BERGMAN COMPLEXES, COXETER ARRANGEMENTS, AND GRAPH ASSOCIAHEDRA

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Abstract

Tropical varieties play an important role in algebraic geometry. The Bergman complex $\mathcal{B}(M)$ and the positive Bergman complex $\mathcal{B}^+(M)$ of an oriented matroid M generalize to matroids the notions of the tropical variety and positive tropical variety associated to a linear ideal. Our main result is that if $\mathcal{A}$ is a Coxeter arrangement of type Φ with corresponding oriented matroid M_Φ, then $\mathcal{B}^+(M_\Phi)$ is dual to the graph associahedron of type Φ, and $\mathcal{B}(M_\Phi)$ equals the nested set complex of $\mathcal{A}$.

Résumé. Les variétés tropicales jouent un rôle important en géométrie algébrique. Le complexe de Bergman $\mathcal{B}(M)$ et le complexe de Bergman positif $\mathcal{B}^+(M)$ d'un matroïde orienté M étendent aux matroïdes les notions de variété tropicale et de variété tropicale positive associées à un idéal linéaire. Notre résultat principal est que si $\mathcal{A}$ est un arrangement de Coxeter de type Φ, et si M_Φ est le matroïde orienté correspondant, alors $\mathcal{B}^+(M_\Phi)$ est le dual de l'associaèdre du graphe de type Φ, et $\mathcal{B}(M_\Phi)$ est le complexe des ensembles imbriqués de $\mathcal{A}$.

1. Introduction

In this paper we study the Bergman complex and the positive Bergman complex of a Coxeter arrangement, and we relate them to the nested set complexes that arise in De Concini and Procesi's wonderful arrangement models [8, 9], and to the graph associahedra introduced by Carr and Devadoss [6], by Davis, Januszkiewicz, and Scott [7], and by Postnikov [14].

The *Bergman complex* of a matroid is a pure polyhedral complex which can be associated to any matroid. It was first defined by Sturmfels [18] in order to generalize to matroids the notion of a tropical variety associated to a linear ideal. The Bergman complex can be described in terms of the lattice of flats of the matroid, and is homotopy equivalent to a wedge of spheres, as shown by Ardila and Klivans [1].

The *positive Bergman complex* $\mathcal{B}^+(M)$ of an oriented matroid M is a subcomplex of the Bergman complex of the underlying unoriented matroid $\underline{M}$. It generalizes to oriented matroids the notion of the positive tropical variety associated to a linear ideal. $\mathcal{B}^+(M)$ depends on a choice of acyclic orientation of M, and as one varies this acyclic orientation, one gets a covering of the Bergman complex of $\underline{M}$. The positive Bergman complex can be described in terms of the Las Vergnas face lattice of M and it is homeomorphic to a sphere, as shown by Ardila, Klivans, and Williams [2].

Graph associahedra are polytopes which generalize the associahedron, which were discovered independently by Carr and Devadoss [6], by Davis, Januszkiewicz, and Scott [7], and by Postnikov [14]. There is an intrinsic tiling by associahedra of the Deligne-Knudsen-Mumford compactification of the real moduli space of curves $\overline{M_0^n(\mathbb{R})}$, a space which is related to the Coxeter complex of type A. The motivation for Carr and Devadoss' work was the desire to generalize this phenomenon to all simplicial Coxeter systems.

Let $\mathcal{A}_\Phi$ be the Coxeter arrangement corresponding to the (possibly infinite, possibly non-crystallographic) root system Φ associated to a Coxeter system (W, S) with diagram Γ; see Section 4 below. Choose a region R of the arrangement, and let M_Φ be the oriented matroid associated to $\mathcal{A}_\Phi$ and R. In this paper we prove:

Theorem 1.1. *The positive Bergman complex $\mathcal{B}^+(M_\Phi)$ of the arrangement $\mathcal{A}_\Phi$ is dual to the graph associahedron $P(\Gamma)$.*

In particular, the cellular sphere $\mathcal{B}^+(M_\Phi)$ is actually a simplicial sphere, and a *flag* (or *clique*) complex.

This result is also related to the *wonderful model of a hyperplane arrangement* and to *nested set complexes*. The wonderful model of a hyperplane arrangement is obtained by blowing up the non-normal crossings of the arrangement, leaving its complement unchanged. De Concini and Procesi [8] introduced this model in order to study the topology of this complement. They showed that the *nested sets* of the arrangement encode the underlying combinatorics. Feichtner and Kozlov [9] gave an abstract notion of the *nested set complex* for any meet-semilattice, and Feichtner and Müller [10] studied its topology. Recently, Feichtner and Sturmfels [11] studied the relation between the Bergman fan and the nested set complexes (see Section 5 below).

In this paper we also prove:

Theorem 1.2. *The Bergman complex $\mathcal{B}(M_\Phi)$ of $\mathcal{A}_\Phi$ equals its nested set complex.*

In particular, the cell complex $\mathcal{B}(M_\Phi)$ is actually a simplicial complex.

2. The Bergman complex and the positive Bergman complex

Our goal in this section is to explain the notions of the Bergman complex of a matroid and the positive Bergman complex of an oriented matroid which were studied in [1] and [2]. In order to do so we must review a certain operation on matroids and oriented matroids.

Definition. Let M be a matroid or oriented matroid of rank r on the ground set $[n]$, and let $\omega \in \mathbb{R}^n$. Regard ω as a weight function on M, so that the weight of a basis $B = \{b_1, \dots, b_r\}$ of M is given by $\omega_B = \omega_{b_1} + \omega_{b_2} + \cdots + \omega_{b_r}$. Let B_ω be the collection of bases of M having minimum ω-weight. (If M is oriented, then bases in B_ω inherit orientations from bases of M.) This collection is itself the set of bases of a matroid (or oriented matroid) which we call M_ω.

It is not obvious that M_ω is well-defined. However, when M is an unoriented matroid, we can see this by considering the matroid polytope of M: the face that minimizes the linear functional ω is precisely the matroid polytope of M_ω. For a proof that M_ω is well-defined when M is oriented, see [2].

Notice that M_ω will not change if we translate ω or scale it by a positive constant. We can therefore restrict our attention to the sphere $S^{n-2} := \{\omega \in \mathbb{R}^n : \omega_1 + \cdots + \omega_n = 0\,, \omega_1^2 + \cdots + \omega_n^2 = 1\}$. The Bergman complex of M will be a certain subset of this sphere.

The matroid M_ω depends only on a certain flag associated to ω.

Definition. Given $\omega \in \mathbb{R}^n$, let $\mathcal{F}(\omega)$ denote the unique flag of subsets

$$\emptyset = F_0 \subset F_1 \subset \cdots \subset F_k \subset F_{k+1} = [n] \tag{1}$$

such that ω is constant on each set $F_i \setminus F_{i-1}$ and satisfies $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$. We call $\mathcal{F}(\omega)$ the *flag* of ω, and we say that the *weight class* of ω or of the flag $\mathcal{F}$ is the set of vectors ν such that $\mathcal{F}(\nu) = \mathcal{F}$.

It is shown in [1] that M_ω depends only on the flag $\mathcal{F} := \mathcal{F}(\omega)$; specifically

$$M_\omega = \bigoplus_{i=1}^{k+1} F_i / F_{i-1} \tag{2}$$

where F_i/F_{i-1} is obtained from the matroid restriction of M to F_i by quotienting out the flat F_{i-1}. Hence we we also refer to this oriented matroid M_ω as $M_{\mathcal{F}}$.

Definition/ Theorem 2.1. [1] *The Bergman complex of a matroid M on the ground set $[n]$ is the set*

$$\begin{aligned}\mathcal{B}(M) &= \{\omega \in S^{n-2} \ : \ M_{\mathcal{F}(\omega)} \textit{ has no loops}\} \\ &= \{\omega \in S^{n-2} \ : \ \mathcal{F}(\omega) \textit{ is a flag of flats of } M\}\end{aligned}$$

Since the matroid M_ω depends only on the weight class that ω is in, the Bergman complex of M is the disjoint union of the weight classes of flags $\mathcal{F}$ such that $M_{\mathcal{F}}$ has no loops. We say that the weight class of a flag $\mathcal{F}$ is *valid* for M if $M_{\mathcal{F}}$ has no loops.

There are two polyhedral subdivisions of $\mathcal{B}(M)$, one of which is clearly finer than the other.

Definition. The *fine subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into valid weight classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $\mathcal{F}(u) = \mathcal{F}(v)$. The *coarse subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into M_ω-equivalence classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $M_u = M_v$.

The fine subdivision gives the following corollary of Theorem 2.1.

Corollary 2.2. [1] *Let M be a matroid. The fine subdivision of the Bergman complex $\mathcal{B}(M)$ is a geometric realization of $\Delta(L_M - \{\hat{0}\,,\hat{1}\,\})$, the order complex of the proper part of the lattice of flats of M. It follows that $\mathcal{B}(M)$ is homotopy equivalent to a wedge of spheres.*

There are positive analogues of all of the above definitions and theorems. First we must give the definition of *positive covectors* and *positive flats*.

Definition. Let M be an acyclic oriented matroid on the ground set $[n]$. We say that a covector $v \in \{+,-,0\}^n$ of M is *positive* if each of its entries is $+$ or 0. We say that a flat of M is *positive* if it is the 0-set of a positive covector.

Observation 2.3. *If M is the acyclic oriented matroid corresponding to a hyperplane arrangement $\mathcal{A}$ and a specified region R, then the positive flats are in correspondence with the faces of R.*

For example, consider the braid arrangement A_3, consisting of the six hyperplanes $x_i = x_j, 1 \leq i < j \leq 4$ in $\mathbb{R}^4$. Figure 1 illustrates this arrangement, when intersected with the hyperplane $x_4 = 0$ and the sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Let R be the region specified by the inequalities $x_1 \geq x_2 \geq x_3 \geq x_4$, and let M_{A_3} be the oriented matroid corresponding to the arrangement A_3 and the region R. Then the positive flats are $\emptyset, 1, 4, 6, 124, 16, 456$ and 123456.

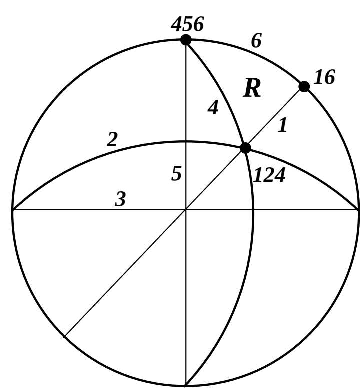

FIGURE 1. The braid arrangement A_3.

Definition/ Theorem 2.4. [2] *The positive Bergman complex of M is*

$$\begin{aligned} \mathcal{B}^+(M) &= \{\omega \in S^{n-2} \ : \ M_{\mathcal{F}(\omega)} \textit{ is acyclic}\} \\ &= \{\omega \in S^{n-2} \ : \ \mathcal{F}(\omega) \textit{ is a flag of positive flats of } M\} \end{aligned}$$

Within each equivalence class of the coarse subdivision of $\mathcal{B}(M)$, the vectors ω give rise to the same unoriented M_ω. Since the orientation of M_ω is inherited from that of M, they also give rise to the same oriented matroid M_ω. Therefore each coarse cell of $\mathcal{B}(M)$ is either completely contained in or disjoint from $\mathcal{B}^+(M)$. Thus $\mathcal{B}^+(M)$ inherits the coarse and the fine subdivisions from $\mathcal{B}(M)$, and each subdivision of $\mathcal{B}^+(M)$ is a subcomplex of the corresponding subdivision of $\mathcal{B}(M)$.

Recall that the *Las Vergnas face lattice* $\mathcal{F}_{\ell v}(M)$ is the lattice of positive flats of M, ordered by containment. Note that the lattice of positive flats of the oriented matroid M sits inside L_M, the lattice of flats of M. By Observation 2.3, if M is the oriented matroid of the arrangement $\mathcal{A}$ and the region R, then $\mathcal{F}_{\ell v}(M)$ is the face poset of R.

Corollary 2.5. [2] *Let M be an oriented matroid. Then the fine subdivision of $\mathcal{B}^+(M)$ is a geometric realization of $\Delta(\mathcal{F}_{\ell v}(M) - \{\,\hat{0}\,,\hat{1}\,\})$, the order complex of the proper part of the Las Vergnas face lattice of M. It follows that the positive Bergman complex of an oriented matroid is homeomorphic to a sphere.*

Therefore $\mathcal{B}^+(M)$ is one of the spheres in $\mathcal{B}(M)$.

Example 2.6. *Let M be the oriented matroid from Figure 1. The positive flats of M are $\{\emptyset, 1, 4, 6, 16, 124, 456, 123456\}$. The lattice of positive flats of M is shown in bold in Figure 2, within the lattice of flats of M.*

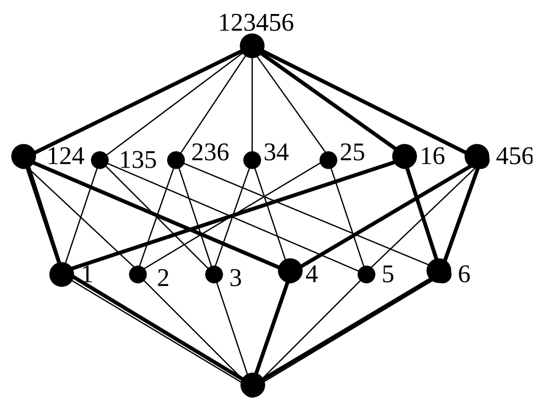

FIGURE 2. The lattice of positive flats within the lattice of flats.

We close this section with some observations about when two flags of flats in M correspond to the same cell of the coarse subdivision of $\mathcal{B}(M)$. Recall that the *connected components* of matroid M are the equivalence classes for the following equivalence relation on the ground set E of M: say $e \sim e'$ for two elements e, e' in E whenever they lie in a common circuit of M, and then take the transitive closure of $\sim$. Recall also that every connected component is a flat of M, and M decomposes (uniquely) as the direct sum of its connected components.

Definition. To each flag $\mathcal{F}$ of flats of a matroid M indexed as in (1), associate a forest $T_{\mathcal{F}}$ of rooted trees, in which each vertex v is labelled by a flat $F(v)$, as follows:

- For each connected component F of the matroid M, create a rooted tree (as specified below) and label its root vertex with F.
- For each vertex v already created, and already labelled by some flat $F(v)$ which is a connected component of some flat F_j in the flag $\mathcal{F}$, create children of v labelled by each of the connected components of F_{j-1} which are contained *properly* in $F(v)$.

Alternatively, one can construct the forest $T_{\mathcal{F}}$ by listing all the connected components of all the flats in $\mathcal{F}$, and partially ordering them by inclusion.

Proposition 2.7. *For any flag $\mathcal{F}$ of flats in a matroid M, the labelled forest $T_{\mathcal{F}}$ determines the matroid $M_{\mathcal{F}}$.*

In general, the converse of this proposition does *not* hold; one can have $M_{\mathcal{F}} = M_{\mathcal{F}'}$ without $T_{\mathcal{F}} = T_{\mathcal{F}'}$. For example (cf. [11, Example 1.2]), in the matroid M on ground set $E = \{1, 2, 3, 4, 5\}$ having rank 3 and circuits $\{123, 145, 2345\}$, the two flags

$$\mathcal{F} := (\emptyset \subset 1 \subset 123 \subset 12345)$$
$$\mathcal{F}' := (\emptyset \subset 1 \subset 145 \subset 12345).$$

exhibit this possibility.

However, we can give at least one nice hypothesis that allows one to reconstruct $T_{\mathcal{F}}$ from $M_{\mathcal{F}}$. Given a base B of a matroid M on ground set E, and any element $e \in E\backslash B$, there is a unique circuit of M contained in $B \cup \{e\}$, called the *basic circuit* $\mathrm{circ}(B, e)$. Note that the flat spanned by $\mathrm{circ}(B, e)$ will always be a connected flat.

Definition. Say that a base B of a matroid M is *circuitous* if every connected flat spanned by a subset of B is spanned by the basic circuit $\mathrm{circ}(B, e)$ for some $e \in E\backslash B$.

Note that the basic circuit $\mathrm{circ}(B, e)$ spanning the connected flat F must be $(F \cap B) \cup e$. Before we state our proposition, we give two useful lemmas.

Lemma 2.8. *Let F be a flat in a matroid, spanned by some independent set I. Then every connected component of F is spanned by some subset of I, namely, by the intersection of that component with I.*

Lemma 2.9. *Let $F \subset G$ be flats of a matroid that are spanned by subsets of a circuitous base B. If G is connected, then G/F is also connected.*

Proposition 2.10. *Let B be a circuitous base of a matroid M. Then for any two flags $\mathcal{F}, \mathcal{F}'$ of flats spanned by subsets of B, one has $M_{\mathcal{F}} = M_{\mathcal{F}'}$ if and only if $T_{\mathcal{F}} = T_{\mathcal{F}'}$.*

It will turn out that the simple roots Δ of a root system Φ always form a circuitous base for the associated matroid M_{Φ}; see Proposition 4.3(iii) below.

Remark 2.11. When the matroid M is connected, the forest $T_{\mathcal{F}}$ constructed above is a rooted tree. It coincides with the tree constructed by Feichtner and Sturmfels in [11, Proposition 3.1] when they choose the minimal building set for their lattice. In this way, Proposition 2.7 follows from [11, Theorem 4.4].

3. Graph associahedra

Graph associahedra are polytopes which generalize the associahedron, which were discovered independently by Carr and Devadoss [6], Davis, Januszkiewicz, and Scott [7], and Postnikov [14]. There is an intrinsic tiling by associahedra of the Deligne-Knudsen-Mumford compactification of the real moduli space of curves $\overline{M_0^n}(\mathbb{R})$, a space which is related to the Coxeter complex of type A. The motivation for Carr and Devadoss' work was the desire to generalize this phenomenon to all Coxeter systems.

In order to define graph associahedra, we must introduce the notions of tubes and tubings. We follow the presentation of [6].

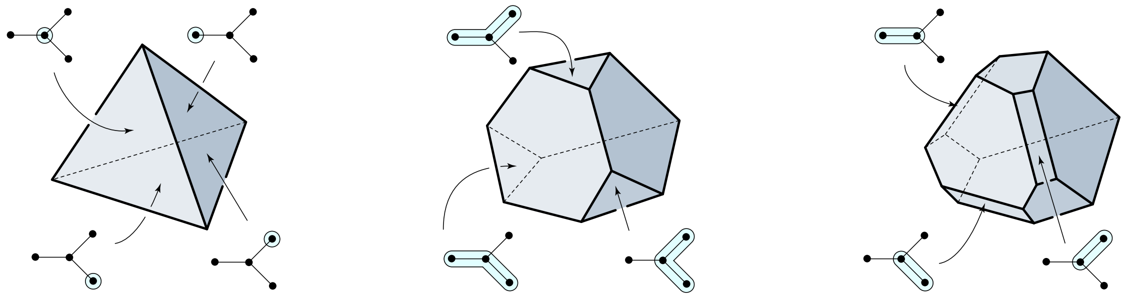

FIGURE 3. $P(D_4)$ ©Satyan Devadoss

Definition. Let Γ be a graph. A *tube* is a proper nonempty set of nodes of Γ whose induced graph is a proper, connected subgraph of Γ. There are three ways that two tubes can interact on the graph:

- Tubes are *nested* if $t_1 \subset t_2$.
- Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.
- Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in Γ.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of Γ is a set of tubes of Γ such that every pair of tubes in T is compatible. A k-*tubing* is a tubing with k tubes.

Graph-associahedra are defined via a construction which we will now describe.

Definition. Let Γ be a graph on n nodes. (Note that Γ need not be the graph of a Coxeter system.) Let Δ_Γ be the $n-1$ simplex in which each facet corresponds to a particular node. Note that each proper subset of nodes of Γ corresponds to a unique face of Δ_Γ, defined by the intersection of the faces associated to those nodes. The empty set corresponds to the face which is the entire polytope Δ_Γ. For a given graph Γ, truncate faces of Δ_Γ which correspond to 1-tubings in increasing order of dimension (i.e. first truncate vertices, then edges, then 2-faces, ...). The resulting polytope $P(\Gamma)$ is the *graph associahedron* of Carr and Devadoss.

Figure 3 illustrates the construction of the graph associahedron of a Coxeter diagram of type D_4. We start with a simplex, whose four facets correspond to the vertices of the diagram. In the first step, we truncate three of the vertices, to obtain the second polytope shown. We then truncate three of the edges, to obtain the third polytope shown. In the final step, we truncate the four facets which all correspond to tubes. This step is not shown in Figure 3, since it does not affect the combinatorial type of the polytope.

When the graph Γ is the n-element chain, the polytope $P(\Gamma)$ is the associahedron A_{n-1}. One can see this by considering an easy bijection between valid tubings and parenthesizations of a word of length $n-1$, as illustrated in Figure 4.

We thank Satyan Devadoss for allowing us to reproduce in our Figures 3 and 4, two of his figures from [6].

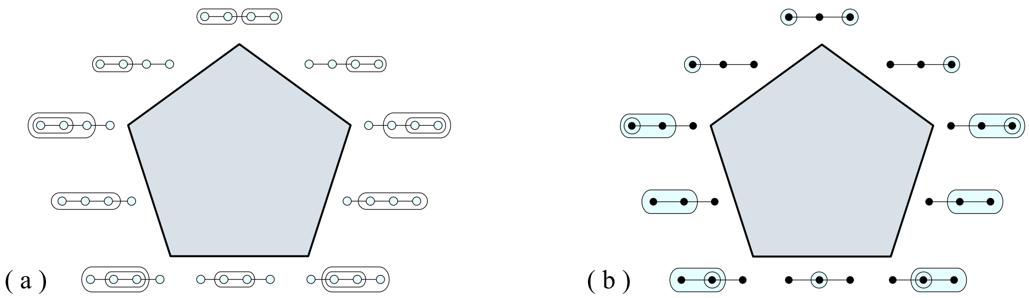

FIGURE 4. The associahedron A_2 is the graph associahedron of a 3-element chain. ©Satyan Devadoss

Carr and Devadoss proved that the face poset of $P(\Gamma)$ can be described in terms of valid tubings.

Theorem 3.1. [6] *The face poset of $P(\Gamma)$ is isomorphic to the set of valid tubings of Γ, ordered by reverse containment: $T < T'$ if T is obtained from T' by adding tubes.*

Corollary 3.2. [6] *When Γ is a path with $n-1$ nodes, $P(\Gamma)$ is the associahedron A_n of dimension n. When Γ is a cycle with $n-1$ nodes, $P(\Gamma)$ is the cyclohedron W_n.*

4. The positive Bergman complex of a Coxeter arrangement

In this section we prove that the positive Bergman complex of a Coxeter arrangement of type Φ is dual to the graph associahedron of type Φ. More precisely, both of these objects are homeomorphic to spheres of the same dimension, and their face posets are dual. We begin by reviewing our conventions about Coxeter systems and the related arrangements and matroids.

A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = 1,$$

where $m(s,s) = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. In case no relation occurs for a pair s, s', we make the convention that $m(s,s') = \infty$. We will always assume that S is finite.

Note that to specify a Coxeter system (W, S), it is enough to draw the corresponding *Coxeter diagram* Γ: this is a graph on vertices indexed by elements of S, with vertices s and s' joined by an edge labelled $m(s,s')$ whenever this number (∞ allowed) is at least 3.

Remark 4.1. In what follows, the reader should note that nothing will turn out to depend on the edge labels $m(s,s')$ of Γ; the positive Bergman complex, the Bergman complex, or the graph associahedron associated with Γ will depend only upon the undirected graph underlying Γ.

Although an arbitrary Coxeter system (W, S) need not have a faithful representation of W as a group generated by orthogonal reflections with respect to a positive definite inner product, there exists a reasonable substitute, called its *geometric representation* [12, Sec.

5.3, 5.13], which we recall here. Let $V := \mathbb{R}^{|S|}$ with a basis of *simple roots* $\Delta := \{\alpha_s : s \in S\}$. Define an $\mathbb{R}$-valued bilinear form $(\cdot, \cdot)$ on V by

$$(\alpha_s, \alpha_{s'}) := -\cos\left(\frac{\pi}{m(s,s')}\right)$$

and let s act on V by the "reflection" that fixes $\alpha_s^\perp$ and negates α_s:

$$s(v) := v - 2(v, \alpha_s)\alpha_s.$$

This turns out to extend to a faithful representation of W on V, and one defines the *root system* Φ and *positive roots* Φ^+ by

$$\Phi := \{w(\alpha_s) : w \in W, s \in S\}$$
$$\Phi^+ := \{\alpha \in \Phi : \alpha = \sum_{s\in S} c_s\alpha_s \text{ with } c_s \geq 0\}$$

It turns out that $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^- := -\Phi^+$. We use M_Φ to denote the matroid represented by Φ^+ in V, which is of finite rank $r = |S|$, but has ground set E of possibly (countably) infinite cardinality. Its lattice of flats L_{M_Φ} may be infinite, although of finite rank r, and is well-known (see, e.g. [3]) to be isomorphic to the poset of *parabolic subgroups*

$$\{wW_Jw^{-1} : w \in W, J \subseteq S\}$$

ordered by inclusion. In other words, every flat F is spanned by $w(\Phi_J^+)$ for some standard parabolic subroot system Φ_J^+ and $w \in W$.

Definition. Given a root $\alpha \in \Phi$, expressed uniquely in terms of the simple roots Δ as $\alpha = \sum_{s\in S} c_s\alpha_s$, define the *support* of α (written $\operatorname{supp}\alpha$) to be the vertex-induced subgraph of the Coxeter diagram Γ on the set of vertices $s \in S$ for which $c_s \neq 0$.

We will need the following well-known lemma about supports of roots. A proof of its first assertion for the Coxeter systems associated to Kac-Moody Lie algebras can be found in [13, Lemma 1.6]; we will need the assertion in general.

Lemma 4.2. *Let (W,S) be an arbitrary Coxeter system with Coxeter graph Γ. Then for any root $\alpha \in \Phi$ the graph $\operatorname{supp}\alpha$ is connected, and conversely, every connected subgraph Γ' of Γ occurs as $\operatorname{supp}\alpha$ for some positive root α.*

If one wants to think of the oriented matroid M_Φ as the oriented matroid of a hyperplane arrangement (as opposed to the oriented matroid of the configuration of vectors Φ^+), one must work with the contragredient representation V^* [12, 5.13]. Let $\{\delta_s : s \in S\}$ denote the basis for V^* dual to the basis of simple roots Δ for V. Then the *(closed) fundamental chamber* R is the nonnegative cone spanned by $\{\delta_s : s \in S\}$ inside V^*. The *Tits cone* is the union $\bigcup_{w\in W} w(R)$, a (possibly proper, not necessarily closed nor polyhedral) convex cone inside V^*. Every positive root $\alpha \in \Phi^+$ gives an oriented hyperplane H_α in V^* with nonnegative half-space $\{f \in V^* : f(\alpha) \geq 0\}$. These hyperplanes and half-spaces decompose the Tits cone[1] into (closed) regions that turn out to be simplicial cones which are exactly

[1]We should point out that when W is infinite, only part of the hyperplane or its nonnegative half-space lies inside the Tits cone, so we only consider their intersection with the Tits cone.

the images $w(R)$ as w runs through W; the tope (maximal covector) in the oriented matroid M_Φ associated to $w(R)$ will have the sign $+$ on the roots $\Phi^+ \cap w^{-1}(\Phi^+)$ and the sign $-$ on the roots $\Phi^+ \cap w^{-1}(\Phi^-)$.

Proposition 4.3. *Let (W,S) be an arbitrary Coxeter system, with root system Φ and Coxeter diagram Γ.*

(i) *Positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$.*
(ii) *Connected positive flats in the oriented matroid M_Φ correspond to subsets $J \subset S$ such that the vertex-induced subgraph Γ_J is connected, that is, to tubes in Γ.*
(iii) *The simple roots Δ form a circuitous base for the matroid M_Φ.*
(iv) *If $F \subset G$ are flats in M_Φ with G connected, then the matroid quotient G/F is connected.*

Proof. (i): The hyperplanes bounding the base region/tope R are $\{H_{\alpha_s} : s \in S\}$, so positive flats are those spanned by sets of the form $\{\alpha_s : s \in J\}$ for subsets $J \subset S$. We denote such a positive flat by $\mathrm{cl}(J)$.
(ii): Let $J \subset S$ with subgraph Γ_J, and consider its associated positive flat $\mathrm{cl}(J)$. The first assertion of Lemma 4.2 shows that $\mathrm{cl}(J)$ will not be connected if Γ_J is disconnected. To see this, represent the flat $\mathrm{cl}(J)$ by a matrix in which the rows correspond to simple roots of $\mathrm{cl}(J)$, i.e. vertices of Γ_J, and the columns express each positive root in $\mathrm{cl}(J)$ as a combination of simple roots. By permuting columns, one can obtain a matrix which is a block-direct sum of two smaller matrices, and hence $\mathrm{cl}(J)$ will not be connected.

On the other hand, if Γ_J is connected, then the second assertion of Lemma 4.2 shows that there is a positive root α with $\mathrm{supp}\,\alpha = \Gamma_J$, and consequently $\{\alpha_s : s \in J\} \cup \{\alpha\}$ gives a circuit in M_Φ spanning this flat, so it is connected.
(iii): This follows from the argument in (ii); given $J \subset S$ with Γ_J connected, the basic circuit $\mathrm{circ}(\Delta, \alpha)$ where $\mathrm{supp}\,\alpha = \Gamma_J$ spans the connected flat corresponding to J.
(iv): Let F, G correspond to the parabolic subgroups uW_Ju^{-1}, vW_Kv^{-1}, or equivalently, assume they are spanned by $u\Phi_J^+, v\Phi_K^+$. One can make the following reductions:

- Translating by v^{-1}, one can assume that v is the identity.
- Since (W_K, K) itself forms a Coxeter system with root system Φ_K, one can assume $M_\Phi = G$ and $K = S$. In particular, M_Φ is connected.
- Replacing the Coxeter system (W,S) by the system (W, uSu^{-1}), one can assume that u is the identity.

In other words, F is the positive flat corresponding to some subgraph Γ_J of Γ, and we must show M_Φ/F is a connected matroid. This is a consequence of (iii) and Lemma 2.9. □

We now give our main result.

Theorem 1.1. *Let (W,S) be an arbitrary Coxeter system, with root system Φ, Coxeter diagram Γ, and associated oriented matroid M_Φ. Then the face poset of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$ is dual to the face poset of the graph associahedron $P(\Gamma)$.*

Proof. By Theorem 3.1, we need to show that the face poset of (the coarse subdivision of) $\mathcal{B}^+(M_\Phi)$ is equal to the poset of tubings of Γ, ordered by containment. We begin by describing a map Ψ from flags of positive flats to tubings of Γ.

By Proposition 4.3, positive flats of M_Φ correspond to subsets $J \subset S$ or subgraphs Γ_J of the Coxeter graph Γ. Furthermore, a positive flat is connected if and only if Γ_J is a tube, and hence an arbitrary positive flat corresponds to a disjoint union of compatible tubes, no two of which are nested. Since an inclusion of flats corresponds to an inclusion of the subsets J, a flag $\mathcal{F}$ of positive flats corresponds to a nested chain of such unions of non-nested compatible tubes, that is, to a tubing $\Psi(\mathcal{F})$. Furthermore, in this correspondence, inclusion of flags corresponds to containment of tubings.

We claim that the map from flags to tubings is surjective. Given some tubing of Γ, linearly order its tubes $J_1, \ldots, J_k$ by any linear extension of the inclusion partial ordering, and then the flag $\mathcal{F}$ of positive flats having F_i spanned by $\{\alpha_s : s \in J_1 \cup J_2 \cup \cdots \cup J_i\}$ will map to this tubing.

Lastly, we show that Ψ is actually a well-defined injective map when regarded as a map on cells of the coarse subdivision of $\mathcal{B}^+(M_\Phi)$. To do so, it is enough to show that two flags $\mathcal{F}, \mathcal{F}'$ of positive flats give the same tubing if and only if $M_{\mathcal{F}}$ and $M_{\mathcal{F}'}$ coincide. By Lemma 4.3(iv) and Proposition 2.10, we need to show that $\Psi(\mathcal{F})$ and $\Psi(\mathcal{F}')$ coincide if and only if $T_{\mathcal{F}}$ and $T_{\mathcal{F}'}$ coincide. But this is clear, because by construction, the rooted forest $T_{\mathcal{F}}$ ignores the ordering within the flag, and only records the data of the tubes which appear, that is, the tubing.

□

Corollary 4.4. *The Bergman complex and the positive Bergman complex of a Coxeter arrangement $\mathcal{A}$ are both simplicial. The latter is furthermore a flag simplicial sphere.*

Another corollary of our proof is a new realization for the positive Bergman complex of a Coxeter arrangement: we can obtain it from a simplex by a sequence of stellar subdivisions.

5. The Bergman complex of a Coxeter arrangement

Nested set complexes are simplicial complexes which are the combinatorial core of De Concini and Procesi's subspace arrangement models [8], and of the resolution of singularities in toric varieties [9]. We now recall the definition of the minimal nested set complex of a meet-semilattice L, which we will simply refer to as **the** nested set complex of L, and denote $\mathcal{N}(L)$.

Say an element y of L is *irreducible* if the lower interval $[\hat{0}, y]$ cannot be decomposed as the product of smaller intervals of the form $[\hat{0}, x]$. The nested set complex $\mathcal{N}(L)$ of L is a simplicial complex whose vertices are the irreducible elements of L. A set X of irreducibles is *nested* if for any antichain $\{x_1, \ldots, x_k\}$ in X, $x_1 \vee \cdots \vee x_k$ is not irreducible. These nested sets are the faces of $\mathcal{N}(L)$.

If M is a matroid and L_M is its lattice of flats, we will also call $\mathcal{N}(L_M)$ the nested set complex of M, and denote it $\mathcal{N}(M)$. (Recall that the irreducible elements of L_M are the connected flats of M.) It turns out that when we are considering the oriented matroid M_Φ of a Coxeter arrangement of type Φ, the Bergman complex $\mathcal{B}(M_\Phi)$ and the nested set complex $\mathcal{N}(M_\Phi)$ are equal.

To prove this theorem, we use a result of Feichtner and Sturmfels [11]. They showed that, for any matroid M, the order complex of $\mathcal{N}(M)$ refines the coarse subdivision of the

Bergman complex $\mathcal{B}(M)$ and is refined by its fine subdivision. Moreover, they proved the following theorem.

Theorem 5.1. [11] *The nested set complex $\mathcal{N}(M)$ equals the Bergman complex $\mathcal{B}(M)$ if and only if the matroid G/F is connected for every pair of flats $F \subset G$ in which G is connected.*

Combining their Theorem 5.1 with Proposition 4.3(iv) immediately yields the following result.

Theorem 1.2. *For any Coxeter system (W, S) and associated root system Φ, the coarse subdivision of the Bergman complex $\mathcal{B}(M_\Phi)$ of the Coxeter arrangement of type Φ is equal to the nested set complex $\mathcal{N}(M_\Phi)$.*

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References

[1] F. Ardila and C. Klivans. *The Bergman complex of a matroid and phylogenetic trees.* Preprint, arXiv:math.CO/0311370, 2003.

[2] F. Ardila, C. Klivans, and L. Williams. *The positive Bergman complex of an oriented matroid*, to appear in the European Journal of Combinatorics.

[3] H. Barcelo and E. Ihrig, *Lattices of parabolic subgroups in connection with hyperplane arrangements*, J. Algebraic Combin. **9** (1999), 5–24.

[4] A. Björner and F. Brenti, *The combinatorics of Coxeter groups*, in preparation.

[5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. *Oriented matroids.* Encyclopedia of Mathematics and Its Applications, vol. 46. Cambridge University Press, Cambridge, 1993.

[6] M. Carr, S. Devadoss. *Coxeter complexes and graph associahedra*, preprint.

[7] M. Davis, T. Januszkiewicz, and R. Scott, *Fundamental groups of minimal blow-ups*, Adv. Math. **177** (2003) 115-179.

[8] C. De Concini, C. Procesi. *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) 1, no. 3 (1995), 459-494.

[9] E. Feichtner, D. Kozlov. *Incidence combinatorics of resolutions*, to appear in Selecta Math. (N.S.).

[10] E. Feichtner, I. Müller, *On the topology of nested set complexes*, to appear in Proc. Amer. Math. Soc.

[11] E. Feichtner, B. Sturmfels. *Matroid polytopes, nested sets and Bergman fans*, preprint.

[12] J. Humphreys. *Reflection groups and Coxeter groups.* Cambridge University Press, 1990.

[13] V. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, 1990.

[14] A. Postnikov. *Permutahedra, associahedra, and beyond*, in preparation.

[15] D. Speyer and B. Sturmfels. *The tropical Grassmannian.* Advances in Geometry, **4** (2004), no.3, p. 389-411.

[16] D. Speyer and L. Williams. *The tropical totally positive Grassmannian,* to appear in the Journal of Algebraic Combinatorics.

[17] R. P. Stanley. *Enumerative combinatorics, vol.1.* Cambridge University Press, New York, 1986.

[18] B. Sturmfels. *Solving systems of polynomial equations.* CBMS Regional Conference Series in Mathematics, vol. 97. American Mathematical Society, Providence, RI, 2002.

[19] G. Ziegler. *Lectures on polytopes.* Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

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