EHRHART POLYNOMIALS OF CYCLIC POLYTOPES

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Abstract. The Ehrhart polynomial of an integral convex polytope counts the number of lattice points in dilates of the polytope. In [1], the authors conjectured that for any cyclic polytope with integral parameters, the Ehrhart polynomial of it is equal to its volume plus the Ehrhart polynomial of its lower envelope and proved the case when the dimension $d = 2$. In our article, we prove the conjecture for any dimension.

1. Introduction

For any *integral convex polytope* P, that is, a convex polytope whose vertices have integral coordinates, any positive integer $m \in \mathbb{N}$, we denote by $i(P, m)$ the number of lattice points in mP, where $mP = \{mx \mid x \in P\}$ is *the mth dilate polytope* of P. In our paper, we will focus on a special class of polytopes, cyclic polytopes, which are defined in terms of the moment curve:

Definition 1.1. The **moment curve** in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$

Let $T = \{t_1, \dots, t_n\}_<$ be a linearly ordered set. Then the **cyclic polytope** $C_d(T) = C_d(t_1, \dots, t_n)$ is the convex hull $\mathrm{conv}\{v_d(t_1), v_d(t_2), \dots, v_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i), 1 \le i \le n$, on the moment curve.

The main theorem in our article is the one conjecured in [1, Conjecture 1.5]:

Theorem 1.2. *For any integral cyclic polytope $C_d(T)$,*

$$i(C_d(T), m) = \mathrm{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$i(C_d(T), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(mC_k(T)) = \sum_{k=0}^{d} \mathrm{Vol}_k(C_k(T)) m^k,$$

where $\mathrm{Vol}_k(mC_k(T))$ *is the volume of* $mC_k(T)$ *in k-dimensional space, and we let* $\mathrm{Vol}_0(mC_0(T)) = 1$.

One direct result of Theorem 1.2 is that $i(C_d(T), m)$ is always a polynomial in m. This result was already shown by Eugène Ehrhart for any integral polytope in 1962 [2]. Thus, we call $i(P, m)$ the Ehrhart polynomial of P when P is an integral polytope. There is much work on the coefficients of Ehrhart polynomials. For instance it's well known that the leading and second coefficients of $i(P, m)$ are the normalized volume of P and one half of the normalized volume of the boundary of P. But there is no known explicit method of describing all the coefficients of Ehrhart polynomials of general integral polytopes. However, because of some special properties that cyclic polytopes have, we are able to calculate the Ehrhart polynomial of cyclic polytopes in the way described in Theorem 1.2.

In this paper, we use a standard triangulation decomposition of cyclic polytopes, and careful counting of lattice points to reduce Theorem 1.2 to the case n=d+1, (Theorem 2.9). We then prove Theorem 2.9 with the use of certain linear transformations and decompositions of polytopes containing our cyclic polytopes.

2. Statements and Proofs

All polytopes we will consider are full-dimensional, so for any convex polytope P, we use d to denote both the dimension of the ambient space $\mathbb{R}^d$ and the dimension of P. Also, We denote by ∂P and $I(P)$ the boundary and the interior of P, respectively.

For simplicity, for any region $R \subset \mathbb{R}^d$, we denote by $\mathcal{L}(R) := R \cap \mathbb{Z}^d$ the set of lattice points in R.

Consider the projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ that forgets the last coordinate. In [1, Lemma 5.1], the authors showed that the inverse image under π of a lattice point $y \in C_{d-1}(T) \cap \mathbb{Z}^{d-1}$ is a line that intersects the boundary of $C_d(T)$ at integral points. By a similar argument, it's easy to see that this is true when we replace the cyclic polytopes by their dilated polytopes. Note that $\pi(mC_d(T)) = mC_{d-1}(T)$, so for any lattice point y in $mC_{d-1}(T)$ the inverse image under π intersects the boundary at lattice points.

Definition 2.1. For any x in a real space, let $l(x)$ denote the last coordinate of x.

For any polytope $P \subset \mathbb{R}^d$ and any point $y \in \mathbb{R}^{d-1}$, let $\rho(y, P) = \pi^{-1}(y) \cap P$ be the intersection of P with the inverse image of y under π. Let $p(y, P)$ and $n(y, P)$ be the point in $\rho(y, P)$ with the largest and smallest last coordinate, respectively. If $\rho(y, P)$ is the empty set, i.e., $y \notin \pi(P)$, then let $p(y, P)$ and $n(y, P)$ be empty sets as well. Clearly, $p(y, P)$ and $n(y, P)$ are on the boundary of P. Also, we let $\rho^+(y, P) = \rho(y, P) \setminus n(y, P)$, and for any $S \subset \mathbb{R}^{d-1}$, $\rho^+(S, P) = \cup_{y \in S} \rho^+(y, P)$.

Define $PB(P) = \bigcup_{y \in \pi(P)} p(y, P)$ to be the *positive boundary* of P; $NB(P) = \cup_{y \in \pi(P)} n(y, P)$ to be the *negative boundary* of P and $\Omega(P) = P \setminus NB(P) = \rho^+(\pi(P), P) = \cup_{y \in \pi(P)} \rho^+(y, P)$ to be the *nonnegative part* of P.

For any facet F of P, if F has an interior point in the positive boundary of P, (it's easy to see that $F \subset PB(P)$) then we call F a *positive facet* of P and define the sign of F as $+1$: $\operatorname{sign}(F) = +1$. Similarly, we can define the *negative facets* of P with associated sign -1.

By the argument we gave before Definition 2.1, π induces a bijection of lattice points between $NB(mC_d(T))$ and $\pi(mC_d(T)) = mC_{d-1}(T)$. Hence, Theorem 1.2 is equivalent to the following Proposition:

Proposition 2.2. $\operatorname{Vol}(mC_d(T)) = |\mathcal{L}(\Omega(mC_d(T)))|$.

From now on, we will consider any polytopes or sets as *multisets* which allow *negative multiplicities.* We can consider any element of a multiset as a pair (x, m), where m is the multiplicity of element x. (A multiplicity zero for an element x is used when x does not appear at all in the multiset.) Then for any multisets M_1, M_2 and any integers m, n and i, we define the following operators:

(i) Scalar product: $iM_1 = i \cdot M_1 = \{(x, im) \mid (x, m) \in M_1\}$.
(ii) Addition: $M_1 \oplus M_2 = \{(x, m+n) \mid (x, m) \in M_1, (x, n) \in M_2\}$.
(iii) Subtraction: $M_1 \ominus M_2 = M_1 \oplus ((-1) \cdot M_2)$.

It's clear that the following holds:

Lemma 2.3.
a) $\forall R_1, \ldots, R_k \subset \mathbb{R}^d$, $\forall i_1, \ldots, i_k \in \mathbb{Z} : \mathcal{L}(\bigoplus_{j=1}^k i_j R_j) = \bigoplus_{j=1}^k i_j \mathcal{L}(R_j)$.
b) *For any polytope* $P \subset \mathbb{R}^d$, $\forall R_1, \ldots, R_k \subset \mathbb{R}^{d-1}$, $\forall i_1, \ldots, i_k \in \mathbb{Z}$:

$$\rho^+ \left(\bigoplus_{j=1}^k i_j R_j, P \right) = \bigoplus_{j=1}^k i_j \rho^+(R_j, P).$$

Let P be a convex polytope. For any y an interior point of $\pi(P)$, since π is a continous open map, the inverse image of y contains an interior point of P. Thus $\pi^{-1}(y)$ intersects the boundary of P exactly twice. For any y a boundary point of $\pi(P)$, again because π is an open map, we have that $\rho(y, P) \subset \partial P$, so $\rho(y, P) = \pi^{-1}(y) \cap \partial P$ is either one point or a line segment. We hope that $\rho(y, P)$ always has only one point, so we define the following polytopes and discuss several properties of them.

Definition 2.4. We call a convex polytope P a *nice* polytope with respect to π if for any $y \in \partial\pi(P)$, $|\rho(y, P)| = 1$ and for any lattice point $y \in \pi(P)$, $\pi^{-1}(y)$ intersects ∂P at lattice points.

Lemma 2.5. *A nice polytope P has the following properties:*
(i) *For any* $y \in I(\pi(P))$, $\pi^{-1}(y) \cap \partial P = \{p(y, P), n(y, P)\}$. *In particular, if* y *is a lattice point, then* $p(y, P)$ *and* $n(y, P)$ *are each lattice points.*
(ii) *For any* $y \in \partial\pi(P)$, $\pi^{-1}(y) \cap \partial P = \rho(y, P) = p(y, P) = n(y, P)$, *so* $\rho^+(y, P) = \emptyset$. *In particular, when* y *is a lattice point,* $\rho(y, P)$ *is a lattice point as well.*
(iii) $\mathcal{L}$ *and* ρ^+ *commute: for any* $R \subset \mathbb{R}^{d-1}$, $\mathcal{L}(\rho^+(R, P)) = \rho^+(\mathcal{L}(R), P)$.
(iv) *Let* R *be a region containing* $I(\pi(P))$. *Then*

$$\Omega(P) = \rho^+(R, P) = \bigoplus_{y \in R} \rho^+(y, P).$$

Moreover,

$$|\mathcal{L}(\Omega(P))| = \sum_{y \in \mathcal{L}(R)} l(p(y, P)) - l(n(y, P)).$$

(By convention, if $y \notin \pi(P)$, *we let* $l(p(y, P)) - l(n(y, P)) = 0$.*)*
(v) *If* P *is decomposed into nice polytopes* $P_1, \ldots, P_k$, *i.e.,* $P = P_1 \cup \cdots \cup P_k$ *and* $I(P_i) \cap I(P_j) = \emptyset$ *for any distinct* i, j, *then* $\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i)$, *so* $\mathcal{L}(\Omega(P)) = \bigoplus_{i=1}^k \mathcal{L}(\Omega(P_i))$.
(vi) *The set of facets of* P *are partitioned into the set of positive facets and the set of negative facets, i.e., every facet is either positive or negative but not both.*

Proof. The first three and last properties are immediately true. And the fourth one follows directly from the second one. The fifth property can be checked by considering the definition of Ω. □

By using these properties, we are able to give the following proposition about a nice convex polytope:

Proposition 2.6. *Let P be a nice convex polytope with respect to π such $\pi(P)$ is also nice, and all the points in P have nonnegative last coordinate. Suppose further that for any facet F of P, $\pi(F)$ is a nice polytope with respect to π. Then*

$$\Omega(P) = \bigoplus_{F:\ \text{a facet of } P} \operatorname{sign}(F)\rho^+(\Omega(\pi(F)), \operatorname{conv}(F, \pi(F))),$$

where $\operatorname{conv}(F, \pi(F))$ *denotes the convex hull of the set* $F \cup \{(y', 0)' \mid y \in \pi(F)\}$, *i.e. the region between F and its projection onto the hyperplane* $\{(x_1, \dots, x_d)' \mid x_d = 0\}$. *(Note, for any vector v, we use v' to denote its transpose. So for a vertical vector y, $(y', 0)'$ is just the vector obtained from y by attaching a zero to the bottom of y.)*

Proof. A special case of Lemma 2.5/(iv) is when $R = \Omega(\pi(P))$, so we have

$$\Omega(P) = \rho^+(\Omega(\pi(P)), P) = \bigoplus_{y \in \Omega(\pi(P))} \rho^+(y, P).$$

Now for any points a and b, we use $(a, b]$ to denote the half-open line segment between a(excluding) and b(including). Then, $\rho^+(y, P) = (n(y, P), p(y, P)] = (((y', 0)', p(y, P)] \ominus ((y', 0)', n(y, P)])$. Therefore,

$$\begin{aligned}\Omega(P) &= \bigoplus_{y \in \Omega(\pi(P))} (((y', 0)', p(y, P)] \ominus ((y', 0)', n(y, P)]) \\ &= \left(\bigoplus_{y \in \Omega(\pi(P))} ((y', 0)', p(y, P)]\right) \bigoplus \left(\bigoplus_{y \in \Omega(\pi(P))} (-1) \cdot ((y', 0)', n(y, P)]\right).\end{aligned}$$

Let $F_1, F_2, \dots, F_\ell$ be all the positive facets of P and $F_{\ell+1}, \dots, F_k$ be all the negative facets. Then it's clear that $\pi(F_1) \cup \pi(F_2) \cup \cdots \cup \pi(F_\ell)$ and $\pi(F_{\ell+1}) \cup \cdots \cup \pi(F_k)$ both give a decomposition of $\pi(P)$. Therefore by Lemma 2.5/(v), we have that $\Omega(\pi(P)) = \bigoplus_{i=1}^{\ell} \Omega(\pi(F_i)) = \bigoplus_{j=\ell+1}^{k} \Omega(\pi(F_j))$. Hence,

$$\begin{aligned}\bigoplus_{y \in \Omega(\pi(P))} ((y', 0)', p(y, P)] &= \bigoplus_{i=1}^{\ell} \bigoplus_{y \in \Omega(\pi(F_i))} ((y', 0)', p(y, P)] \\ &= \bigoplus_{i=1}^{\ell} \rho^+(\Omega(\pi(F_i)), \operatorname{conv}(F_i, \pi(F_i))).\end{aligned}$$

Similarly, we will have

$$\bigoplus_{y \in \Omega(\pi(P))} (-1) \cdot ((y', 0)', n(y, P)] = \bigoplus_{j=\ell+1}^{k} (-1)\rho^+(\Omega(\pi(F_j)), \operatorname{conv}(F_j, \pi(F_j))).$$

Thus, by putting them together, we get

$$\Omega(P) = \bigoplus_{F\ :\ \text{a facet of } P} \operatorname{sign}(F)\rho^+(\Omega(\pi(F)), \operatorname{conv}(F, \pi(F))).$$

$\square$

In the last proposition, we used a new notation $\text{conv}(F, \pi(F))$ to denote certain polytopes. For polytopes that can be written in this way, we have the following lemma, whose proof is trivial:

Lemma 2.7. *Let H be a hyperplane in $\mathbb{R}^d$ such that $\pi(H) = \mathbb{R}^{d-1}$. Let $S_1 \subset S_2$ be two convex polytopes inside H and the last coordinates of all of their points are non-negative. Then for any $y \in \pi(S_1)$, $\rho^+(y, \text{conv}(S_1, \pi(S_1))) = \rho^+(y, \text{conv}(S_2, \pi(S_2)))$.*

Having discussed some properties of nice polytopes with respect to π, we come back to the dilated cyclic polytopes which are our main interest and show that they are nice:

Lemma 2.8. *$mC_d(T)$ is a nice polytope with respect to π.*

Proof. We already argued that $mC_d(T)$ satisfies the second condition to be nice. So it's left to check that $|\rho(y, C_d(T))| = 1$ for any $y \in \partial C_{d-1}(T)$.

Let $y = (y_1, y_2, \ldots, y_{d-1})'$ and suppose y is on a facet F of $mC_{d-1}(T)$ and without loss of generality, let $m\nu_{d-1}(t_1), m\nu_{d-1}(t_2), \ldots, m\nu_{d-1}(t_{d-1})$ be the $d-1$ vertices of F. Then there exist $\lambda_1, \ldots, \lambda_{d-1} \in \mathbb{R}_{\geq 0}$ such that $y = \sum_{j=1}^{d-1} \lambda_j m\nu_{d-1}(t_j)$ and $\sum_{j=1}^{d-1} \lambda_j = 1$.

Let $x \in \pi^{-1}(y) \cap mC_d(T)$. There exist $\lambda'_1, \ldots, \lambda'_n \in \mathbb{R}_{\geq 0}$ such that $x = \sum_{j=1}^{n} \lambda'_j m\nu_d(t_j)$ and $\sum_{j=1}^{n} \lambda'_j = 1$. Then $y = \pi(x) = \sum_{j=1}^{n} \lambda'_j m\nu_{d-1}(t_j)$. Since y is on the facet F, $\lambda'_j = 0$ unless $1 \leq j \leq d-1$. Thus $y = \sum_{j=1}^{d-1} \lambda'_j m\nu_{d-1}(t_j)$ and $\sum_{j=1}^{d-1} \lambda'_j = 1$. Therefore $\lambda_j = \lambda'_j, 1 \leq j \leq d-1$. Hence $x = \sum_{j=1}^{d-1} \lambda_j m\nu_d(t_j)$ is the only point in $\pi^{-1}(y) \cap mC_d(T)$. $\square$

We know that for any cyclic polytope $C_d(T)$ with $n = |T| > d+1$, we can decompose it into $n-d$ cyclic polytopes $P_1 \cup \cdots \cup P_{n-d}$, which is a triangulation of $C_d(T)$ and where P_i's are all defined by $(d+1)$-element integer sets. E.g., the pulling triangulation of [4] has this property. Therefore by the fourth property in Lemma 2.5, we have that $\mathcal{L}(\Omega(C_d(T))) = \bigcup_{i=1}^{n-d} \mathcal{L}(\Omega(P_i))$. Thus $|\mathcal{L}(\Omega(P))| = \sum_{i=1}^{k} |\mathcal{L}(\Omega(P_i))|$. Note that we also have $\text{Vol}(C_d(T)) = \sum_{i=1}^{n-d} \text{Vol}(P_i)$. We conclude that to prove Proposition 2.2, it is enough to prove the following:

Theorem 2.9. *For any integer sets T with $n = |T| = d+1$, $\text{Vol}(mC_d(T)) = |\mathcal{L}(\Omega(mC_d(T)))|$.*

Definition 2.10. A map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is *structure perserving* if it preserves volume and it commutes with the following operations:

(i) $\mathcal{L}$: taking lattice points of a region $R \subset \mathbb{R}^d$;
(ii) conv : taking the convex hull of a collection of points;
(iii) Ω : taking the nonnegative part of a convex polytope;
(iv) PB : taking the positive boundary of a convex polytope;
(v) NB : taking the negative boundary of a convex polytope.

Remark 2.11. Here φ commuting with conv implies (or is equivalent to) that for any set of points $x_1, \ldots, x_k \in \mathbb{R}^d$, and for any $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^{\geq 0}$ with $\sum_{i=1}^{k} \lambda_i = 1$, $T(\sum_{i=1}^{k} \lambda_i x_i) = \sum_{i=1}^{k} \lambda_i T(x_i)$. Therefore φ is an affine transformation, which can be defined by a $d \times d$ matrix A and a vector $u \in \mathbb{R}^d : T(x) = Ax + u$. Moreover, φ commuting with PB and NB implies that φ preserves the positive facets and negative facets of a convex polytope.

Lemma 2.12. *Let A be a $d \times d$ integral lower triangular matrix with 1's on its diagonal, and u be an integral vector in $\mathbb{R}^d$. Then $\varphi : x \mapsto Ax + u$ gives a map which is structure preserving, and so does φ^{-1}. Therefore, φ is a bijection from $\mathbb{Z}^d$ to $\mathbb{Z}^d$. Hence, for any subset $S \in \mathbb{R}^d$, $|\mathcal{L}(S)| = |\mathcal{L}(\varphi(S))|$.*

Moreover, for any $y \in \mathbb{R}^{d-1}$, if we define $\tilde{\varphi}(y) = \tilde{A}y + \tilde{u}$, where $\tilde{A}$ is the right upper $(d-1)\times(d-1)$ matrix of A and $\tilde{u} = \pi(u)$, then $\rho^+(\tilde{\varphi}(y), \varphi(P)) = \varphi(\rho^+(y, P))$, for any polytope P.

Proof. The determinant of A is 1, hence φ is volume preserving. It's easy to check that φ commutes with $\mathcal{L}$ and conv. To show that φ commutes with Ω, PB and NB, it suffices to show that for all $x_1, x_2 \in \mathbb{R}^d$ with $\pi(x_1) = \pi(x_2)$ and $l(x_1) > l(x_2)$, then $\pi(\varphi(x_1)) = \pi(\varphi(x_2))$ and $l(\varphi(x_1)) > l(\varphi(x_2))$. This is not hard to check using the fact that A is a lower triangular matrix with 1's on its diagonal. Hence, φ is structure preserving.

Note that φ^{-1} maps x to $A^{-1}x - A^{-1}u$. But we know that A^{-1} is also an integral lower triangular matrix with 1's on its diagonal and $-A^{-1}u$ is an integral vector. So φ^{-1} is structure preserving as well.

It's clear that $\tilde{\varphi} = \pi \circ \varphi \circ \pi^{-1}$, which implies that $\pi^{-1} \circ \tilde{\varphi} = \varphi \circ \pi^{-1}$. So we have

$$x \in \varphi(\rho^+(y, P)) \Leftrightarrow \varphi^{-1}(x) \in \rho^+(y, P) = \pi^{-1}(y) \cap P$$
$$\Leftrightarrow x \in \varphi(\pi^{-1}(y)) \cap \varphi(P) = \pi^{-1}(\tilde{\varphi}(y)) \cap \varphi(P) \Leftrightarrow x \in \rho^+(\tilde{\varphi}(y), \varphi(P)).$$

□

Now for any real numbers $r_1, r_2, \ldots, r_d$, we consider the $d \times d$ lower triangular matrices

$$A_{r_1,\ldots,r_d}(i,j) = \begin{cases} (-1)^{i-j} e_{i-j}(r_1, \ldots, r_i), & i \geq j \\ 0, & i < j \end{cases}$$

and

$$B_{r_1,\ldots,r_d}(i,j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \& i < d \\ (-1)^{i-j} e_{i-j}(r_1, \ldots, r_i), & j \neq i = d \end{cases}$$

where $e_k(r_1, \ldots, r_l) = \sum_{i_1 < i_2 < \cdots < i_k} r_{i_1} r_{i_2} \ldots r_{i_k}$ is the kth elementary symmetric function in $r_1, \ldots, r_l$.

For simplicity, we allow a map originally defined on $\mathbb{R}^d$ to work in higher dimension, by applying the map to the first d coordinates. Then it's not hard to see that $A_{r_1,\ldots,r_d} = A_{r_1,\ldots,r_{d-1}} B_{r_1,\ldots,r_d} = B_{r_1,\ldots,r_d} A_{r_1,\ldots,r_{d-1}}$.

We also define vectors

$$u_{r_1,\ldots,r_d} = \begin{pmatrix} -r_1 \\ r_1 r_2 \\ -r_1 r_2 r_3 \\ \vdots \\ (-1)^d r_1 r_2 \ldots r_d \end{pmatrix} = \begin{pmatrix} -e_1(r_1) \\ e_2(r_1, r_2) \\ -e_3(r_1, r_2, r_3) \\ \vdots \\ (-1)^d e_d(r_1, r_2, \ldots, r_d) \end{pmatrix},$$

and

$$v_{r_1,\ldots,r_d} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^d r_1 r_2 \ldots r_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^d e_d(r_1, r_2, \ldots, r_d) \end{pmatrix}.$$

Similarly, we allow the addition operation between two vectors of different dimensions by adding the lower dimension one to the first corresponding coordinates of the higher one. Thus, $u_{r_1,\ldots,r_d} = u_{r_1,\ldots,r_{d-1}} + v_{r_1,\ldots,r_d}$.

Now we define maps $\varphi_{r_1,\ldots,r_d} : x \mapsto A_{r_1,\ldots,r_d}x + u_{r_1,\ldots,r_d}$ and $\phi_{r_1,\ldots,r_d} : x \mapsto B_{r_1,\ldots,r_d}x + v_{r_1,\ldots,r_d}$. Unlike $\varphi_{r_1,\ldots,r_d}$, $\phi_{r_1,\ldots,r_d}$ does not depend on the order of r_i's. In other words, for any permutation $\sigma \in S_d$, $\phi_{r_1,\ldots,r_d} = \phi_{r_{\sigma(1)},\ldots,r_{\sigma(d)}}$.

Note that $\phi_{r_1,\ldots,r_d}$ only changes the dth coordinate of a vector, so we have the following lemma:

Lemma 2.13. $\varphi_{r_1,\ldots,r_d} = \varphi_{r_1,\ldots,r_{d-1}} \circ \phi_{r_1,\ldots,r_d}$.

Remark 2.14. When we consider $\varphi_{r_1,\ldots,r_d}$ and $\phi_{r_1,\ldots,r_d}$ operating on the moment curve, we have

$$\varphi_{r_1,\ldots,r_d}(\nu_d(t)) = A_{r_1,\ldots,r_d}\begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix} + u_{r_1,\ldots,r_d} = \begin{pmatrix} (t-r_1) \\ (t-r_1)(t-r_2) \\ \vdots \\ (t-r_1)(t-r_2)\cdots(t-r_d) \end{pmatrix},$$

$$\phi_{r_1,\ldots,r_d}(\nu_d(t)) = B_{r_1,\ldots,r_d}\begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix} + v_{r_1,\ldots,r_d} = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^{d-1} \\ (t-r_1)(t-r_2)\cdots(t-r_d) \end{pmatrix}.$$

Remark 2.15. When $r_1, \ldots, r_d$ are integers, $\varphi_{r_1,\ldots,r_d}, \phi_{r_1,\ldots,r_d}$ and their inverse maps are structure preserving by Lemma 2.12.

Now by using ϕ's (or φ's), we are able to determine the sign of the facets of dilated cyclic polytopes:

Proposition 2.16. *Let $P = mC_d(T)$, where $m \in \mathbb{N}$ and $T = \{t_1, t_2, \ldots, t_n\}_<$ an integral ordered set. Let F be a facet of P determined by vertices $\nu_d(t_{i_1}), \nu_d(t_{i_2}), \ldots, \nu_d(t_{i_d})$. Let k be the smallest element of the set $\{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_d\}$, then* $\operatorname{sign}(F) = (-1)^{d-k}$. *In particular, when $|T| = n = d+1$, let F_k be the facet of P determined by all the vertices of P except $\nu_d(t_{i_k})$, then for $k \in [d], \operatorname{sign}(F_k) = \operatorname{sign}(\sigma_k)$, where $\sigma_k = (k, k+1, \cdots, d) \in S_d$ and* $\operatorname{sign}(F_{d+1}) = -1$.

Proof. We first consider the case when $m = 1$, i.e. P is a cyclic polytope. Without loss of generality, we assume that $i_1 < i_2 < \cdots < i_d$. Consider the polytope $Q = \phi_{t_{i_1},\ldots,t_{i_d}}(P)$. For $j = 1, 2, \ldots, n$, the last coordinate of the vertex of Q which mapped from $\nu_d(t_j)$ is $l(\phi_{t_{i_1},\ldots,t_{i_d}}(\nu_d(t_j))) = (t_j - t_{i_1})(t_j - t_{i_2})\cdots(t_j - t_{i_d})$. Hence the last coordinates of the vertices of $\phi_{t_{i_1},\ldots,t_{i_d}}(F)$ are all 0's. So $\phi_{t_{i_1},\ldots,t_{i_d}}(F)$ is on the hyperplane obtained by setting the last coordinate to 0. Since k is the smallest element not in $\{i_1, \ldots, i_d\}$, $i_1 = 1, i_2 = 2, \ldots, i_{k-1} = k-1, i_k > k$. So $t_k - t_{i_l} > 0$ when $l = 1, 2, \ldots, k-1$; and $t_k - t_{i_l} < 0$ when $l = k, k+1, \ldots, d$. Therefore $\operatorname{sign}(l(\phi_{t_{i_1},\ldots,t_{i_d}}(\nu_d(t_k))) = (-1)^{d-k+1}$. By using Gale's evenness condition [3], it's not hard to see that $\operatorname{sign}(l(\phi_{t_{i_1},\ldots,t_{i_d}}(\nu_d(t_l))) = (-1)^{d-k+1}$, for all $l \notin \{i_1, \ldots, i_d\}$. Thus we can conclude that $l(\phi_{t_{i_1},\ldots,t_{i_d}}(P))$ is nonnegative if $d-k$ is odd, and is nonpositive if $d-k$ is even. Hence $\phi_{t_{i_1},\ldots,t_{i_d}}(F)$ and F are negative facets if $d-k$ is odd, and positive facets if $d-k$ is even. So $\operatorname{sign}(F) = (-1)^{d-k}$. For $n = d+1$, it's easy to see that $\operatorname{sign}(\sigma_k) = (-1)^{d-k} = \operatorname{sign}(F_k)$.

For $m > 1$, we just need to consider the map $x \mapsto B_{t_{i_1},\ldots,t_{i_d}} x + m v_{t_{i_1},\ldots,t_{i_d}}$ instead of $\phi_{t_{i_1},\ldots,t_{i_d}}$, and then we will have similar results.

□

Lemma 2.17. *For all $d \in \mathbb{R}^+$, for all $s_1, \ldots, s_d \in \mathbb{N}$, let $x_0 = 1$ and $P_{s_1,\ldots,s_d} = \{(x_1,\ldots,x_d) \in \mathbb{R}^d \mid \forall i \in [d] : 0 \le x_i \le s_i x_{i-1}\}$, $R_{s_1,\ldots,s_d} = \Omega(P_{s_1,\ldots,s_d})$. Then $R_{s_1,\ldots,s_d} = P_{s_1,\ldots,s_d} \cap \{x_d > 0\}$ and for all $d \ge 2 : R_{s_1,\ldots,s_d} = \rho^+(R_{s_1,\ldots,s_{d-1}}, P_{s_1,\ldots,s_d})$.*

Moreover, the vertices of $P_{s_1,\ldots,s_d}$ are

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_1 s_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} s_1 \\ s_1 s_2 \\ s_1 s_2 s_3 \\ \vdots \\ s_1 s_2 \cdots s_d \end{pmatrix}$$

and the positive boundary of $P_{s_1,\ldots,s_d}$ is just the convex hull of the first $d-1$ vertices and the last one. Note the first $d-1$ vertices span a $(d-2)$-dimensional space $\{(x_1,\ldots,x_d)' \mid x_d = x_{d-1} = 0\}$. Hence $PB(P_{s_1,\ldots,s_d})$ is in the hyperplane spanned by this $(d-2)$-dimensional space and the last vertex.

Proof. The first result is immediate by considering the definition of Ω.

We have $R_{s_1,\ldots,s_{d-1}} \subset P_{s_1,\ldots,s_{d-1}}$, so

$$\begin{aligned} \rho^+(R_{s_1,\ldots,s_{d-1}}, P_{s_1,\ldots,s_d}) \subset \rho^+(P_{s_1,\ldots,s_{d-1}}, P_{s_1,\ldots,s_d}) &= \rho^+(\pi(P_{s_1,\ldots,s_d}), P_{s_1,\ldots,s_d}) \\ &= \Omega(P_{s_1,\ldots,s_d}) = R_{s_1,\ldots,s_d}. \end{aligned}$$

But for $x = (x_1,\ldots,x_d) \in R_{s_1,\ldots,s_d}$, we have that $x_d > 0$ which implies that $s_d x_{d-1} > 0$, so $x_{d-1} > 0$. Therefore $\pi(x) \in R_{s_1,\ldots,s_{d-1}}$. Thus, $x \in \rho^+(R_{s_1,\ldots,s_{d-1}}, P_{s_1,\ldots,s_d})$. Now we can conclude that $R_{s_1,\ldots,s_d} = \rho^+(R_{s_1,\ldots,s_{d-1}}, P_{s_1,\ldots,s_d})$.

□

Theorem 2.18. *Let $d \in \mathbb{N}$ and $T = \{t_1, t_2, \ldots, t_{d+1}\}_<$ be an integral ordered set, then*

$$\Omega(C_d(T)) = \bigoplus_{\sigma \in S_d} \operatorname{sign}(\sigma) \varphi^{-1}_{t_{\sigma(1)},\ldots,t_{\sigma(d)}} (R_{t_{d+1}-t_{\sigma(1)},\ldots,t_{d+1}-t_{\sigma(d)}}).$$

Proof. We proceed by induction on d. When $d = 1$, $C_d(T)$ is just the interval $[t_1, t_2]$. Then the only element $\sigma \in S_1$ is the identity map. $R_{t_2-t_1} = (0, t_2 - t_1]$. And $\varphi_{t_1} : x \mapsto x - t_1$, so $\varphi^{-1}_{t_1} : x \mapsto x + t_1$. Thus $\varphi^{-1}_{t_1}((0, t_2 - t_1]) = (t_1, t_2] = \Omega([t_1, t_2])$.

Now we assume the theorem is true for dimensions less than d, and we will prove the case of dimension $d (\ge 2)$. Let $P = \phi_{t_1,\ldots,t_d}(C_d(T))$, and let $v_i = \phi_{t_1,\ldots,t_d}(\nu_d(t_i))$, $i \in [d+1]$, be the vertices of P. Then for $i \in [d]$, $v_i = \begin{pmatrix} \nu_{d-1}(t_i) \\ 0 \end{pmatrix}$ and for $i = d+1$, $v_{d+1} = \begin{pmatrix} \nu_{d-1}(t_{d+1}) \\ \prod_{i=1}^d (t_{d+1} - t_i)) \end{pmatrix}$. Since $\prod_{i=1}^d (t_{d+1} - t_i)) > 0$, the last coordinates of all the points in P are nonnegative. By Proposition 2.6, we have that

$$\Omega(P) = \bigoplus_{F\text{: a facet of } P} \operatorname{sign}(F) \rho^+(\Omega(\pi(F)), \operatorname{conv}(F, \pi(F))).$$

As in Proposition 2.16, we let F_k be the facet of $C_d(T)$ determined by all the vertices of $C_d(T)$ except $\nu_d(t_{i_k})$, then

$$\Omega(P) = \bigoplus_{k\in[d+1]} \operatorname{sign}(\phi_{t_1,\dots,t_d}(F_k))\rho^+(\Omega(\pi(\phi_{t_1,\dots,t_d}(F_k))), \operatorname{conv}(\phi_{t_1,\dots,t_d}(F_k), \pi(\phi_{t_1,\dots,t_d}(F_k)))).$$

For $k = d+1$, $\tilde{F} = \phi_{t_1,\dots,t_d}(F_{d+1}) = \operatorname{conv}(\{v_i\}_{i=1}^d)$ is on the hyperplane $H_0 = \{(x_1,\dots,x_d)' \in \mathbb{R}^d \mid x_d = 0\}$. So $\operatorname{conv}(\tilde{F}, \pi(\tilde{F}))$ is just $\tilde{F}$. Thus $\rho^+(\Omega(\pi(\tilde{F})), \operatorname{conv}(\tilde{F}, \pi(\tilde{F})))$ is an empty set.

And for $k \in [d]$, by Proposition 2.16, $\operatorname{sign}(F_k) = \operatorname{sign}(\sigma_k)$, where $\sigma_k = (k, k+1, \cdots, d) \in S_d$. Let $T_k = T \setminus \{t_k\}$, then $\pi(\phi_{t_1,\dots,t_d}(F_k)) = \pi(F_k) = C_{d-1}(T_k)$, because $\phi_{t_1,\dots,t_d}$ just changes the last coordinates. It's easy to see that

$$\operatorname{conv}(\phi_{t_1,\dots,t_d}(F_k), \pi(\phi_{t_1,\dots,t_d}(F_k))) = \operatorname{conv}(\{v_i\}_{i\neq k} \cup \{v'_{d+1}\}),$$

where $v'_{d+1} = \begin{pmatrix} \nu_{d-1}(t_{d+1}) \\ 0 \end{pmatrix}$ is the projection of v_{d+1} to the hyperplane H_0.

Hence,

$$\Omega(P) = \bigoplus_{k\in[d]} \operatorname{sign}(\sigma_k)\rho^+(\Omega(C_{d-1}(T_k)), \operatorname{conv}(\{v_i\}_{i\neq k} \cup \{v'_{d+1}\})).$$

For any $k \in [d]$, $T_k = \{t_{\sigma_k(1)}, t_{\sigma_k(2)}, \dots, t_{\sigma_k(d-1)}, t_{d+1}\}_<$. By the induction hypothesis, we have that

$$\Omega(C_{d-1}(T_k)) = \bigoplus_{\tau\in S_{d-1}} \operatorname{sign}(\tau)\varphi^{-1}_{t_{\sigma_k(\tau(1))},\dots,t_{\sigma_k(\tau(d-1))}}(R_{t_{d+1}-t_{\sigma_k(\tau(1))},\dots,t_{d+1}-t_{\sigma_k(\tau(d-1))}}).$$

So,

$$\begin{aligned} &\operatorname{sign}(\sigma_k)\phi^{-1}_{t_1,\dots,t_d}(\Omega(C_{d-1}(T_k))) \\ = &\bigoplus_{\tau\in S_{d-1}} \operatorname{sign}(\sigma_k)\operatorname{sign}(\tau)\varphi^{-1}_{t_{\sigma_k(\tau(1))},\dots,t_{\sigma_k(\tau(d-1))}}(R_{t_{d+1}-t_{\sigma_k(\tau(1))},\dots,t_{d+1}-t_{\sigma_k(\tau(d-1))}}) \\ = &\bigoplus_{\sigma\in S_d:\sigma(d)=k} \operatorname{sign}(\sigma)\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(R_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)}}). \quad (\text{let } \sigma = \sigma_k\tau) \end{aligned}$$

Let H_k be the hyperplane determined by $\phi_{t_1,\dots,t_d}(F_k)$, and $H_k^+ = \{x \in H_k \mid l(x) \geq 0\}$. We claim that for all $\sigma \in S_d$ with $\sigma(d) = k$, we have

$$\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}})) \subset H_k^+.$$

Given this, we can pick a convex polytope $S_k \subset H_k$, such that

a) The last coordinates of the points in S_k are nonnegative;
b) S_k contains $\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}))$, for all $\sigma \in S_d$ with $\sigma(d) = k$;
c) S_k contains $\phi_{t_1,\dots,t_d}(F_k)$.

Note that $\pi(H_k)$ contains $\pi(\phi_{t_1,\dots,t_d}(F_k)) = \pi(F_k) = C_{d-1}(T_k)$, which has dimension $d-1$. So $\pi(H_k) = \mathbb{R}^{d-1}$.

Hence, by Lemma 2.7

$$
\begin{aligned}
& \Omega(C_d(T)) \\
= \;& \phi_{t_1,\dots,t_d}^{-1}(\Omega(P)) \\
= \;& \bigoplus_{k\in[d]} \operatorname{sign}(\sigma_k)\phi_{t_1,\dots,t_d}^{-1}(\rho^+(\Omega(C_{d-1}(T_k)), \operatorname{conv}(\{v_i\}_{i\neq k}\cup\{v'_{d+1}\}))) \\
= \;& \bigoplus_{k\in[d]} \operatorname{sign}(\sigma_k)\phi_{t_1,\dots,t_d}^{-1}(\rho^+(\Omega(C_{d-1}(T_k)), \operatorname{conv}(S_k,\pi(S_k)))) \\
= \;& \bigoplus_{k\in[d]} \operatorname{sign}(\sigma_k)\phi_{t_1,\dots,t_d}^{-1}(\rho^+(\bigoplus_{\tau\in S_{d-1}} \operatorname{sign}(\tau)\varphi^{-1}_{t_{\sigma_k(\tau(1))},\dots,t_{\sigma_k(\tau(d-1))}}(R_{t_{d+1}-t_{\sigma_k(\tau(1))},\dots,t_{d+1}-t_{\sigma_k(\tau(d-1))}}), \\
& \qquad\qquad \operatorname{conv}(S_k,\pi(S_k)))) \\
= \;& \bigoplus_{k\in[d]} \bigoplus_{\sigma\in S_d,\sigma(d)=k} \operatorname{sign}(\sigma)\phi_{t_1,\dots,t_d}^{-1}(\rho^+(\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(R_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)}}), \\
& \qquad\qquad \varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}))) \\
= \;& \bigoplus_{\sigma\in S_d} \operatorname{sign}(\sigma)\phi_{t_1,\dots,t_d}^{-1}\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(\rho^+(R_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)}}, \\
& \qquad\qquad P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}})) \\
= \;& \bigoplus_{\sigma\in S_d} \operatorname{sign}(\sigma)\varphi^{-1}_{t_{\sigma(1)},\dots,t_{\sigma(d)}}(R_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}).
\end{aligned}
$$

Thus the claim implies the theorem.

Showing the claim is equivalent to showing that

$$PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}) \subset \varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(H_k^+).$$

Both $\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}$ and its inverse only work on the first $d-1$ coordinates of any point in $\mathbb{R}^d$. Thus $\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(H_k^+)$ is just $\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(H_k)\cap\{x\in\mathbb{R}^d \mid l(x)\geq 0\}$. But it's clear that $PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}})$ is in $\{x\in\mathbb{R}^d \mid l(x)\geq 0\}$. So it's enough to show that

$$PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}) \subset \varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(H_k).$$

By Lemma 2.17, $PB(P_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}})$ lies in the hyperplane H which is spanned by $\{(x_1,\dots,x_d)' \mid x_d = x_{d-1} = 0\}$ and $\begin{pmatrix} t_{d+1}-t_{\sigma(1)} \\ (t_{d+1}-t_{\sigma(1)})(t_{d+1}-t_{\sigma(2)}) \\ (t_{d+1}-t_{\sigma(1)})(t_{d+1}-t_{\sigma(2)})(t_{d+1}-t_{\sigma(3)}) \\ \vdots \\ (t_{d+1}-t_{\sigma(1)})(t_{d+1}-t_{\sigma(2)})\cdots(t_{d+1}-t_{\sigma(d)}) \end{pmatrix}$.

So we need show that $\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(H_k) = H$. Since H_k is the hyperplane containing $\phi_{t_1,\dots,t_d}(F_k)$, it's enough to show that $\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d-1)}}(\phi_{t_1,\dots,t_d}(F_k)) = \varphi_{t_{\sigma(1)},\dots,t_{\sigma(d)}}(F_k)$ is contained in H. However, $F_k = \operatorname{conv}(\nu_d(T_k))$. Meanwhile, by remark 2.14, we have

$$\varphi_{t_{\sigma(1)},\dots,t_{\sigma(d)}}(\nu_d(t)) = \begin{pmatrix} (t-t_{\sigma(1)}) \\ (t-t_{\sigma(1)})(t-t_{\sigma(2)}) \\ \vdots \\ (t-t_{\sigma(1)})(t-t_{\sigma(2)})\cdots(t-t_{\sigma(d)}) \end{pmatrix}.$$

Since $\sigma(d) = k$, for any $i \in [d], i \neq k$, $\varphi_{t_{\sigma(1)},\ldots,t_{\sigma(d)}}(\nu_d(t_i))$ has the last two coordinates equal to 0. And for $i = d+1$, $\varphi_{t_{\sigma(1)},\ldots,t_{\sigma(d)}}(\nu_d(t_{d+1}))$ is exactly the last vertex of $P_{t_{d+1}-t_{\sigma(1)},\ldots,t_{d+1}-t_{\sigma(d-1)},t_{d+1}-t_{\sigma(d)}}$, which completes the proof the claim and hence the theorem.

□

Remark 2.19. If we define $\varphi_{m,r_1,\ldots,r_d} : x \mapsto A_{r_1,\ldots,r_d}x + mu_{r_1,\ldots,r_d}$, then similarly we can prove that

$$\Omega(mC_d(T)) = \bigoplus_{\sigma \in S_d} \operatorname{sign}(\sigma)\varphi^{-1}_{m,t_{\sigma(1)},\ldots,t_{\sigma(d)}}(mR_{t_{d+1}-t_{\sigma(1)},\ldots,t_{d+1}-t_{\sigma(d)}}).$$

Corollary 2.20.

$$\mathcal{L}(\Omega(mC_d(T))) = \bigoplus_{\sigma \in S_d} \operatorname{sign}(\sigma)\mathcal{L}(\varphi^{-1}_{m,t_{\sigma(1)},\ldots,t_{\sigma(d)}}(mR_{t_{d+1}-t_{\sigma(1)},\ldots,t_{d+1}-t_{\sigma(d)}})).$$

Hence,

$$|\mathcal{L}(\Omega(mC_d(T)))| = \sum_{\sigma \in S_d} \operatorname{sign}(\sigma)|\mathcal{L}(mR_{t_{d+1}-t_{\sigma(1)},\ldots,t_{d+1}-t_{\sigma(d)}})|.$$

It's easy to see that $mR_{s_1,\ldots,s_d} = R_{ms_1,s_2,\ldots,s_d}$. Moreover,

$$|L(R_{s_1,\ldots,s_d})| = \sum_{x_1=1}^{s_1} \sum_{x_2=1}^{s_2x_1} \cdots \sum_{x_n=1}^{s_nx_{n-1}} 1.$$

Therefore, it's natural to look at the following:

Lemma 2.21. *For any nonnegative integers $a_1, a_2, \ldots, a_n$, let*

$$h(a_1, a_2, \ldots, a_n) = \sum_{x_1=1}^{a_1} \sum_{x_2=1}^{a_2x_1} \cdots \sum_{x_n=1}^{a_nx_{n-1}} 1.$$

Then the only highest degree term of h is $\frac{1}{n!}a_1^n a_2^{n-1} a_3^{n-2} \ldots a_n$. This is also true when we consider h as a polynomial just in the variable a_1.

Proof of Lemma 2.21: We will prove it by induction on n.

When $n = 1$, $h(a_1) = \sum_{x_1=1}^{a_1} 1 = a_1$. Thus the lemma holds.

Assume the lemma is true for n, and note that $h(a_1, a_2, \ldots, a_{n+1}) = \sum_{x_1=1}^{a_1} h(a_2x_1, a_3, \ldots, a_{n+1})$. By assumption, $\frac{1}{n!}a_2^n a_3^{n-1} \ldots a_{n+1}x_1^n$ is the only highest degree term of $h(a_2x_1, a_3, \ldots, a_{n+1})$ when we consider it as polynomial both in $y = a_2x_1, a_3, \ldots, a_{n+1}$ and in y. This implies that $\frac{1}{n!}a_2^n a_3^{n-1} \ldots a_{n+1}x_1^n$ is the only highest degree term of $h(a_2x_1, a_3, \ldots, a_{n+1})$ when we consider it both in $a_2, a_3, \ldots, a_{n+1}$ and in x_1. Then our lemma immediately follows from the fact that the highest degree term of $\sum_{x_1=1}^{a_1} x_1^n$ is $\frac{1}{n+1}a_1^{n+1}$.

□

Proposition 2.22. *For any nonnegative integers $a_1, a_2, \ldots, a_n$, let $\mathcal{H}_m(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) h(ma_{\sigma(1)}, a_{\sigma(2)} \ldots, a_{\sigma(n)})$. Then*

$$\mathcal{H}_m(a_1, a_2, \ldots, a_n) = \frac{m^n}{n!} \prod_{i=1}^{n} a_i \prod_{1 \le i < j \le n} (a_i - a_j).$$

Proof of Proposition 2.22:
Clearly if any of a_i's is 0, then $\mathcal{H}_m(a_1,\dots,a_n)=0$. Also for $1\le i<j\le n$, $\mathcal{H}_m$ changes sign when we switch a_i and a_j, i.e.,

$$\mathcal{H}_m(\dots,a_i,\dots,a_j,\dots)=-\mathcal{H}_m(\dots,a_j,\dots,a_i,\dots).$$

Therefore, $\mathcal{H}_m(a_1,\dots,a_n)$ must be a multiple of

$$\prod_{i=1}^{n} a_i \prod_{1\le i<j\le n}(a_i-a_j),$$

which has degree $\frac{1}{2}n(n+1)$.

So now it's enough to show that $\mathcal{H}_m(a_1,\dots,a_n)$ is of degree $\frac{1}{2}n(n+1)$ and the coefficient of $a_1^n a_2^{n-1} a_3^{n-2}\dots a_n$ in $\mathcal{H}_m(a_1,\dots,a_n)$ is $\frac{m^n}{n!}$, which follows from Lemma 2.21. □

Proof of Theorem 2.9: By Corollary 2.20,

$$\begin{aligned}
|\mathcal{L}(\Omega(mC_d(T)))| &= \sum_{\sigma\in S_d}\operatorname{sign}(\sigma)|\mathcal{L}(mR_{t_{d+1}-t_{\sigma(1)},\dots,t_{d+1}-t_{\sigma(d)}})| \\
&= \mathcal{H}_m(t_{d+1}-t_{\sigma(1)},t_{d+1}-t_{\sigma(2)},\dots,t_{d+1}-t_{\sigma(d)}) \\
&= \frac{m^d}{d!}\prod_{i=1}^{d}(t_{d+1}-t_i)\prod_{1\le i<j\le d}(t_i-t_j) \\
&= \frac{m^d}{d!}\prod_{1\le i<j\le d+1}(t_i-t_j)=\operatorname{Vol}(mC_d(T)).
\end{aligned}$$

□

As we argued earlier in our paper, the proof of Theorem 2.9 completes the proof of Proposition 2.2 and thus proof of our main Theorem 1.2.

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