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Sheared Tableaux and bases for the symmetric functions

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Abstract.
We study the operation of shearing *Schur functions, which yields a new family of bases for the space of symmetric functions. In the course of this study, we derive some interesting combinatorial results and inequalities on the Littlewood-Richardson coefficients of sheared Schur functions.*
Résumé. *Nous étudions l'opération de trancher les fonctions de Schur, qui nous donne une nouvelle famille de bases pour l'espace des fonctions symétriques. Pendent cette étude, nous dérivons des résultats et des inégalités combinatoires intéressants. Ces résultats décrivant les coefficients Littlewood-Richardson des fonctions Schur tranchées.*

1. Introduction

Suppose n is a positive integer and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0)$ is a partition of n, denoted $\lambda \vdash n$. The *(Young) diagram* associated to λ is a diagram made of rows of boxes, in which the kth row has λ_k boxes. If $m \leq n$ and μ is a partition of m such that the Young diagram of μ is contained within the Young diagram of λ, we define the *skew diagram* λ/μ to be the set of all boxes contained in the diagram of λ, but not contained in the diagram of μ. For emphasis, we shall refer to Young diagrams which are not skew as *normal* diagrams.

A *Young tableau* T is a Young diagram in which the boxes have been filled with positive integers. If the rows of T are weakly increasing and the columns of T are strictly increasing, then T is said to be *semistandard*. The *content* $c(T)$ of T is the weak composition of nonnegative integers $(\gamma_1, \ldots, \gamma_m)$ for which γ_i is the number of i's in T.

Let Λ denote the graded algebra of symmetric functions. Λ has a well-known basis consisting of *Schur functions*, denoted s_λ and indexed by normal Young diagrams $\lambda \vdash n$, for all positive integers n. We define s_λ as

$$s_\lambda = \sum_T x^{\mathrm{c(T)}}$$

where T runs over all semistandard tableaux with shape λ, and $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$. We can also define the *skew Schur function* $s_{\lambda/\mu}$ in precisely the same fashion.

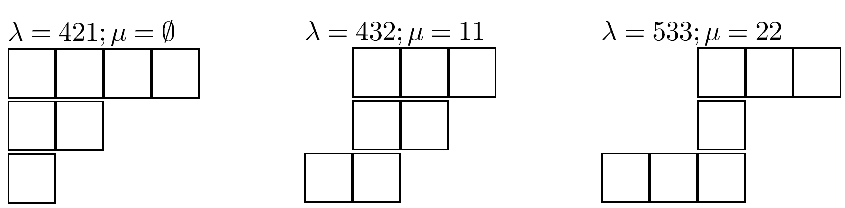

FIGURE 1. Normal, skew, and ribbon diagrams

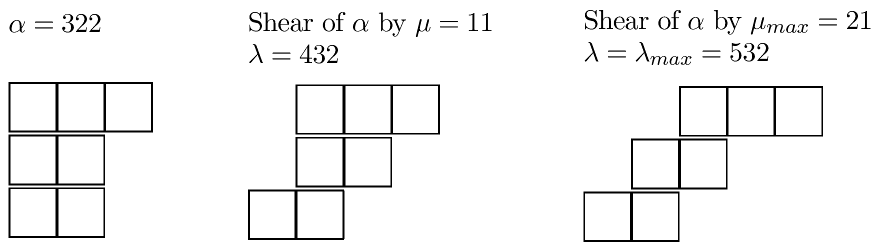

FIGURE 2. Shearings of a partition α

A diagram λ is said to be *connected* if any two adjacent rows of λ share a common horizontal edge. We define a *ribbon diagram* to be a connected diagram which does not contain a two-by-two sub-diagram, and define *ribbon tableaux* and *ribbon Schur functions* accordingly. Please refer to Figure 1 for examples of normal, skew, and ribbon diagrams.

Ribbons are of particular interest as they play a fundamental rule in the Murnaghan-Nakayama rule (see [**4**], chapter 7). In addition, the recent papers [**1**] and [**2**] link ribbons to the Fock space representation of $U_q(\widehat{\mathfrak{sl}_n})$.

In this paper, we provide a new basis of the symmetric functions which consists of ribbon Schur functions. This basis is obtained from the normal Schur functions by the process of *shearing.* Moreover, the change-of-basis matrix from the normal Schur functions to the shear basis has some interesting combinatorial properties. In particular, the entries of this matrix are Littlewood-Richardson coefficients, some of which are explicitly computed to be 0 or 1. Further results on symmetric functions and Littlewood-Richardson coefficients can be found in [**4**].

2. Shearing

Suppose $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$. Define the *maximal shear of* α to be the ribbon diagram with rows of length $\alpha_1, \ldots, \alpha_n$.

If the maximal shear of α is the skew diagram λ_{max}/μ_{max}, and μ is any diagram contained in μ_{max}, consider the skew diagram λ/μ, where $\lambda = \mu + \alpha = (\alpha_1 + \mu_1, \ldots, \alpha_k - 1 + \mu_k - 1, \alpha_k)$. If λ/μ is a connected diagram, we say that μ is a *shearing diagram* for λ. We define the *shear of* α *by* μ, denoted $\mathrm{Shear}_\mu(\alpha)$, to be the skew diagram λ/μ.

While the maximal shears of normal diagrams α are the primary objects of interest, our main theorem works for any shearing of α. For an example of sheared Young diagrams, see Figure 2.

Let $P(n)$ be the set of all partitions of n. For the remainder of this section, fix a function $M : P(n) \to \bigcup_{i=0}^{\infty} P(i)$ which maps each $\alpha \vdash n$ to a shearing diagram μ for α. To simplify the notation, for a fixed partition α of n, we will abuse notation and write $\mathrm{Shear}(\alpha)$ or $s_{\lambda/\mu}$ in place of $s_{\mathrm{Shear}_M(\alpha)}$.

We will prove the following:

Theorem 1. The set of skew Schur functions

$$\left\{\mathrm{Shear}(\lambda) \middle| n \in \mathbb{Z}^+, \lambda \vdash n\right\},$$

forms a basis for Λ.

Applying Theorem 1 using maximal shears gives the following:

Corollary. The set of ribbon Schur functions in which the rows are weakly decreasing in length form a basis of Λ.

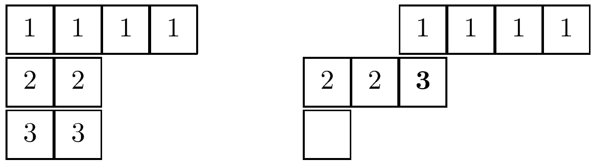

FIGURE 3. An ill-fated attempt to create a tableau T of shape $521/21$ and content 43

To prove this theorem, we shall restrict ourselves to the set of symmetric functions of a fixed degree n, denoted Λ^n. The Schur functions of degree n, $\{s_\lambda | \lambda \vdash n\}$, form a basis for Λ^n, and represent the trivial shear $M = 0$. We shall demonstrate that the set of Schur functions

$$\{\text{Shear}(\lambda) | \lambda \vdash n\}$$

also form a basis for Λ^n, by creating a change-of-basis matrix from the Schur function basis to $\{Shear(\lambda)\}$. The theorem follows immediately, since $\Lambda = \bigoplus_{i=1}^{\infty} \Lambda^i$.

We express λ/μ in terms of the Schur function basis using Littlewood-Richardson coefficients:

$$\text{Shear}(\alpha) = \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu$$

Following [**3**], we say that a word $a_1 a_2 \dots a_n$ is a reverse lattice partition if, in any of the suffixes $a_k a_{k+1} \dots a_n$, the number of l's is at least as large as the number of $(l+1)$'s. By the Littlewood-Richardson rule, the coefficient $c^{\lambda}_{\mu\nu}$ counts the number of semistandard tableaux T for which

(a) T has shape λ/μ and content ν,
(b) the row word of T is a reverse lattice partition.

In particular, if $\nu = \alpha$, the first row of T must end with a 1 by condition (2), and must be weakly increasing by condition (1). Therefore, the first row of T is made up of α_1 ones. However, the content of T is equal to α – that is, there are only α_1 ones in T. Hence, the rest of T must be filled with integers greater or equal to 2. Repeating this argument for the rest of the rows in λ/μ, we find that the only way to construct T is to fill the ith row with the number i. We have proven

Lemma 1. $c^{\lambda}_{\mu\alpha} = 1$.

Now, suppose that $\alpha \not\preceq \nu$, where $\preceq$ is the dominance ordering of partitions: $(\beta_1, \dots, \beta_k) \preceq (\gamma_1, \dots, \gamma_j)$ means $\sum_{i=1}^{t} \beta_i \leq \sum_{i=1}^{t} \gamma_i$ for each $t \leq \max\{k, j\}$. Suppose further that we have constructed a tableau T of shape Shear(α) satisfying conditions (1) and (2) above.

Observe that row t of T contains only numbers which are less than or equal to t. This can be seen by induction on t. The base case asserts that the first row of T contains only ones, which has already been shown in the context of Lemma 1. Now suppose that the first $t-1$ rows contain only numbers which are less than or equal to $t-1$. The largest element t_1 of row t must occur at the right end of row t; in order to satisfy the requirement that the row word of T be a reverse lattice partition, $t_1 = t$ (otherwise, the number of t_1's leads the number of t's at the right end of row t). See Figure 3 for an example.

Let t_0 be the first t for which $\sum_{i=1}^{t} \alpha_i > \sum_{i=1}^{t} \nu_i$. The sum on the left side of the inequality counts the boxes in the first t_0 rows of λ/μ, whereas the sum on the right side of the inequality counts the boxes in the first t_0 rows of ν.

Let t_1 be the largest element of row t_0 in the diagram λ/μ. Since T is to be filled with the content of ν, it follows that $t_1 > t_0$, a contradiction. So such a tableau T cannot exist after all. We have proven

Lemma 2. If $\alpha \not\preceq \nu$, then $c^{\lambda}_{\mu\nu} = 0$.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

FIGURE 4. The change–of–basis matrix M for maximal shears of s_λ, $n = 7$

Proof of Theorem 1: Let us write $c_\nu^{\mathrm{Shear}(\alpha)}$ for $c_{\mu\nu}^\lambda$, in order to emphasize the connection between α and ν. Let M be the matrix $[c_\nu^{\mathrm{Shear}(\alpha)}]$ in which the rows are ordered lexicographically by α, and the columns are ordered lexicographically by ν. Observe that M gives the coordinates of each Shear(α) in terms of the normal Schur function basis s_ν.

Since the dominance ordering is a strengthening of the lexicographic ordering, Lemma 2 implies that M is upper triangular, while Lemma 1 implies that M has ones on its main diagonal (for a concrete example of M, see Figure 4). Therefore, M is invertible, so the chosen set of sheared Schur functions must form a basis for Λ^n. □

3. Shears of a single diagram α

Let us now fix a partition α of n. Suppose μ and η are two shearing diagrams for α such that μ is contained in η. We say that $\mathrm{Shear}_\eta(\alpha)$ is a *relative shearing* of $\mathrm{Shear}_\mu(\lambda)$ if the skew diagram η/μ is a shearing of some normal diagram – that is, if the rows of η/μ are weakly decreasing in length. We have the following theorem:

Theorem 2. If $\mathrm{Shear}_\eta(\alpha)$ is a relative shearing of $\mathrm{Shear}_\mu(\alpha)$, then

$$c_\nu^{\mathrm{Shear}_\mu(\alpha)} \le c_\nu^{\mathrm{Shear}_\eta(\alpha)}.$$

Proof. In order to prove the theorem, suppose we have constructed a Young tableau T_μ with shape $\mathrm{Shear}_\mu(\alpha)$ and content ν which meets conditions (1) and (2) of the Littlewood-Richardson rule. Let T_η be the Young tableau with shape $\mathrm{Shear}_\eta(\alpha)$ such that the ith row of T_η is equal to the ith row of T_μ. If we show that T_η also satisfies conditions (1) and (2), then the Littlewood-Richardson rule will imply the theorem.

Observe that the row word of T_μ is equal to the row word of T_η, and thus the rows of T_η are weakly increasing. So we need only check that T_η has strictly increasing columns.

Let $\Delta = (\eta_r - \mu_r) - (\eta_{r+1} - \mu_{r+1})$ be the number of boxes that row r moves with respect to row $r+1$ as we pass from $\mathrm{Shear}_\mu(\alpha)$ to $\mathrm{Shear}_\eta(\alpha)$. Because $\mathrm{Shear}_\eta(\alpha)$ is a relative shearing of $\mathrm{Shear}_\mu(\alpha)$, we know that $\Delta \ge 0$.

Suppose the element a lies in row r of T_η, and b lies directly below a in row $r+1$ of T_η. There are at least Δ elements to the left of b; take the rightmost of these and label them $b_1, \ldots, b_\Delta$. Observe that a lies

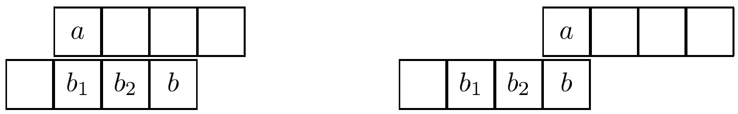

FIGURE 5. Two adjacent rows of T_μ and T_η; $\Delta = 2$

above b_1 in T_μ. Because T_μ is semistandard, we have that $a < b_1 \leq \cdots \leq b_\Delta \leq b$ (see Figure 5). Thus the columns of T_η are strictly increasing. $\square$

4. Vertical shearing

In the preceeding development, all of our shears have been *horizontal*, in the sense that we have obtained a shear of the tableau α by shifting some of the rows of α to the right. We could equally well shift some of the *columns* of α downward, to obtain a *vertical shearing* of α.

A proof similar to that of Theorem 1 can be employed to show that any set of vertical shears of the normal Schur functions also yields a new basis for λ. In the proof, the change of basis matrix M becomes lower triangular. Likewise, Theorem 2 also holds with relative shears being replaced with relative vertical shears.

5. Further work

Aside from exploring connections to the active ribbon-based research areas mentioned, the most pressing issue which arises from this work is to study the change-of-basis matrix M in greater detail. The author has computed M for $n \leq 8$, using an algorithm for computing Littlewood-Richardson coefficents. These small matrices M are fairly sparse and have small entries. It seems likely that the best way to obtain further information about shearing is to sharpen these observations.

Computations of M grow quickly intractable as n increases, so it would also be worthwhile to look for shear-specific algorithms for computing Littlewood-Richardson coefficients of sheared Schur functions. Theorem 2, in particular, suggests that to compute M for $\mathrm{Shear}_\eta(\alpha)$, we could proceed iteratively. First, one would find shearing functions $\mu_0 = \emptyset \subseteq \mu_1 \subseteq \ldots \subseteq \mu_k = \eta$, where each μ_i is a relative shearing of μ_{i-1}. Then, one would compute the corresponding matrices M_i for $\mathrm{Shear}_{\mu_i}(\alpha)$. Hopefully, if enough intermediary μ_i are used, the changes between M_i and M_{i+1} will be small. Of course, in order for this to work, we would need to sharpen Theorem 2 considerably, providing at least an *upper* bound for the Littlewood-Richardson coefficients in question.

References

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[3] Bruce E. Sagan, *The symmetric group: Representations, combinatorial algorithms, and symmetric functions*, Springer-Verlag New York Inc., New York, 2001.

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