n^B$. $T_n^S$ posesses an unrefinable chain of left modular elements, which implies that it is EL-shellable. Further, the order complex of any interval is either contractible or homotopic to a single sphere.*

We define an equivalence relation $\sim_S$ on $T_n^B$ as follows: two non-identical triangulations are equivalent iff they differ in that one of them, say $T$, is not in $T_n^S$, and the other is the triangulation obtained by removing the diameter of $T$ and replacing it with the other possible diameter.

An equivalence relation $\sim$ on a lattice $L$ is said to be a congruence relation if the lattice operations pass to equivalence classes. In this case, there is an induced lattice structure on the equivalence classes (see [**Gr**]).

**Lemma 7.3.** *The relation $\sim_S$ on $T_n^B$ is a congruence relation.*

Since the equivalence classes of $\sim_S$ each contain a single element of $T_n^S$, the induced lattice structure on $T_n^B/\sim_S$ gives rise to a lattice structure on $T_n^S$.

(One could also define a partial order on $T_n^S$ by considering the order induced by its inclusion in $T_n^B$. It turns out that the order defined in this way coincides with the order we have already defined.)

It is immediate that the property of being left modular passes to equivalence classes, so $T_n^S$ has a maximal chain of left modular elements, and is therefore EL-shellable. This maximal chain is shorter than that of $T_n^B$, because $S_{i,n-1} \sim_S S_{i,*}$ for $i \in S$.

It is easy to see that the join irreducibles of $T_n^S$ are those $W_{i,t}$ such that either $i \notin S$ or $t \neq n-1$; again, they are in bijection with the elements of the left modular chain.

As in the type $B$ case, the result on homotopy types of intervals follows by showing that there is at most one decreasing chain in any interval in $T_n^S$.

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