THE HIVE MODEL AND THE POLYNOMIAL NATURE OF STRETCHED LITTLEWOOD-RICHARDSON COEFFICIENTS

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ABSTRACT. The hive model is used to explore the properties of both ordinary and stretched Littlewood-Richardson coefficients. The latter are polynomials in the stretching parameter t. It is shown that these may factorise, and that they can then be expressed as products of certain primitive polynomials. It is further shown how to determine a sequence of linear factors $(t+m)$ of the primitive polynomials, as well as bounds on their degree which are conjectured to be exact.

RÉSUMÉ. Nous utilisons le modèle des ruches pour étudier les propriétés des cœfficients dilatés de Litlewood-Richardson et des polynômes associés aux cœfficients de Littlewood-Richardson dilatés. Nous montrons que les uns et les autres peuvent se factoriser : ils s'écrivent comme des produits de coefficients (resp. polynômes) primitifs. En outre, nous montrons comment établir une suite de facteurs linéaires $(t+m)$ des polynômes primitifs , et proposons les bornes supérieure de leur degrés.

1. Introduction

Littlewood-Richardson coefficients, $c^{\nu}_{\lambda\mu}$, are interesting combinatorial objects [LR]. They are indexed by partitions λ, μ and ν, and they count the number of Littlewood-Richardson tableaux of skew shape ν/λ and weight μ. They are therefore non-negative integers. Although it is a non-trivial matter to determine whether or not $c^{\nu}_{\lambda\mu}$ is non-zero, it turns out that this is the case if and only if $|\lambda|+|\mu|=|\nu|$ and certain partial sums of the parts of λ, μ and ν satisfy what are known as Horn inequalities.

Multiplying all the parts of the partitions λ, μ and ν by a stretching parameter t, with t a positive integer, gives new partitions $t\lambda$, $t\mu$ and $t\nu$. The corresponding stretched Littlewood-Richardson coefficient are known to be polynomial in the stretching parameter t [DW2, R]. Such an LR-polynomial is defined by

$$P^{\nu}_{\lambda\mu}(t)=c^{t\nu}_{t\lambda,t\mu}, \tag{1.1}$$

and has a generating function of the form

$$F^{\nu}_{\lambda\mu}(z)=\frac{G^{\nu}_{\lambda\mu}(z)}{(1-z)^{d+1}}=\sum_{t=0}^{\infty}c^{t\nu}_{t\lambda,t\mu}\ z^t \tag{1.2}$$

where d is the degree of $P^{\nu}_{\lambda\mu}(t)$, and $G^{\nu}_{\lambda\mu}$ is a polynomial in z of degree $g\le d$.

For example, in the case $\lambda=(4,3,3,2,1)$, $\mu=(4,3,2,2,1)$ and $\nu=(7,4,4,3,2,1)$ one finds $c^{\nu}_{\lambda\mu}=13$ and

$$P^{\nu}_{\lambda,\mu}(t)=\frac{1}{10080}(t+1)(t+2)(t+3)(t+4)(t+5)(5t+21)(t^2+2t+4), \tag{1.3}$$

with

$$F^{\nu}_{\lambda\mu}(z)=\frac{1+4z+12z^2+3z^3}{(1-z)^9}. \tag{1.4}$$

It is the intention here to try to shed some light on the nature of the LR-polynomials and their generating functions. In particular we concentrate on the possible factorisation of any particular LR-polynomial as a product of simpler LR-polynomials, the degree d of an LR-polynomial, and the number of its linear factors $(t+m)$. We do not explore two particular conjectures [KTT1] to the effect that the non-zero coefficients of the polynomial $P^{\nu}_{\lambda\mu}(t)$ are always positive rational numbers, while those of $G^{\nu}_{\lambda\mu}(z)$ are all positive integers.

Our approach is based largely on the use of a hive model [BZ2, KT, B] which allows Littlewood-Richardson coefficients to be evaluated through the enumeration of integer points of certain rational polytopes. Before defining hives, puzzles and plans, that are the combinatorial constructs to be used in this context, it is worth recalling some definitions and properties of Littlewood-Richardson coefficients and LR-polynomials.

2. Definitions and properties

Let n be a fixed positive integer, let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ be a vector of indeterminates, and let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ be a partition of weight $|\lambda|$ and of length $\ell(\lambda)\leq n$. Thus $\lambda_k\in\mathbb{Z}^+$ for $k=1,2,\ldots,n$, with while $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{\ell(\lambda)}>0.$ and $\lambda_k=0$ for $k>\ell(\lambda)$.

Definition 2.1. *For each partition λ with $\ell(\lambda)\leq n$ there corresponds a Schur function $s_\lambda(\mathbf{x})$ defined by*

$$s_\lambda(\mathbf{x})=\frac{\left|x_i^{n+\lambda_j-j}\right|_{1\leq i,j\leq n}}{\left|x_i^{n-j}\right|_{1\leq i,j\leq n}} \tag{2.1}$$

Choosing n sufficiently large, the Littlewood-Richardson coefficients may be defined by

Definition 2.2.

$$s_\lambda(\mathbf{x})\ s_\mu(\mathbf{x})=\sum_\nu c^{\nu}_{\lambda\mu}\ s_\nu(\mathbf{x}). \tag{2.2}$$

where the summation is over all partitions ν.

The expansion (2.2) may be effected by means of the Littlewood-Richardson rule [LR] which states that $c^{\nu}_{\lambda\mu}$ is the number of Littlewood-Richardson skew tableaux of shape ν/μ and weight λ obtained by numbering the boxes of the skew Young diagram $F^{\nu/\mu}$ with λ_i entries i for $i=1,2,\ldots,n$ that are weakly increasing across rows, strictly increasing down columns and satisfy the lattice permutation rule.

To specify the necessary and sufficient conditions on λ, μ and ν for $c^{\nu}_{\lambda\mu}$ to be non-zero it is convenient to introduce the notion of partial sums of the parts of a partition and some other notational devices.

Let n be a fixed positive integer and $N=\{1,2,\ldots,n\}$. Then for any positive integer $r\leq n$ and any subset $I=\{i_1,i_2,\ldots,i_r\}$ of N of cardinality $\#I=r$, the partial sum indexed by I of any partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ of length $\ell(\lambda)\leq n$ is defined to be

$$ps\,(\nu)_I=\nu_{i_1}+\nu_{i_2}+\cdots+\nu_{i_r} \tag{2.3}$$

If $i_1<i_2<\cdots<i_r$, let $\tilde{I}=(i_r,\ldots,i_2,i_1)$. It follows that if $\delta_r=(r,r-1,\ldots,1)$ then $\alpha(I)=\tilde{I}-\delta_r$ is a partition of length $\ell(\alpha(I))\leq r$.

With this notation, building on a connection with the Horn conjecture [H] regarding eigenvalues of Hermitian matrices, the following theorem has been established by Klyachko [K], Knutson and Tao [KT], Knutsen, Tao and Woodward [KTW] and others. A comprehensive review of these devlopments has been provided by Fulton [F].

Theorem 2.3 (Horn inequalities)**.** *Let λ, μ and ν be partitions of lengths $\ell(\lambda),\ell(\mu),\ell(\nu)\leq n$. Then $c^{\nu}_{\lambda\mu}>0$ if and only if $|\nu|=|\lambda|+|\mu|$ and for all $r=1,2,\ldots,n$*

$$ps\,(\nu)_K\leq ps\,(\lambda)_I+ps\,(\mu)_J \tag{2.4}$$

for all triples $(I,J,K)\in R^n_r$, where R^n_r is the set of triples (I,J,K) with $\#I=\#J=\#K=r$ such that if $\alpha(I)=\tilde{I}-\delta_r$, $\beta(J)=\tilde{J}-\delta_r$ and $\gamma(K)=\tilde{K}-\delta_r$ then $c^{\gamma(K)}_{\alpha(I)\beta(J)}=1$.

Unfortunately, even for comparatively small values of r it is not a trivial matter to identify all partitions $\alpha(I), \beta(J), \gamma(K)$, and hence (I, J, K), such that the Littlewood-Richardson coefficient $c^{\gamma(K)}_{\alpha(I)\beta(J)} = 1$. We will return to this problem only after the hive model has been introduced.

Turning instead to stretched Littlewood-Richardson coefficients, the fact that all the above partial sum conditions are linear and homogeneous in the various parts of λ, μ and ν ensures the validity of the following:

Theorem 2.4 (Saturation Condition). [KT, B, DW1] *For all positive integers t*

$$c^{t\nu}_{t\lambda,t\mu} > 0 \iff c^{\nu}_{\lambda\mu} > 0. \tag{2.5}$$

Furthermore, it has been established by a variety of means that

Theorem 2.5 (Polynomial Condition). [DW2, R] *For all partitions λ, μ and ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists a polynomial $P^{\nu}_{\lambda\mu}(t)$ in t such that $P^{\nu}_{\lambda\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ for all $t \in \mathbb{N}$.*

As a special case of these conditions we have the following conjecture now established as a theorem:

Theorem 2.6 (Fulton's Conjecture). [KTW] *For all positive integers t*

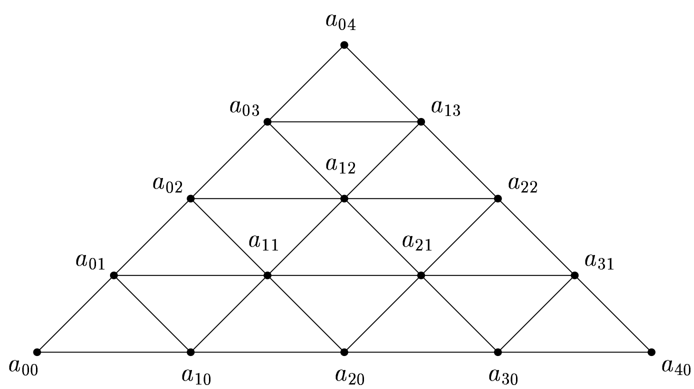
$$c^{t\nu}_{t\lambda,t\mu} = 1 \iff c^{\nu}_{\lambda\mu} = 1. \tag{2.6}$$

Thus, if $c^{\nu}_{\lambda\mu} = 1$, the corresponding polynomial $P^{\nu}_{\lambda\mu}(t) = 1$.

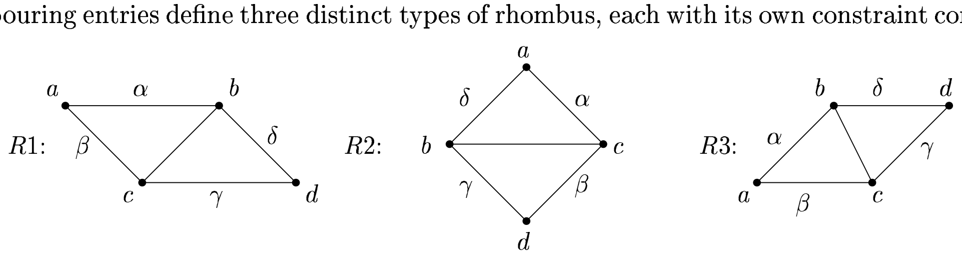
3. The hive model

The hive model arose out of the triangular arrays of Berenstein and Zeleviinsky used to specify individual contributions to Littlewood-Richardson coefficients [BZ2]. The model was then taken up by Knutsen and Tao in a manner described in an exposition by Buch [B].

An n-hive is an array of numbers a_{ij} with $0 \le i, j, i + j \le n$ placed at the vertices of an equilateral triangular graph. Typically, for $n = 4$ their arrangement is as shown below:



Such an n-hive is said to be an integer hive if all of its entries are non-negative integers. Neighbouring entries define three distinct types of rhombus, each with its own constraint condition.



In each case, with the labelling as shown, the hive condition takes the form:

$$b + c \geq a + d \tag{3.1}$$

In what follows we make use of edge labels more often than vertex labels. Each edge in the hive is labelled by means of the difference, $\epsilon = q - p$, between the labels, p and q, on the two vertices connected by this edge, with q always to the right of p. In all the above cases, with this convention, we have $\alpha + \delta = \beta + \gamma$, and the hive conditions take the form:

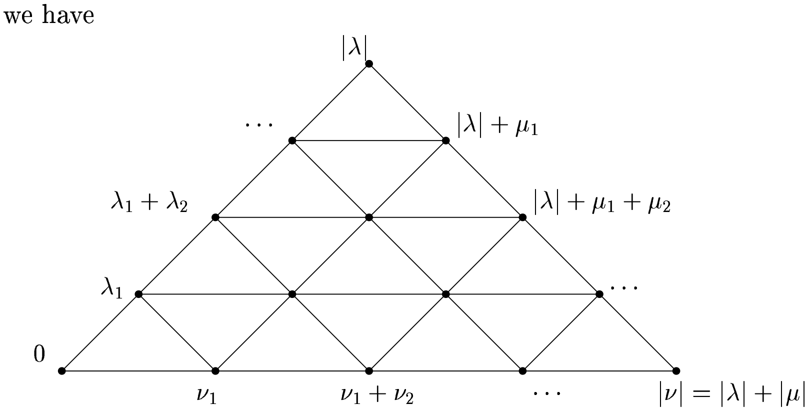
$$\alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta, \tag{3.2}$$

where, of course, either one of the conditions $\alpha \geq \gamma$ or $\beta \geq \delta$ is sufficient to imply the other.

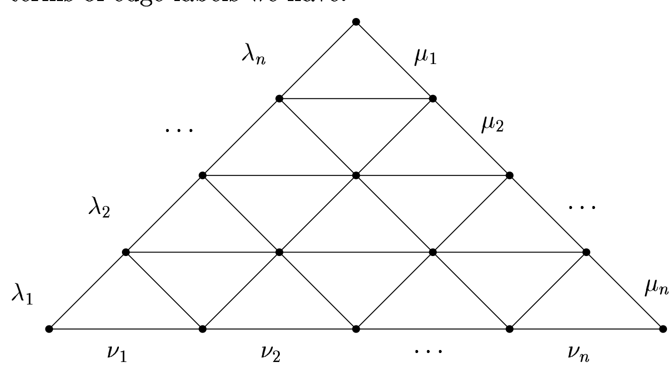
In order to enumerate contributions to Littlewood-Richardson coefficients, we require the following:

Definition 3.1. *An LR-hive is an integer n-hive, for some positive integer n, satisfying the hive conditions (3.1), or equivalently (3.2) for all its constituent rhombi of type R1, R2 and R3, with border labels determined by partitions λ, μ and ν, for which $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$, in such a way that $a_{00} = 0$, $a_{0,i} = ps\,(\lambda)_i$, $a_{j,n-j} = |\lambda| + ps\,(\mu)_j$ and $a_{k,0} = ps\,(\nu)_k$, for $i, j, k = 1, 2, \ldots, n$,*

Schematically, we have



Alternatively, in terms of edge labels we have:

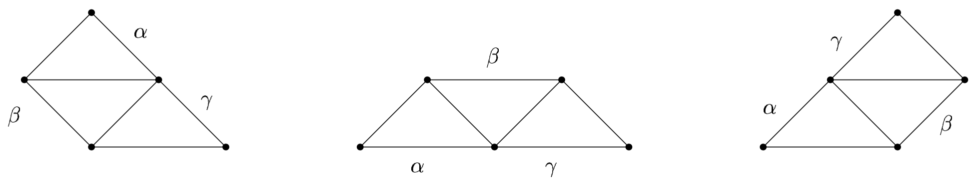


Proposition 3.2. [B] *The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with border labels determined as above by λ, μ and ν.*

Proof There exists a bijection between Littlewood-Richardson diagrams, D, of shape determined by ν/λ and of weight μ and LR-hives, H with border labels specified by λ, μ and ν. An illustration of this bijection is given below for a typical Littlewood-Richardson diagram, D, in the case $n = 3$, $\lambda = (3, 2)$, $\mu = (2, 1)$ and $\nu = (4, 3, 1)$. In D, which has overall shape ν, the portion of shape λ has been signified by entries 0, while the other entries correspond to the parts of the weight μ arranged in accordance with the Littlewood-Richardson rules. The first step is to form a sort of generalised Gelfand-Zetlin pattern G by writing down a list of partitions describing the shapes of subdiagrams of D formed by restricting the entries to be no more than k for $k = 3, 2, 1, 0$. Then one adds a diagonal of zeros and forms cumulative sums to arrive at an array Z. The lower right triangular portion of Z is then reoriented to give an LR-hive H, where for display purposes the hive edges have been omitted.

$$
\begin{aligned}
&D = \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\ \hline 0&0&2& \\ \cline{1-3} 1& & & \\ \cline{1-1}\end{array}
\quad \Longleftrightarrow \quad
G = \begin{array}{cccccc}
4 & 3 & 1 & & & \\
 & 4 & 3 & 1 & & \\
 & & 4 & 2 & 1 & \\
 & & & 3 & 2 & 0
\end{array} \\
(3.3)\qquad &\Longleftrightarrow \quad Z = \begin{array}{ccccccc}
0 & 4 & 7 & 8 & & & \\
 & 0 & 4 & 7 & 8 & & \\
 & & 0 & 4 & 6 & 7 & \\
 & & & 0 & 3 & 5 & 5
\end{array}
\quad \Longleftrightarrow \quad
H = \begin{array}{ccccccc}
 & & & 5 & & & \\
 & & 5 & & 7 & & \\
 & 3 & & 6 & & 8 & \\
0 & & 4 & & 7 & & 8
\end{array}
\end{aligned}
$$

When expressed in terms of edge labels, the hive conditions (3.2) for all constituent rhombi of types R1, R2 and R3 imply that in every LR-hive the edge labels along any line parallel to the north-west, north-east and southern boundaries of the hive are weakly decreasing in the north-east, south-west and easterly directions, respectively. This can be seen from the following 5-vertex sub-diagrams.



The edge conditions on the overlapping pairs of rhombi (R1,R2), (R1,R3) and (R2,R3) in the above diagrams give in each case $\alpha \geq \beta$ and $\beta \geq \gamma$, so that $\alpha \geq \gamma$ as claimed. This is of course consistent with the fact that edges of the three north-west, north-east and southern boundaries of each LR-hive are specified by partitions λ, μ and ν, respectively.

4. Factorisation

It was noted in the work of Berenstein and Zelevinsky [BZ1] that some Kostka coefficients may factorise. Although rather easy to prove using semistandard tableaux, this factorisation property may be established through the use of K-hives [KTT1]. A full account of this is presented here in the poster session [KTT2]. The same methods may then be used to show that some Littlewood-Richardson coefficients may also factorise.

In order to state a conjecture for the precise conditions under which $c^{\nu}_{\lambda\mu}$ factorises it is convenient to introduce some further notation. As usual let n be a fixed positive integer and let $N = \{1,2,\ldots,n\}$. Then for any $I = \{i_1,i_2,\ldots,i_r\} \subseteq N$, with $i_1 < i_2 < \cdots < i_r$ and $1 \leq r \leq n$, let $\tilde{I} = N \backslash I$ be the complement of I in N. In addition, for any partition or weight $\kappa = (\kappa_1,\kappa_2,\ldots,\kappa_n)$ let $\kappa_I = (\kappa_{i_r},\ldots,\kappa_{i_2},\kappa_{i_1})$. With this notation we make the following:

Conjecture 4.1. *Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(mu), \ell(\nu) \leq n$. If $c^{\nu}_{\lambda\beta} > 0$ and there exists proper subsets $I,J,K \subset N$ with $(I,J,K) \in R^n_r$ such that $ps\,(\mu)_K = ps\,(\lambda)_I + ps\,(\mu)_J$ then*

$$c^{\nu}_{\lambda\mu} = c^{\nu_K}_{\lambda_I\mu_J}\; c^{\nu_{\tilde{K}}}_{\lambda_{\tilde{I}}\mu_{\tilde{J}}}. \tag{4.1}$$

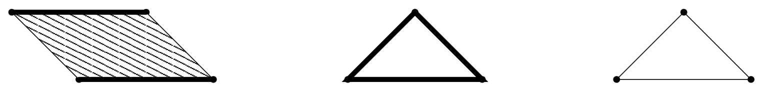
This means that if any one of Horn's inequalities (2.4) is an equality for $1 \leq r < n$ then $c^{\nu}_{\lambda\mu}$ factorises. We say that a Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is *primitive* if it cannot be factorised, that is to say all Horn's inequalities (2.4) are strict inequalities. Repeated use of the above conjecture would allow any non-vanishing Littlewood-Richardson coefficient to be written as a product of primitive Littlewood-Richardson coefficients. Furthermore, since the partial sum conditions are preserved under scaling by any positive number t, we have the following:

Conjecture 4.2. *Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(mu), \ell(\nu) \leq n$. If $c^{\nu}_{\lambda\beta} > 0$ and there exists proper subsets I, J and K of N with $(I,J,K) \in R^n_r$ such that $ps\,(\mu)_K = ps\,(\lambda)_I + ps\,(\mu)_J$*

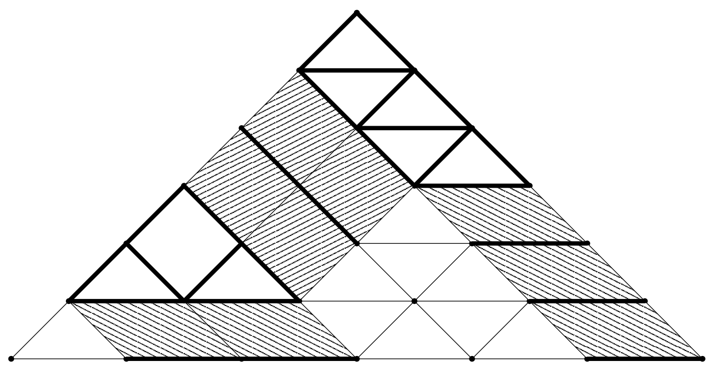
then

$$P^{\nu}_{\lambda\mu}(t) = P^{\nu_K}_{\lambda_I\mu_J}(t)\ P^{\nu_{\bar{K}}}_{\lambda_{\bar{I}}\mu_{\bar{J}}}(t). \tag{4.2}$$

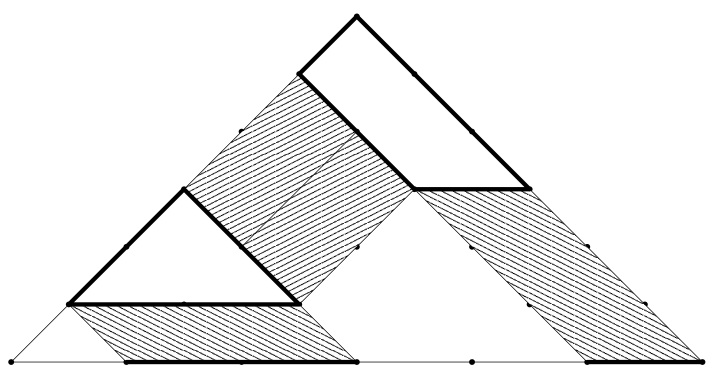
The origin of these conjectures is a study of the properties of certain puzzles introduced by Knutson *at al* [KTW]. These are triangular diagrams on a hive lattice consisting of three elementary pieces: a dark triangle, a light triangle and a shaded rhombus with its edges either dark or light according as they are to the right or left, respectively of an acute angle of the rhombus, when viewed from its interior:



The puzzle is to put these together, oriented in any manner, so as to form a hive shape with all the edges matching. For example, one such puzzle takes the form shown below:

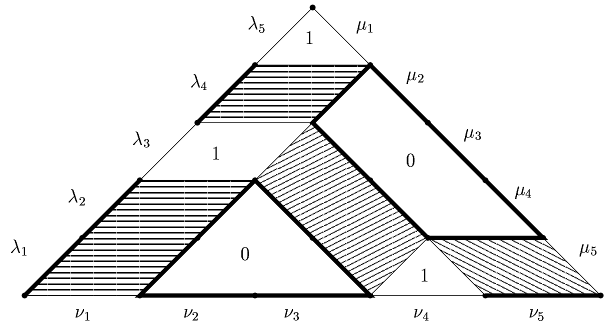


As pointed out by Danilov and Koshevoy [DK], this can be simplified, without loss of information, to give a labyrinth or hive plan by deleting all interior edges of the three types of region: shaded corridors in the form of parallelograms consisting of rhombi of just one type, either R1, or R2 or R3, and dark rooms and light rooms that are convex polygons consisting solely of just dark triangles and just light triangles, respectively.



It is a remarkable fact [KTW] that for each positive integer $r \leq n$ and triple $(I, J, K) \in R^n_r$, there exists a unique puzzle, and correspondingly a unique hive plan of the above type. In this hive plan the dark edges on the boundary are those specified by I, J and K. In connection with the above Conjecture 4.1, the thick edges on the boundary of each LR-hive are then labelled by the parts of λ_I, μ_J andν_K, and the thin edges by the parts of $\lambda_{\tilde{I}}$, $\mu_{\tilde{J}}$ and $\nu_{\tilde{k}}$.

To take a different example with $n = 5$, $r = 3$, $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$, $K = \{2, 3, 5\}$, we have $\alpha(I) = (1, 0, 0)$, $\beta(J) = (1, 1, 1)$ and $\gamma(K) = (2, 1, 1)$. The fact that $(I, J, K) \in R^5_3$ then follows from the observation that $c^{211}_{1,111} = 1$. Superposing the corresponding hive plan on the LR-hives with boundaries specified by λ, μ and ν then gives



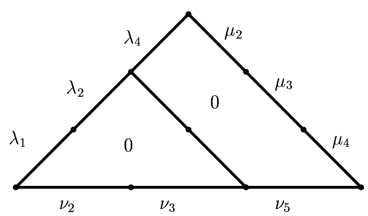
The validity of Conjecture 4.1 would imply that if

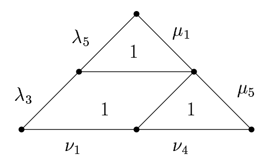
$$\nu_2 + \nu_3 + \nu_5 = \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4. \tag{4.3}$$

then $c^{\nu}_{\lambda\mu}$ must factorise as follows

$$c^{\nu}_{\lambda\mu} = c^{(\nu_2,\nu_3,\nu_5)}_{(\lambda_1,\lambda_2,\lambda_4),(\mu_2,\mu_3,\mu_4)}\ c^{(\nu_1,\nu_4)}_{(\lambda_3,\lambda_5),(\mu_1,\mu_5)} \tag{4.4}$$

To see how this comes about one just deletes the corridors from the initial LR-hive and glues together all the dark rooms, labelled 0, and all the light rooms, labelled 1, to create two smaller LR-hives, as shown below:





To prove the validity of such a factorisation, one has to show that the equality (4.3) leads to a bijection between all the large LR-hives and all pairs of small LR-hives obtained by the deletion and glueing processes. The proof is a matter of confirming that in such a case all the relevant hive conditions are satisfied not only after carrying out the deletion and glueing, but also after carrying out their inverses, cutting a pair of small LR-hives and inserting corridors to create a large LR-hive. A procedure for confirming this will be discussed in the accompanying talk.

An explicit illustration of the result is provided by the case $\lambda = (7,5,3,0,0)$, $\mu = (7,4,2,0,0)$ and $\nu = (8,8,8,2,2)$ for which $\lambda_I = (7,5,0)$, $\mu_J = (4,2,0)$ and $\nu_K = (8,8,2)$ with $\lambda_{\bar{I}} = (3,0)$, $\mu_{\bar{J}} = (7,0)$ and $\nu_{\bar{K}} = (8,2)$. In this case

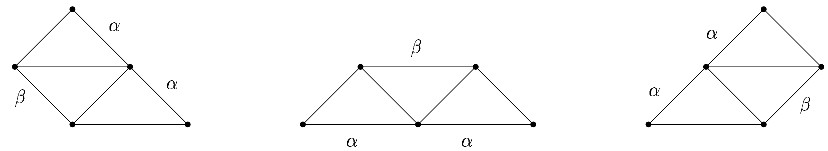
$$c^{88822}_{753,742} = c^{882}_{75,42}\ c^{82}_{3,7} = 1 \cdot 1 = 1. \tag{4.5}$$

Although this trivial result may not appear very exciting, we shall see that it has some important features. First of all the Littlewood-Richardson coeficient has turned out to be 1. It follows from Fulton's Conjecture that the corresponding stretched LR-polynomial is also just 1, that is to say its degree is 0. The question is, could this have been predicted, and what more generally can one say about the degrees of such polynomials? It is in this setting that the above factorisation is important.

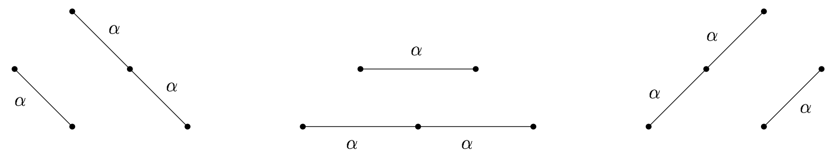
5. Degrees of stretched polynomials

Even given a certain amount of factorisation that reduces the evaluation of stretched Littlewood-Richardson polynomials to that of calculating these LR-polynomials in the primitive case, their evaluation may be combinatorially formidable. In any given case a knowledge of the degree of the polynomial would be extremely advantageous. Here we establish an upper bound on this degree by means of the following rather innocuous looking obervation.

Taking $\alpha = \gamma$ in each of the 5-vertex diagrams encountered earlier, gives



In each case the rhombus constraints give $\alpha \geq \beta \geq \alpha$ so that we must have $\beta = \alpha$. This result can be displayed more simply by suppressing all the labels on the vertices of the hives and inserting an edge between pairs of vertices whose labels differ by the same integer α. This gives the diagrams:

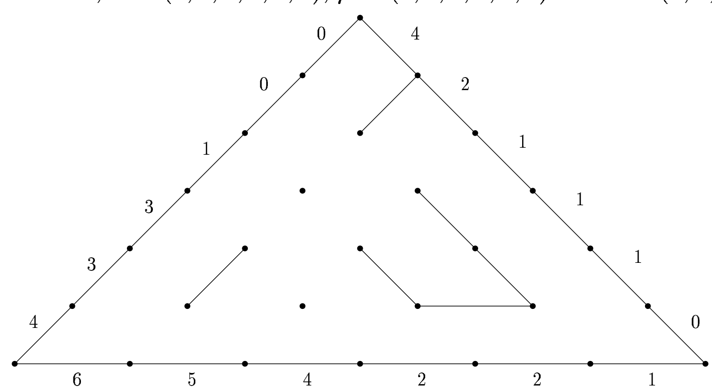


where in each case the equality of neighbouring differences α in a linear sequence of three vertices forces an identical difference α between two vertices in the neighbouring line.

Applying these notions to our LR-hives with boundaries of length n and with border labels determined by λ, μ and ν, it follows from the above that any equalities of successive parts of these partitions propagate as equalities of differences in hive entries within each LR-hive. To be more precise let all the λ-boundary edges be labelled by the parts of λ. If any sequence of parts of λ share the same value, say α, then we can identify an equilateral sub-hive having the sequence of equally labelled edges as one boundary, with its other boundaries parallel to the μ and ν-boundaries of the original hive. Within this sub-hive all the vertices along lines parallel to the λ-boundary are to be connected by edges indicating that in any LR-hive the differences in values between neighbouring entries along these lines are all α.

This process is to be repeated first for all sequences of equal edge labels along the λ-boundary, and then for all sequences of equal edge labels along the μ and ν boundaries. Finally, all neighbouring vertices on all three boundaries are to be connected by edges. In this way we arrive at a skeletal graph $G_{n;\lambda\mu\nu}$ of the hive.

For example, for $n = 6$, $\lambda = (4,3,3,1,0,0)$, $\mu = (4,2,1,1,1,0)$ and $\nu = (6,5,4,2,2,1)$ we have



The important of such skeletal graphs is that they indicates constraints on LR-hive entries that are implied by the specification of the boundary labels. These constraints on the interior vertex labels reduce the total number of degrees of freedom of such labels. This leads to the following:

Proposition 5.1. *Let λ, μ and ν be partitions such that $c^{\nu}_{\lambda\mu} > 0$. Let $\deg(P(t))$ be the degree of the corresponding stretched LR-polynomial $P(t) = c^{t\nu}_{t\lambda,t\mu}$. Let $d(G_{n;\lambda\mu\nu})$ be the number of connected interior components of the graph $G_{n;\lambda\mu\nu}$ that are not connected to the boundary. Then*

$$\deg(P(t)) \leq d(G_{n;\lambda\mu\nu}). \tag{5.1}$$

Proof The application of the stretching parameter t leaves $G_{n;\lambda\mu\nu}$ unaltered, so that the number of degrees of freedom in assigning entries to the stretched LR-hives is still $d(G_{n;\lambda\mu\nu})$. For each interior connected component that is not connected to the boundary we can select any one convenient vertex. The value a_{ij} of each such selected interior vertex label may or may not be fixed by the hive constraints. However, it will be subject to linear inequalities of the form $p \leq a_{ij} \leq q$ arising from the hive conditions. As the boundary vertex and edge labels are scaled by t, then all the parameters specifying these linear inequalities are also scaled by t to give $tp \leq a_{ij} \leq tq$. Hence, in enumerating all possible LR-hives in the stretched case, the freedom in assigning a_{ij} gives rise to a contribution to $P_{\lambda\mu}(t)$ that is at most linear in t. It follows that the degree of this polynomial is at most $d(G_{n;\lambda\mu\nu})$. □

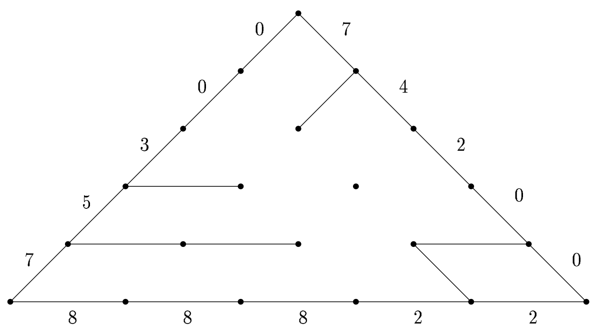
Unfortunately, the interior edges arising from sequences of equal parts in λ, μ and ν may intersect at common vertices. This makes it difficult to arrive at a formula for $d(G_{n;\lambda\mu\nu})$.

In the above example for $n = 6$ with $\lambda = (4,3,3,1,0,0)$, $\mu = (4,2,1,1,1,0)$ and $\nu = (6,5,4,2,2,1)$ we have $d(G_{n;\lambda\mu\nu}) = 4$ and the corresponding polynomial is given by

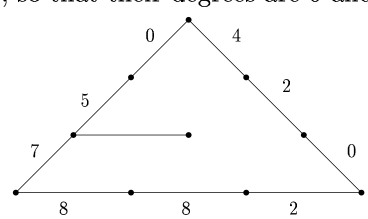
$$P^{\nu}_{\lambda\mu}(t) = \frac{1}{24}(t+1)(t+2)(t+3)(t+4), \tag{5.2}$$

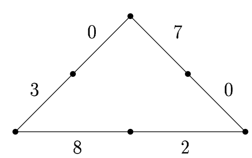
so that in this case the above bound on the polynomial degree is staurated.

On the otherhand, for our previous example in the case $n = 5$ and $\lambda = (7,5,3,0,0)$, $\mu = (7,4,2,0,0)$ and $\nu = (8,8,8,2,2)$, the graph $G_{n;\lambda\mu\nu}$ takes the form:



As can be seen there is just one interior vertex that is not connected by means of edges to the boundary. It follows that $d(G_{n;\lambda\mu\nu}) = 1$. However, in this case we know that $c^{\nu}_{\lambda\mu} = 1$ so that by Fulton's Conjecture we have $P^{\nu}_{\lambda\mu}(t) = 1$and the degree of the LR-polynomial is 0. Thus the above degree bound is not saturated. The explanation for this can be seen in the skeletal diagrams of the factors arising in this non-primitive case. In these skeletal diagrams all vertices are connected to the boundary, so that their degrees are 0 and the corresponding LR-polynomials are both 1.





Encourgaged by these results and many other examples we conjecture that in the primitive case the bound in Proposition 5.1 is saturated, that is we have:

Conjecture 5.2. *If* $c^{\nu}_{\lambda\mu}$ *is primitive and* $P(t) = c^{t\nu}_{t\lambda,t\mu}$ *then*

$$\deg(P(t)) = d(G_{n;\lambda\mu\nu}), \tag{5.3}$$

and conversely, if $\deg(P(t)) < d(G_{n;\lambda\mu\nu})$ *then* $c^{\nu}_{\lambda\mu}$ *is not-primitive.*

6. Linear factors

It will have been noted that in our illustrative examples (1.3) and (5.2) the stretched Littlwood-Richardson polynomials $P^{\nu}_{\lambda\mu}(t)$ contain factors $(t+m)$ for some sequence of values $m = 1, 2, \ldots, M$ for some positive integer M. This is no accident since $P^{\nu}_{\lambda\mu}(t)$ is nothing other than an Ehrhart quasi-polynomial $i(\mathcal{P}, t)$ of a rational complex polytope $\mathcal{P}$ defined by the set of linear inequalities corresponding to the LR-hive conditions. This quasi-polynomial is actually a polynomial, but whether this is the case or not, the reciprocity theorem for Ehrhart quasi-polynomials [S] states that $i(\mathcal{P},t)$ is defined for all integers t and that for $t=-m$ with m a positive integer $i(\mathcal{P},-m)=(-1)^d\,\bar{i}(\mathcal{P},m)$, where d is the dimension of the polytope $\mathcal{P}$, and $\bar{i}(\mathcal{P},m)$ is the number of integer points inside $m\mathcal{P}$. This number of integer points may be zero, thereby giving rise to a zero of $P(t)=i(\mathcal{P},t)$ at $t=-m$. Moreover, if this number is zero for $m=M$ and non-zero for $m=M+1$ it follows from its geometric interpretation that it is zero for $m=1,2,\ldots,M$ and non-zero for all $m>M$. In such a case $P^{\nu}_{\lambda\mu}(t)$ necessarily contains $(t+1)(t+2)\cdots(t+M)$ as a factor.

In this section we describe one particular approach to the determination of M, based on a conjecture regarding the continuation of $P^{\nu}_{\lambda\mu}(t)$ to negative integer values $t=-m$. For t a positive integer, we certainly have

$$s_{t\lambda}(x_1,x_2,\ldots,x_n)=\frac{\left|x_i^{t\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|}. \tag{6.1}$$

This may be readily extended to the case $t=-m$ with m a positive integer, to give

$$s_{-m\lambda}(x_1,x_2,\ldots,x_n)=\frac{\left|x_i^{-m\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|}=\frac{\left|x_i^{-m\lambda_{n-k+1}-n+k}\right|}{\left|x_i^{-n+k}\right|}, \tag{6.2}$$

where first x_i^{n-1} has been extracted as a common factor from the ith row of each determinant for $i=1,2,\ldots,n$ and cancelled from numerator and denominator, and then j replaced by $k=n-j+1$ with an appropriate reversal of order of the columns in both determinants. If we now set $\overline{x}_i=x_i^{-1}$ for $i=1,2,\ldots,n$. this gives

$$s_{-m\lambda}(x_1,x_2,\ldots,x_n)=\frac{\left|\overline{x}_i^{m\lambda_{n-k+1}+n-k}\right|}{\left|\overline{x}_i^{n-k}\right|}=s_{m\lambda_n,\ldots,m\lambda_2,m\lambda_1}(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_n). \tag{6.3}$$

To simplify the notation, for any partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ let $\tilde{\lambda}=(\lambda_n,\ldots,\lambda_2,\lambda_1)$ be the vector obtained by reversing the order of its parts. Then reverting to the indeterminates $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ we have:

$$s_{m\tilde{\lambda}}(\mathbf{x})=s_{m\lambda_n,\ldots,m\lambda_2,m\lambda_1}(x_1,x_2,\ldots,x_n)=\frac{\left|x_i^{m\lambda_{n-k+1}+n-k}\right|}{\left|x_i^{n-k}\right|}. \tag{6.4}$$

This allows us to make the following definition, which amounts to an extension of stretched Littlewood-Richardson coefficients to the domain of a negative stretching parameter $t=-m$:

Definition 6.1. *For any λ, μ and ν such that $c^{\nu}_{\lambda\mu}>0$, let $c^{-m\nu}_{-m\lambda,-m\mu}=c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}}$, for any positive integer m, where*

$$s_{m\tilde{\lambda}}(\mathbf{x})\ s_{m\tilde{\mu}}(\mathbf{x})=\sum_{\nu} c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}}\ s_{m\tilde{\nu}}(\mathbf{x}). \tag{6.5}$$

With this definition, the consideration of numerous examples, suggests the validity of the following:

Conjecture 6.2. *Let $c^{\nu}_{\lambda\mu}>0$ be primitive and let the corresponding LR-polynomial be $P^{\nu}_{\lambda\mu}(t)$. Then the value of this LR-polynomial at negative integer values $t=-m$ coincides with the corresponding negatively stretched Littlewood-Richardson coefficients, that is to say*

$$P^{\nu}_{\lambda\mu}(-m)=c^{-m\nu}_{-m\lambda,-m\mu}. \tag{6.6}$$

To exploit this it is necessary that Schur functions such as $s_{m\tilde{\lambda}}(\mathbf{x})$, as defined by (6.4), be standardised. This may be carried out by reordering the columns of the numerator determinant. However there are two quite different possible outcomes: either $s_{m\tilde{\lambda}}(\mathbf{x})=0$ or $s_{m\tilde{\lambda}}(\mathbf{x})=\eta_\rho\,s_\rho(\mathbf{x})$ for some partition ρ with $\eta_\rho=\pm1$. Similar results apply to $s_{m\tilde{\mu}}(\mathbf{x})$ and $s_{m\tilde{\nu}}(\mathbf{x})$. The validity of the above conjecture would then imply that

$$P^{\nu}_{\lambda\mu}(-m)=\eta^{\rho}_{\lambda\mu}c^{\tau}_{\rho\sigma} \tag{6.7}$$

where $\eta^{\nu}_{\lambda\mu} = 0$ if any one of $s_{m\tilde{\lambda}}(\mathbf{x})$, $s_{m\tilde{\mu}}(\mathbf{x})$ or $s_{m\tilde{\nu}}(\mathbf{x})$ is identically zero, and is ± 1 in all other cases, while ρ, σ and τ are defined by the identities $s_{m\tilde{\lambda}}(\mathbf{x}) = \eta_\rho\, s_\rho(\mathbf{x})$, $s_{m\tilde{\mu}}(\mathbf{x}) = \eta_\sigma\, s_\sigma(\mathbf{x})$ and $s_{m\tilde{\nu}}(\mathbf{x}) = \eta_\tau\, s_\tau(\mathbf{x})$.

It follows that we can expect two types of zero of $P^{\nu}_{\lambda\mu}(t)$ for $t = -m$: type (i) associated with $\eta^{\nu}_{\lambda\mu} = 0$ and and type (ii) associated with the vanishing of $c^{\tau}_{\rho\sigma}$.

To see this in an example consider the case $n = 7$, $\lambda = (4,3,3,2,1)$, $\mu = (4,3,2,2,1)$ and $\nu = (7,4,4,4,3,2,1)$ for which $P(t) = P^{\nu}_{\lambda\mu}(t)$ has already been given in (1.3). Here we find that $s_{m\tilde{\lambda}} = 0$ for $m = 1,2$, $s_{m\tilde{\mu}}(\mathbf{x}) = 0$ for $m = 1,2$ and $s_{m\tilde{\nu}}(\mathbf{x}) = 0$ for $m = 1,2,3$. This accounts for the three zeros associated with the factors $(t+1)$, $(t+2)$ and $(t+3)$. For all $m \geq 4$ we have $s_{m\tilde{\lambda}}(\mathbf{x}) = s_\rho(\mathbf{x})$, $s_{m\tilde{\mu}}(\mathbf{x}) = s_\sigma(\mathbf{x})$ and $s_{m\tilde{\nu}}(\mathbf{x}) = s_\tau(\mathbf{x})$ for some partitions ρ, σ and τ. It then remains to be seen whether or not $c^{\tau}_{\rho\sigma} = 0$.

Starting with $m = 4$ it is found that $\rho = (10,9,9,8,6,5,5)$, $\sigma = (10,8,7,7,6,5,5)$ and $\tau = (22,14,14,14,14,12,10)$. Since $\rho_1 + \sigma_1 = 20 < 22 = \tau_1$ it follows that $c^{\tau}_{\rho\sigma} = 0$. This accounts for a factor of $(t+4)$ in $P(t)$. Similarly with $m = 5$ it is found that $\rho = (14,12,12,10,7,5,5)$, $\sigma = (14,12,9,9,7,5,5)$ and $\tau = (19,18,18,18,17,14,11)$. This time since $\rho_1 + \sigma_1 = 28 < 29 = \tau_1$ it again follows that $c^{\tau}_{\rho\sigma} = 0$, thereby accounting for a factor of $(t+5)$ in $P(t)$. On the other hand for $m = 6$ it is found that $\rho = (18,15,15,12,8,5,5)$, $\sigma = (18,14,11,11,8,5,5)$ and $\tau = (36,22,22,22,20,16,12)$. This time it is found that $c^{\tau}_{\rho\sigma} = 3$. This implies that there is no factor $(t+6)$ in $P(t)$. Indeed it is easy to check from (1.3) that $P(-6) = 3$. In the same way we can derive the fact that $P(-7) = 39$ and $P(-8) = 247$, in perfect agreement with (1.3).

Thus, as a corollary of Conjecture 6.2, we are lead by virtue of (6.7) to:

Conjecture 6.3. *If $c_{\lambda\mu} > 0$ is primitive, then the LR-polynomial $P^{\nu}_{\lambda\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ contains a factor $(t+m)$ if and only if either $\eta^{\nu}_{\lambda\mu} = 0$ or $c^{\tau}_{\rho\sigma} = 0$.*

It can be shown that for sufficiently large m both $\eta^{\nu}_{\lambda\mu}$ and $c^{\tau}_{\rho\sigma}$ in (6.7) are positive, provided that the original $c^{\nu}_{\lambda\mu}$ is primitive. Moreover, it can be shown in such a case that if $\eta^{\nu}_{\lambda\mu} c^{\tau}_{\rho\sigma} = 0$ for some psoitive integer m, then the same is true for all smaller positive integers. This means that it is possible to identify M such that the right hand side of (6.7) is zero for all $m \leq M$ and non-zero for all $m > M$. This is entirely, consistent with the remarks made at the beginning of this section regarding the zeros of Ehrhart quasi-polynomials.

As a final conjecture we offer

Conjecture 6.4. *Let λ, μ and ν be partitions such that $c^{\nu}_{\lambda\mu} > 0$. Then $c^{\nu}_{\lambda\mu}$ is primitive if and only if $c^{-m\nu}_{-m\lambda,-m\mu}$ is non-zero for some positive integer m.*

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