ECO's carol

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Among the general methods usually employed in combinatorics (such as generating functions, theory of species, linear operator methods, order theory, incidence algebras, Hopf algebras, umbral calculus, and so on), I will deal with a particular one, which was born in Florence in the last decade of the past century, thanks to the work of a group of researchers in discrete mathematics and theoretical computer science. This method is usually referred to as the *ECO method* (ECO is an acronym, standing for Enumeration of Combinatorial Objects). The roots of the ECO method can be traced back to the paper [CGHK], where the authors study Baxter permutations. For the first time, a combinatorial construction is presented which can be described by a *generating tree*, as it usually happens for many ECO-construction. However, we have to wait for the master degree thesis of Alberto Del Lungo [DL], in 1992, to see the birth of the ECO method, whose first application was of a theoretical computer science nature, concerning the exhaustive generation of directed animals. However it became quickly evident that the range of applicability of such a method was much wider than suggested by a mere, specific problem of generation. In 1995 the first enumerative application of the ECO method was presented [BDLPP1], and various problems connected with the enumeration of k-coloured Motzkin paths were solved. Soon after this first result, a general methodology for plane tree enumeration was settled down [BDLPP2], and finally a general survey was published [BDLPP] where many enumerative examples are treated in great detail.

The ECO method consists basically of a way of constructing the objects of a class of combinatorial structures. Consider a class of objects $\mathcal{O}$ in which a concept of *size* is introduced. This means that a partition $\{\mathcal{O}_n \mid n \in \mathbf{N}\}$ of $\mathcal{O}$ is defined, such that $\mathcal{O}_n$ is the subset of the objects of size n. Our main problem is to determine the numerical sequence $f_n = |\mathcal{O}_n|$. The ECO method gives an answer to this question in several cases, providing a recursive construction of the objects of the class under consideration.

Let $\vartheta : \mathcal{O} \longrightarrow 2^{\mathcal{O}}$ be a function[1] such that, for any $n \in \mathbf{N}$, if $O \in \mathcal{O}_n$, then $\vartheta(O) \subseteq \mathcal{O}_{n+1}$. We say that ϑ is an *ECO operator* when the family of

[1] for a given set A, we denote by 2^A the power set of A.

sets $\{\vartheta(O) \mid O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$. The ECO method consists precisely of the effective construction of an ECO operator $\vartheta = \vartheta_{\mathcal{O}}$ for the class of objects $\mathcal{O}$; it is clear that the enumeration of $\mathcal{O}$ is possible only if such a construction is sufficiently regular. Typically, the construction of an ECO operator allows to find a functional equation satisfied by the (ordinary) generating function of $\mathcal{O}$ (or a recursive relation satisfied by the sequence $f_n = |\mathcal{O}_n|$), whose solution is often provided by the application of suitable analytical tools.

The purpose of my talk is threefold. First of all, I would like to run through the main stages of ECO by showing its birth and rapid development from a historical and chronological point of view. Next I will try to give a detailed survey of what have been done with the ECO method. More precisely, I will show how ECO can be fruitfully used in enumerative and algebraic combinatorics, in bijective combinatorics and in the random and exhaustive generation of combinatorial objects. This will be done by first presenting the algebraic foundations of ECO, mainly interpreting it in a linear algebra context, and then by describing several enumerative and bijective results concerning disparate combinatorial objects, such as plane trees, lattice paths, pattern avoiding permutations, polyominoes, restricted set partitions. In particular, the generating tree associated with a given ECO construction provides a fundamental tool for finding bijections which also preserve several statistics. As far as generation is concerned, I will show how the ECO method naturally suggests a general method to randomly and exhaustively generate several classes of combinatorial objects, depending on the nature of their generating function (rational, algebraic, transcendental). Finally, I intend to provide some ideas for further work, by proposing open problems which naturally arise in this context. This suggestions will be scattered throughout the talk, each of them appearing where it seemed to me more suitable.

References

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