COMBINATORIAL DEFORMATIONS OF THE FULL TRANSFORMATION SEMIGROUP

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Abstract. We define two deformations of the Full Transformation Semigroup algebra. One makes the algebra "as semisimple as possible", while another leads to an eigenvalue result involving Schur functions.

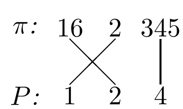
Preliminaries

The *Full Transformation Semigroup on n letters*, denoted T_n, is the semigroup of all set maps $w : [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$ and the multiplication is the usual composition. Such maps can be depicted in several ways; we will most often use one-line notation, for example $w = 214442$ denotes the map sending 1 to 2, 2 to 1, 3 to 4, etc.

Maps in T_n are indexed by triples (π, P, ϕ), where P is the image of the map, π is the set partition of $[n]$ whose blocks are the inverse images of the elements of P, and ϕ is the permutation describing which block is mapped to which element of the image. In what follows, $\pi = \{\pi_1, \pi_2, \ldots\}$ will always denote a set partition of $[n]$ with blocks ordered by increasing smallest element. Similarly in writing $P = \{p_1, p_2, \ldots\}$ a subset of $[n]$ we shall always intend $p_1 < p_2 < \ldots$. Permutations will be written in cycle notation.

With these conventions, we shall let $w_{\pi,P,\phi} \in T_n$ denote the map taking $x \in \pi_i$ to $p_{\phi(i)}$.

Example 1. *For $w = 214442 \in T_6$ we have $\pi(w) = 16|2|345, P(w) = \{1, 2, 4\}$, and $\phi(w) = (12)(3)$, the transposition exchanging 1 and 2 and fixing 3. The permutation ϕ is most easily visualized in the following diagram of w.*



The invertible elements of T_n, i.e., the bijective maps, form a subsemigroup isomorphic to the Symmetric Group S_n. Thus the elements of T_n can be thought of as generalized permutations, and we can ask which of the many combinatorial aspects of the Symmetric Group can be extended in a meaningful way to the Full Transformation Semigroup.

Let $\mathbb{C}T_n$ denote the Full Transformation Semigroup algebra, consisting of complex linear combinations of elements of T_n. $\mathbb{C}T_n$ has a chain of two-sided ideals

$$\mathbb{C}T_n = I_n \supseteq I_{n-1} \supseteq \ldots \supseteq I_1 \supseteq I_0 = 0,$$

where for $1 \leq k \leq n$, I_k as a vector space is the complex span of the maps of rank less than or equal to k (the *rank* of a map is the cardinality of its image). For

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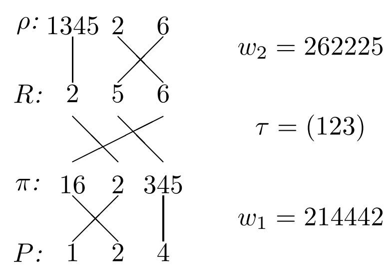
$1 \leq k \leq n$ define the algebras $A_{n,k} = I_k/I_{k-1}$. We can think of $A_{n,k}$ as being the algebra spanned by the maps of rank k, where two maps multiply to zero if their composition has rank less than k. The top quotient $A_{n,n}$ is isomorphic to $\mathbb{C}S_n$, the group algebra of the Symmetric Group, and is therefore semisimple. However $A_{n,k}$ is not semisimple for $k < n$, meaning that the radical $\sqrt{A_{n,k}}$ is non-trivial.

It is known that the irreducible modules for $A_{n,k}$ are indexed by partitions $\lambda \vdash k$. In fact Hewitt and Zuckerman give a calculation in [5] that generates all irreducible matrix representations for $A_{n,k}$. However, their methods are difficult to apply in practice and do not even determine the dimensions of the representations. These dimensions are known, thanks to a more recent character result of Putcha [8]. Regardless of the approach, it is clear that the non-semisimplicity of $A_{n,k}$ causes great difficulties. This has led us to define several deformations of $A_{n,k}$, with the aim of making the algebra generically semisimple.

The first deformation

Let $w_1 = w_{\pi,P,\phi}$ and $w_2 = w_{\rho,R,\psi}$ be two maps of rank k. Notice that in order for the product $w_1 w_2$ to be nonzero in $A_{n,k}$ each element of $R = P(w_2)$ must lie in a different block of $\pi = \pi(w_1)$. In this situation we can associate to the maps w_1 and w_2 the permutation $\tau \in S_k$ defined by the condition $r_i \in \pi_{\tau(i)}$.

Example 2. *Let* $w_1 = 214442$ *and* $w_2 = 262225$*. Then* $P(w_2) = \{2, 5, 6\}$ *and* $\pi(w_1) = 16|2|345$*. The smallest element* 2 *of* $P(w_2)$ *is in the second block of* $\pi(w_1)$*, the next-smallest element* 5 *is in the third block, and the largest* 6 *is in the first block. Thus* $\tau = (123)$*, as can be seen in the following diagram.*



Now define a new multiplication in $A_{n,k}$ by

$$w_1 * w_2 := x^{\mathrm{inv}(\tau)} w_1 w_2,$$

where $\mathrm{inv}(\tau)$ is the number of inversions of τ. It is not difficult to show that this multiplication is associative.

Example 3. *Taking* w_1 *and* w_2 *as above we have*

$$w_1 * w_2 = x^2 w_1 w_2 = x^2 121114.$$

Let $A_{n,k}(x)$ denote the algebra with the multiplication $*$. Setting $x = 1$ recovers the original multiplication in $A_{n,k}$. As we shall see, there is a sense in which $A_{n,k}(x)$ is "as semisimple as possible" for generic x.

Definition 1. *Let* A *be an associative algebra. The* Munn matrix algebra $\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ *as a vector space is the set of all* $m \times n$ *matrices with entries in* A*.* Π *is an* $n \times m$ *matrix over* A*, called the* sandwich matrix*, and multiplication is defined by* $X \cdot Y := X \Pi Y$*.*

Fact 1. *$A_{n,k}$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{n}{k}, S(n,k); \Pi_{n,k})$, where $\Pi_{n,k}$ is the $S(n,k) \times \binom{n}{k}$ sandwich matrix*

$$(\Pi_{n,k})_{\pi,P} = \begin{cases} \tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. *For $n = 4$ and $k = 3$ we have*

$$\Pi_{4,3} = \begin{pmatrix} (id) & (id) & 0 & 0 \\ 0 & (id) & (id) & 0 \\ 0 & 0 & (id) & (id) \\ (id) & 0 & 0 & (123) \\ (id) & 0 & (23) & 0 \\ 0 & (id) & 0 & (12) \end{pmatrix},$$

where the ordering on the columns is $123, 124, 134, 234$, and the ordering on the rows is $1|2|34, 1|23|4, 12|3|4, 14|2|3, 1|24|3, 13|2|4$.

Note that the non-zero entries of $\Pi_{n,k}$ are precisely the permutations τ that arise in the $*$ multiplication. Hence the first deformation preserves the Munn matrix algebra structure.

Proposition 1. *$A_{n,k}(x)$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{n}{k}, S(n,k); \Pi_{n,k}(x))$, where $\Pi_{n,k}(x)$ is the $S(n,k) \times \binom{n}{k}$ sandwich matrix defined by*

$$(\Pi_{n,k}(x))_{\pi,P} = \begin{cases} x^{\mathrm{inv}(\tau)}\tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

When $k = 1$ the parameter x does not show up at all, and the semisimple part of $A_{n,1}$ is only one-dimensional. For the remainder of this section we shall assume $k > 1$.

Since we want our Munn matrix algebra to be semisimple, it is natural to ask in what way the semisimplicity of $\mathcal{A}$ depends on the sandwich matrix Π. The following result of Clifford and Preston ([2], Theorem 5.19) provides an answer.

Theorem 1. *(Clifford and Preston) A Munn matrix algebra of the form $\mathcal{A} = \mathcal{M}(\mathbb{C}G; m, n; \Pi)$ is semisimple if and only if Π is non-singular, i.e., if and only if $m = n$ and Π is a unit in the ring of $m \times m$ matrices over $\mathbb{C}G$.*

Note in particular that for semisimplicity we need the matrices to be square. But our sandwich matrix is $S(n,k) \times \binom{n}{k}$. What can we do?

One idea is to define the *rank* of Π to be the largest non-singular minor of Π. (So, in particular, $\mathrm{rank}(\Pi) \le \min(m,n)$.) A result of McAlister [7] implies that this rank is intimately related to the size of $\sqrt{\mathcal{A}}$. To state McAlister's result we first need to define a technical condition known as *suitability*.

Definition 2. *Let P be an $n \times m$ matrix over A with rank r. Let R and S be permutation matrices over A such that*

$$RPS = \begin{pmatrix} M & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where M is an invertible $r \times r$ submatrix of P, and let

$$Q = S \begin{pmatrix} M^{-1} & 0 \\ 0 & 0 \end{pmatrix} R.$$

Then we say that P is suitable *if $PQP - P \in (\sqrt{A})_{n\times m}$.*

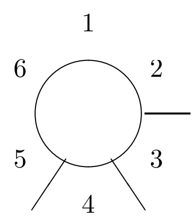
We note here in passing that the suitability condition is trivially satisfied when a matrix has full rank.

Theorem 2. *(McAlister) Let $\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ be a Munn matrix algebra, and let Π be suitable of rank r. Then $\mathcal{A}/\sqrt{\mathcal{A}} \cong \left(A/\sqrt{A}\right)_r$, the algebra of all $r \times r$ matrices with entries in $A/\sqrt{A}$.*

The suitability condition does not hold for the undeformed algebra $A_{n,k}$. But what about for $A_{n,k}(x)$? We show that $A_{n,k}(x)$ has full rank for generic x, by considering the submatrix formed by the rows corresponding to a special set of $\binom{n}{k}$ partitions of $[n]$.

Definition 3. *A set partition π of $[n]$ is* cyclically contiguous *if the blocks of π are intervals, with the possible exception of the first block, which may be of the form $\{1, 2, \ldots, i\} \cup \{j, j+1, \ldots, n\}$, i.e., the union of an initial segment and a terminal segment. If the first block is also an interval, we say that π is* contiguous*. (Note that contiguous implies cyclically contiguous, not the other way around.)*

We use the term "cyclically contiguous" for such a partition π because if we think of the elements of $[n]$ as being arranged in (clockwise) order around a circle, then in a sense *all* of the blocks of π are intervals. For example, $\pi = 1256|3|4$ is cyclically contiguous, as shown in the following diagram.



$\pi = 1256|3|4$

For $k > 1$ there is an obvious bijection between cyclically contiguous partitions of $[n]$ into k blocks and k-subsets of $[n]$. Let $\Pi^c(x)$ be the $\binom{n}{k} \times \binom{n}{k}$ submatrix of $\Pi(x)$ consisting of the rows corresponding to cyclically contiguous partitions. We show that $\Pi^c(x)$ is nonsingular for generic x, and hence $\Pi(x)$ is suitable of rank $\binom{n}{k}$. Thus we have

Theorem 3. *For $k > 1$ and generic x, $A_{n,k}(x)/\sqrt{A_{n,k}(x)} \cong (\mathbb{C}S_k)_{\binom{n}{k}}$.*

Note that the rank of Π cannot be any larger than its width $\binom{n}{k}$. So no deformation of $A_{n,k}$ that preserves the Munn matrix algebra structure can be any more semisimple than $A_{n,k}(x)$.

Corollary 1. *For $k > 1$,*

$$\dim\left(\sqrt{A_{n,k}(x)}\right) = \left(S(n,k) - \binom{n}{k}\right)\binom{n}{k}k!.$$

Since this dimension formula is relatively simple one might hope to find a nice combinatorial basis for $\sqrt{A_{n,k}(x)}$, as Garsia and Reutenauer did in [3] for Solomon's Descent Algebra. So far we have found several families of elements in the radical, but they are not in general independent, and do not form a spanning set.

Let $A^c_{n,k}(x)$ be the subalgebra of $A_{n,k}(x)$ spanned by the maps whose partitions are cyclically contiguous. $A^c_{n,k}(x)$ is also a Munn matrix algebra, with sandwich matrix $\Pi^c(x)$.

Grood [4] has generalized the classical Specht-module construction for the symmetric group (see [6]) to describe the irreducible modules of the *rook monoid* R_n, another semigroup containing the symmetric group. As it turns out, $\mathbb{C}R_n$ has a similar tower of ideals

$$\mathbb{C}R_n = J_n \supseteq J_{n-1} \supseteq \ldots \supseteq J_1 \supseteq J_0 = 0,$$

and defining $B_{n,k} = J_k/J_{k-1}$ we have $B_{n,k} \cong (\mathbb{C}S_k)_{\binom{n}{k}}$. We have extended the first deformation to an algebra $A_{n,k}(x, \mathbf{y})$ with canonical isomorphisms $A^c_{n,k}(1, \mathbf{1}) \cong A^c_{n,k}$ and $A^c_{n,k}(1, \mathbf{0}) \cong B_{n,k}$. We are currently attempting to modify Grood's approach to explicitly construct the irreducible modules for the generic algebra $A^c_{n,k}(x, \mathbf{y})$.

The second deformation

There is another associative multiplication we can define on $A_{n,k}$. The symmetric group S_k acts on the maps of rank k by

$$\sigma w_{\pi,P,\phi} := w_{\pi,P,\sigma\phi}.$$

Now define

$$w_1 \circ w_2 := \sum_{\sigma\in S_k} p_{\rho(\sigma)}\sigma w_1 w_2,$$

where $\rho(\sigma) \vdash k$ is the cycle type of σ, and $p_{\rho(\sigma)}$ is the corresponding power-sum symmetric function in the variables $x_1, \ldots, x_k$.

Example 5. *Let* $w_1 = 1442$ *and* $w_2 = 3134$. *Then* $w_1w_2 = 4142$, *and*

$$\begin{aligned} w_1 \circ w_2 &= p_{1^3}4142 + p_{21}(1412 + 2124 + 4241) + p_3(1214 + 2421) \\ &= (x_1 + x_2 + x_3)^3 4142 + (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)(1412 + 2124 + 4241) \\ &\quad +(x_1^3 + x_2^3 + x_3^3)(1214 + 2421) \end{aligned}$$

Note that if we choose values for the x_i so that $p_1 = 1$ and $p_i = 0$ for all $i \geq 2$, we recover the original multiplication in $A_{n,k}$. Such a specialization of the x_i must exist because the p_i are algebraically independent.

Let $A_{n,k}(\mathbf{x})$ denote the algebra with the multiplication $\circ$. Explicit calculations for small values of n and k suggest that $A_{n,k}(\mathbf{x})$ is no more semisimple than $A_{n,k}$, i.e., that even for generic values of the x_i we have $\dim\sqrt{A_{n,k}(\mathbf{x})} = \dim\sqrt{A_{n,k}}$. However something interesting does come out of this multiplication.

The following fact gives a useful characterization of the radical of an algebra.

Fact 2. *Let A be a finite-dimensional associative algebra and $\{v_1, \ldots, v_n\}$ a basis of A. Identify A as a vector space with $\mathbb{C}^n$, and define the $n \times n$ Gram matrix M for A by $(M)_{i,j} = tr(v_iv_j)$. Then the nullspace of M is $\sqrt{A}$.*

If we define $M_{n,k}(\mathbf{x})$ to be the Gram matrix for $A_{n,k}(\mathbf{x})$ we have

Proposition 2.

$$(M_{n,k}(\mathbf{x}))_{i,j} = \begin{cases} S(n,k)k! \sum_{\lambda\vdash k} \frac{k!}{f^\lambda}\chi^\lambda(\mu)s_\lambda^2 & w_iw_j \text{ induces a permutation of cycle type } \mu \text{ on the image of } w_i \\ 0 & \text{otherwise.} \end{cases}$$

Here for λ a partition of k, χ^λ is the corresponding irreducible character of S_k, $f^\lambda = \chi^\lambda(1)$ its dimension, and s_λ the associated Schur function.

In the sequel we will always normalize the Gram matrix $M_{n,k}(\mathbf{x})$ by dividing by the constant $S(n,k)k!$.

In the semisimple case $k = n$ one can use Frobenius' factorization of the group determinant (see [1]) to derive the following result about the normalized matrix $M_{n,n}(\mathbf{x})$.

Theorem 4. *The eigenvalues of $M_{n,n}(\mathbf{x})$ are $\pm\left(\frac{n!}{f^\lambda}s_\lambda\right)^2$, $\lambda \vdash n$, where the positive values appear with multiplicity $\binom{f^\lambda+1}{2}$ and the negative values appear with multiplicity $\binom{f^\lambda}{2}$.*

Corollary 2. *The algebra $A_{n,n}(\mathbf{x})$ is semisimple if and only if the values of the parameters x_i avoid the zeros of the Schur functions s_λ.*

For $k = 1$ there is always a unique non-zero eigenvalue ns_1^2, but the analogous result for $1 < k < n$ is so far only conjectural.

Conjecture 1. *The non-zero eigenvalues of $M_{n,k}(\mathbf{x})$, $k < n$, are of the form cs_λ^2 for $\lambda \vdash k$, where c is an algebraic scalar. The "multiplicity" of s_λ^2, i.e., the sum of the multiplicities of the cs_λ^2, is $(\binom{n}{k}f^\lambda)^2$ for $\lambda \vdash k$, $\lambda \neq 1^k$, and $\binom{n-1}{k-1}^2$ for $\lambda = 1^k$.*

This conjecture is difficult to check even by computer for $n > 4$. The following table gives some sample data.

n	k	eigenvalues
3	2	$(-12s_{1^2}^2)^1, (-8s_2^2)^2, (-4s_2^2)^1, 0^5, (4s_2^2)^3, (8s_2^2)^2, (12s_{1^2}^2)^3, (16s_2^2)^1$
4	2	$(-32s_{1^2}^2)^3, (-\sqrt{448}s_2^2)^5, (-8s_2^2)^{10}, 0^{39}, (8s_2^2)^{15}, (\sqrt{448}s_2^2)^5, (32s_{1^2}^2)^6, (56s_2^2)^1$

(The conjectured multiplicities come from Putcha's results; they are the dimensions of the irreducible characters for $A_{n,k}$.)

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