POSITIVITY QUESTIONS FOR CYLINDRIC SKEW SCHUR FUNCTIONS

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Abstract. Recent work of A. Postnikov shows that cylindric skew Schur functions, which are a generalisation of skew Schur functions, have a strong connection with a problem of considerable current interest: that of finding a combinatorial proof of the non-negativity of the 3-point Gromov-Witten invariants. After explaining this motivation, we study cylindric skew Schur functions from the point of view of Schur-positivity. Using a result of I. Gessel and C. Krattenthaler, we generalise a formula of A. Bertram, I. Ciocan-Fontanine and W. Fulton, thus giving an expansion of an arbitrary cylindric skew Schur function in terms of skew Schur functions. While we show that no non-trivial cylindric skew Schur functions is Schur-positive, we conjecture that this can be reconciled using the new concept of cylindric Schur-positivity.

Résumé. Les travaux récents de A. Postnikov montrent que les fonctions gauches cylindriques de Schur, qui sont une généralisation des fonctions gauches de Schur, ont un lien étroit avec un problème actuellement très étudié : trouver une preuve combinatoire de la non-negativité des invariants de Gromov-Witten de 3-pointes. Après avoir expliqué cette motivation, nous étudions les fonctions gauches cylindriques de Schur du point de vue de la Schur-positivité. En utilisant un résultat de I. Gessel et C. Krattenthaler, nous généralisons une formule de A. Bertram, I. Ciocan-Fontanine et W. Fulton, donnant ainsi une expansion d'une fonction gauche cylindrique de Schur arbitraire en termes de fonctions gauches de Schur. Tandis que nous prouvons qu'aucune fonction gauche cylindrique de Schur non triviale n'est Schur-positive, nous introduisons le concept de Schur-positivité cylindrique et conjecturons que tout fonction gauche cylindrique de Schur est Schur-positive cylindrique.

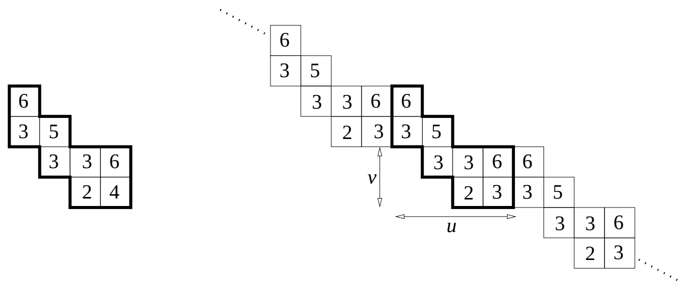
1. Introduction

The Schur functions $s_\lambda(x)$, where λ runs over all partitions, form an important basis for the ring of symmetric functions in infinitely many variables $x = (x_1, x_2, \ldots)$. In particular, the skew Schur function $s_{\lambda/\mu}(x)$ can be expanded in terms of Schur functions as

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu(x),$$

where $c^{\lambda}_{\mu\nu}$ denotes the ubiquitous *Littlewood-Richardson* coefficients. It is well known that Littlewood-Richardson coefficients are non-negative and a skew Schur function is thus one of the most famous examples of a *Schur-positive* function, i.e. a symmetric function whose expansion as a linear combination of Schur functions has all positive coefficients. It is worth mentioning that Schur-positivity has a particular representation-theoretic significance: if a homogeneous symmetric function of degree N is Schur-positive, then it is known to arise as the Frobenius image of some representation of the symmetric group S_N. This is one of the reasons why questions of Schur-positivity have received, and continue to receive, much attention in recent times.

As their name suggests, *cylindric* skew Schur functions are a natural generalisation of skew Schur functions. In particular, cylindric skew Schur functions are symmetric functions, and skew Schur functions are themselves cylindric skew Schur functions. While this is enough to give them combinatorial appeal, cylindric skew Schur functions have recently been shown to play a central role in the study of the quantum cohomology ring of the Grassmannian. More specifically, this contemporary motivation for cylindric skew Schur functions involves the fundamental open problem of finding a combinatorial proof of the non-negativity of the 3-point Gromov-Witten invariants. While it will be our starting point in Section 3, no knowledge of quantum cohomology will be assumed and our emphasis will be combinatorial. Gromov-Witten invariants are connected to the topic of cylindric skew Schur functions



(a) (b)

FIGURE 1.

via a theorem of A. Postnikov [12]. Since cylindric skew Schur functions are symmetric, they can be expanded in terms of Schur functions and Postnikov's theorem states that the Gromov-Witten invariants appear as particular coefficients in this expansion. The fundamental open problem mentioned above then becomes a question about the Schur-positivity of cylindric skew Schur functions. In short, cylindric skew Schur functions play a similar role for the 3-point Gromov-Witten invariants as that played by skew Schur functions for the Littlewood-Richardson coefficients. Rather than tackling the fundamental open problem directly, however, our goal is to give a general study of the Schur-positivity of cylindric skew Schur functions.

At this point, it makes sense to give an informal introduction to cylindric skew shapes, semistandard cylindric tableaux and cylindric skew Schur functions. Suppose we are given a skew shape and a semistandard Young tableau of that shape, such as the one shown in French notation in Figure 1(a). We can think of cylindric skew shapes as coming from skew shapes that have been wrapped around a cylinder so that boxes at the bottom of the rightmost column are now directly to the left of boxes at the top of the leftmost column. We will represent this cylindric skew shape C as a skew shape together with its images under repeated applications of some translation $(-u, v)$, as shown in Figure 1(b). Notice that we have used our previous semistandard Young tableau but have had to modify one of the entries to ensure that the entries continue to be weakly increasing in the rows. The result is an example of a semistandard cylindric tableau. The definition of the cylindric skew Schur function $s_C(x)$ is then completely analogous to that of a skew Schur function. This example motivates the formal introduction that is the subject of Section 2.

The geometric definition of Gromov-Witten invariants, combined with Postnikov's theorem, tells us that cylindric skew Schur functions in a restricted number of variables are Schur-positive. In Section 4, we show that, except for trivial cases, cylindric skew Schur functions are never Schur-positive in infinitely many variables. Since they play a crucial role in our proof of this result, we investigate the class of "cylindric ribbons," determining the form of the Schur expansion of their corresponding cylindric skew Schur functions. We conclude Section 4 with a discussion of the minimum number of variables in which a cylindric skew Schur function will not be Schur-positive.

In Section 5, we develop a tool for expanding cylindric skew Schur functions as a signed sum of skew Schur functions. A result of I. Gessel and C. Krattenthaler [5] serves as the foundation for our tool, while our formulation is inspired by a result of A. Bertram, I. Ciocan-Fontanine and W. Fulton [2].

Since cylindric skew Schur functions are such a natural generalisation of skew Schur functions, one might ask if there is some way to extend the Schur-positivity of skew Schur functions to the cylindric setting. In Section 6, we define cylindric Schur-positivity as an analogue of Schur-positivity and we give evidence in favour of a conjecture that every cylindric skew Schur functions is cylindric Schur-positive.

Before beginning in earnest, we introduce terminology and notation that we will use throughout. We will denote the sets of integers, non-negative integers and positive integers by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{P}$ respectively. For $N \in \mathbb{P}$, we will write $[N]$ to denote the set $\{1, 2, \ldots, N\}$. For symmetric function notation, we will follow [8].

We will allow a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ to have parts equal to zero and we identify λ with the sequence $(\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$. We write $l(\lambda)$ for the number of non-zero parts (length) of λ and $|\lambda|$ for the sum of the parts of λ. We use a^k in the list of parts of a partition to denote a sequence of k a's. For example, $\lambda = (j, 1^k)$ has one part of size j and k parts of size 1. We let $\emptyset$ denote the unique partition with length 0. We can represent a partition λ by its Young diagram, and then the *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ of λ is the partition obtained by reading the column lengths of λ from left to right.

If μ is another partition then we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i; this is equivalent to saying that the diagram of μ is contained in the diagram of λ. If $\mu \subseteq \lambda$, then we define the *skew shape* λ/μ to be the set of boxes in the diagram of λ that remain after we remove those boxes corresponding to the partition μ. A *ribbon* (or *rim hook* or *border strip*) is an edgewise connected skew shape that contains no 2×2 block of boxes. An *n-ribbon* is then simply a ribbon with n boxes.

Acknowledgements

This extended abstract is an abbreviated version of [11], which results from work begun while the author was a graduate student at MIT. I am grateful to my advisor, Richard Stanley, and to Alex Postnikov for several interesting discussions on the topic. François Bergeron and Christophe Reutenauer, my mentors at LaCIM, have both made valuable suggestions, and their expertise and enthusiasm have been of considerable assistance.

2. Cylindric skew Schur functions

Cylindric skew shapes are not a new idea and there are three references in particular that are of great relevance to our work. The first of these is [5], which will play a significant role in Section 5. Semistandard cylindric tableaux, which we will shortly define, appear under the name "proper tableaux" in [2]. The main result of [12] serves as the starting point for our results. For the following introduction to the notation and definition of cylindric skew shapes, we will largely follow [12].

Fix positive integers u and v. We define the *cylinder* $\mathfrak{C}_{vu}$ to be the following quotient of $\mathbb{Z}^2$:

$$\mathfrak{C}_{vu} = \mathbb{Z}^2/(-u, v)\mathbb{Z}.$$

In other words, $\mathfrak{C}_{vu}$ is the quotient of the integer lattice $\mathbb{Z}^2$ modulo a shifting action which sends (i, j) to $(i-u, j+v)$. For $(i, j) \in \mathbb{Z}^2$, we let $\langle i, j\rangle = (i, j) + (-u, v)\mathbb{Z}$ denote the corresponding element of $\mathfrak{C}_{vu}$. $\mathfrak{C}_{vu}$ inherits a natural partial order $\leq_{\mathfrak{C}}$ from $\mathbb{Z}^2$ which is generated by the relations $\langle i, j\rangle <_{\mathfrak{C}} \langle i + 1, j\rangle$ and $\langle i, j\rangle <_{\mathfrak{C}} \langle i, j + 1\rangle$.

Note that this partial order is antisymmetric since u and v are positive. Recall that a subposet Q of a poset P is said to be *convex* if, for all elements $x < y < z$ in P, we have $y \in Q$ whenever we have $x, z \in Q$.

Definition 2.1. A *cylindric skew shape* is a finite convex subposet of the poset $\mathfrak{C}_{vu}$.

Example 2.2. We can regard skew shapes λ/μ as a special case of cylindric skew shapes. Suppose λ/μ fits inside a box of height v and width u. We embed λ/μ in $\mathfrak{C}_{vu}$ by mapping the box in the ith row and jth column of λ/μ to $\langle i, j\rangle$. Figure 2 shows the resulting image of λ/μ in $\mathbb{Z}^2$, with one representative of λ/μ shown in bold. Notice that elements of different representatives of λ/μ are always incomparable in $\mathbb{Z}^2$. Of course, we could also embed λ/μ in $\mathfrak{C}_{v'u'}$ where $v' \geq v$ and $u' \geq u$.

Example 2.3. The class of *cylindric ribbons* will play an important role and they are defined in the analogous way to ribbons in the classical case. As we just did for skew shapes, we will identify any cylindric skew shape with its corresponding set of boxes in $\mathbb{Z}^2$. Note that the skew shapes from the previous example can be edgewise connected when viewed as subsets of $\mathfrak{C}_{vu}$. However, they are not edgewise connected when viewed as subsets of $\mathbb{Z}^2$, as in the figure.

Definition 2.4. A *cylindric ribbon* is a cylindric skew shape which, when viewed as a subset $\mathbb{Z}^2$, is edgewise connected and contains no 2×2 block of boxes.

The cylindric skew shape in Figure 1(b) is an example of a cylindric ribbon once we delete the boxes filled with 6's.

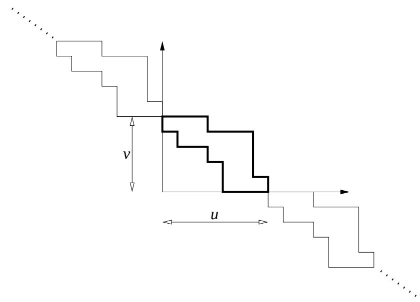


FIGURE 2. Skew shapes are cylindric skew shapes

Suppose C is a cylindric skew shape which is a subposet of the cylinder $\mathfrak{C}_{vu}$. Let us define what we mean by the rows and columns of C. The *p-th row* is the set $\{\langle i,j\rangle \in C \mid j = p\}$ and the *q-th column* is the set $\{\langle i,j\rangle \in C \mid i = q\}$. So the rows only depend on $p \bmod v$ and the columns only depend on $q \bmod u$. Thus the cylinder $\mathfrak{C}_{vu}$ has exactly v rows and u columns.

Definition 2.5. For a cylindric skew shape C, a *semistandard cylindric tableau* of shape C is a map $T : C \to \mathbb{P}$ that weakly increases in the rows of C and strictly increases in the columns.

See Figure 1(b) for an example. We are now ready to define our main object of study.

Definition 2.6. For a cylindric skew shape C, the *cylindric skew Schur function* $s_C(x)$ in the variables $x = (x_1, x_2, \ldots)$ is defined by

$$s_C(x) = \sum_T \prod_{c\in C} x_{T(c)} = \sum_T x_1^{\#T^{-1}(1)} x_2^{\#T^{-1}(2)} \cdots ,$$

where the sums are over all semistandard cylindric tableaux T of shape C.

The terminology "cylindric skew Schur function" is partially justified by the following two observations:

Example 2.7. Because of Example 2.2, skew Schur functions and, in particular, Schur functions are all examples of cylindric skew Schur functions.

Theorem 2.8. *For any cylindric skew shape C, $s_C(x)$ is a symmetric function.*

We omit the proof as it is basically the same as the proof from [1], which also appears as [13, Theorem 7.10.2], that the skew Schur function $s_{\lambda/\mu}(x)$ is symmetric.

3. CYLINDRIC SKEW SCHUR FUNCTIONS FROM GROMOV-WITTEN INVARIANTS

As mentioned in the introduction, there is a good reason for recent interest in cylindric skew Schur functions. This motivation is centred around the main result of [12]. A nice introduction, with emphasis on the context and the importance of Postnikov's result can be found in [14]. Here, however, we merely extract from these two references the minimum amount of background necessary to show how Postnikov's work ties together cylindric skew Schur functions and an open problem of considerable interest.

Given k and n with $n > k \geq 1$, we let Gr_{kn} denote the manifold of k-dimensional subspaces of $\mathbb{C}^n$. Gr_{kn} is a complex projective variety known as the *Grassmann variety* or *Grassmannian.* For a partition λ, we will write $\lambda \subseteq k\times(n-k)$ if the Young diagram for λ has at most k rows and at most $n-k$ columns. In this case, we let $\lambda^\vee$ denote the partition $(n-k-\lambda_k, \ldots, n-k-\lambda_1)$. Given $\lambda, \mu, \nu \subseteq k \times (n-k)$, we let $C_{\mu\nu}^{\lambda,d}$ denote the *(3-point) Gromov-Witten invariant*, defined geometrically as the number of rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties $\Omega_{\lambda^\vee}, \Omega_\mu$ and Ω_ν, provided that this number is finite. This last condition implies that $C_{\mu\nu}^{\lambda,d}$ is defined if $|\mu|+|\nu| = nd+|\lambda|$, and otherwise we set $C_{\mu\nu}^{\lambda,d} = 0$. If $d = 0$, then a degree 0 curve is just a point in Gr_{kn} and we get the geometric interpretation of the Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda = C_{\mu\nu}^{\lambda,0}$. While we do not claim

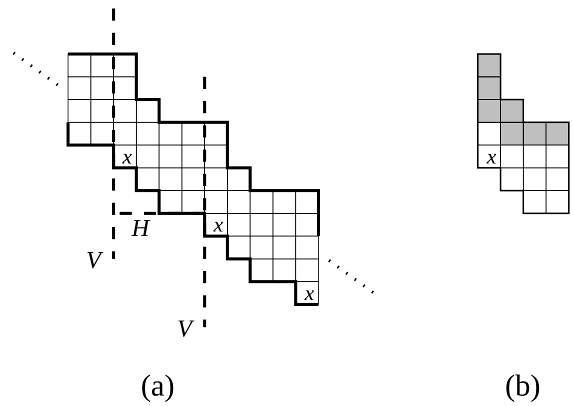


FIGURE 3. Describing C as $\lambda/d/\mu$

that this paragraph is sufficient to give a firm understanding of $C^{\lambda,d}_{\mu\nu}$, we do claim that it is clear from this geometric definition that $C^{\lambda,d}_{\mu\nu} \geq 0$. No algebraic or combinatorial proof of this inequality is known and, as stated in [14], it is a fundamental open problem to find such a proof.

Postnikov's result shows that the Gromov-Witten invariants $C^{\lambda,d}_{\mu\nu}$ appear as the coefficients when we expand certain cylindric skew Schur functions in terms of Schur functions. It follows that improving our understanding of this expansion could lead to a solution of the open problem.

Before stating his result, we need to introduce some notation that will allow us to write any cylindric skew shape in the form $\lambda/d/\mu$, where λ and μ are partitions and $d \in \mathbb{N}$. From this point on, unless otherwise stated, all of our cylindric skew shapes C will be subposets of the cylinder $\mathfrak{C}_{vu}$ with $v = k$ and $u = n - k$.

Suppose we are given any cylindric skew shape C. The process for finding λ, d and μ is best understood from a figure, and we will use the cylindric skew shape shown in Figure 3(a) as a running example. The boxes labelled x are identified, so that $k = 3$ and $n - k = 4$ in this example. First, we must choose a set of representatives for the elements of C. A convenient way to do this is to take the elements between two adjacent representatives of a vertical line V. Now draw a horizontal line segment H running below our representatives of C. We regard the intersection of V and the left end of H in Figure 3(a) as our origin. The partition μ is now the partition whose Young diagram is outlined by H, V and the lower boundary of C. In our example, $\mu = (2, 1)$.

Next, consider just our set of representatives for the elements of C as in Figure 3(b). Define a partition Λ by supposing the resulting skew shape is Λ/μ. Therefore, in our example, $\Lambda = (4, 4, 4, 4, 2, 1, 1)$. If $\Lambda \subseteq k \times (n-k)$ then set $d = 0$, $\lambda = \Lambda$ and we are done. Otherwise, let $\Lambda[-1]$ denote the unique partition ν that makes Λ/ν an n-ribbon with $n-k$ non-empty columns. In other words, $\Lambda[-1]$ is obtained by removing an n-ribbon along the top of Λ, starting in Λ's leftmost column and ending in Λ's rightmost column. It is not difficult to see that such a ribbon always has $k+1$ non-empty rows. In our example, we remove the shaded boxes in Figure 3(b) and $\Lambda[-1] = (4, 4, 4, 1)$. We can see that $\Lambda[-1]$ is well-defined by referring back to Figure 3(a). Effectively what we are doing is removing the cylindric ribbon that runs all the way along the top of C. We see that this cylindric ribbon must have n elements.

Now if $\Lambda[-1] \subseteq k \times (n-k)$, then we set $d = 1$ and $\lambda = \Lambda[-1]$. Otherwise, obtain $\Lambda[-2]$ from $\Lambda[-1]$ in the same way that $\Lambda[-1]$ was obtained from Λ: remove an n-ribbon from the top of $\Lambda[-1]$, starting in the leftmost column and ending in the rightmost column. Repeating this procedure, we can construct $\Lambda[-e]$, stopping as soon as $\Lambda[-e] \subseteq k \times (n-k)$. We then set $d = e$ and $\lambda = \Lambda[-e]$. In our example, we see that $\Lambda[-2] = (3, 3) \subseteq k \times (n-k)$ and so $d = 2$, $\lambda = (3, 3)$ and $\lambda/d/\mu = (3, 3)/2/(2, 1)$.

Remark 3.1. There are several things to note about $\lambda/d/\mu$:

(i) For a given C, $\lambda/d/\mu$ is clearly not unique and depends on our choice of origin.

(ii) μ is not necessarily contained in λ. For example, moving our origin 1 square down and 1 square to the left, the reader is encouraged to verify that $\mu = (4,3,2,1)$, $\Lambda = (4,4,4,4,4,3,2,2)$ and that $\lambda/d/\mu = (3,3)/3/(4,3,2,1)$.

(iii) We always have $\lambda \subseteq k \times (n-k)$ and it is always possible to choose our origin so that $\mu \subseteq k \times (n-k)$.

(iv) We could alternatively have defined λ by saying it is the *n-core* of Λ, where the n-core is defined in the following manner. Given a partition τ, successively remove n-ribbons from τ so that after each ribbon removal, the resulting shape is a partition. Stop when no more n-ribbons can be removed. It is a well-known fact (see, for example, [8, I.1, Example 8]) that the resulting partition λ is independent of the choice of ribbons removed, and λ is said to be the *n-core* of τ.

(v) Our notation $\lambda/d/\mu$ is equivalent to that in [12], but our explanation of it is very different. We choose this description in terms of removal of ribbons because it will be useful in later sections.

For any formal power series f in the variables $x = (x_1, x_2, \ldots)$, we will write $f(x_1, \ldots, x_k)$ to denote the specialization $f(x_1, x_2, \ldots, x_k, 0, 0, \ldots)$. We are finally ready to state [12, Theorem 6.3].

Theorem 3.2. *For any two partitions $\lambda, \mu \subseteq k \times (n-k)$ and a non-negative integer d, we have*

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\mu\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k). \tag{3.1}$$

Since we are restricting to k variables, the left-hand side is a sum over semistandard cylindric tableaux T that map $\lambda/d/\mu$ to the set $[k]$. Since T must increase in the columns of $\lambda/d/\mu$, this implies that $s_{\lambda/d/\mu}(x_1, x_2, \ldots, x_k)$ is non-zero only if all the columns of $\lambda/d/\mu$ contain at most k elements. One can check that this is equivalent to all the rows of $\lambda/d/\mu$ containing at most $n-k$ elements. In this case, we follow Postnikov in saying that $\lambda/d/\mu$ is a *toric* shape. While we take this opportunity to note that toric shapes are the shapes that are most relevant to the Gromov-Witten invariants, we will continue to work with general cylindric skew shapes.

Since will be mostly interested in the case of infinitely many variables $x = (x_1, x_2, \ldots)$, we make a few quick remarks about $s_{\lambda/d/\mu}(x)$ and $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$. First, since all the entries in any column of a semistandard cylindric tableau are distinct, the monomial $x_1^{a_1} x_2^{a_2} \cdots$ appears with coefficient 0 in $s_{\lambda/d/\mu}(x)$ if $a_i > n-k$ for some i. It follows that we have the useful fact that

$$s_{\lambda/d/\mu}(x) = \sum_\nu c_\nu s_\nu(x) = \sum_{\nu : \nu_1 \leq n-k} c_\nu s_\nu(x). \tag{3.2}$$

From this, we conclude

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu : l(\nu) \leq k} c_\nu s_\nu(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu(x_1, \ldots, x_k),$$

explaining why the sum in (3.1) is only over $\nu \subseteq k \times (n-k)$. Finally, we note that $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ is essentially obtained from $s_{\lambda/d/\mu}(x)$ by removing all those terms involving s_ν with $l(\nu) > k$. In fact, in the sections that follow, we will be focusing most of our attention on these terms s_ν with $l(\nu) > k$.

Since we know from the geometric definition of Gromov-Witten invariants that $C_{\mu\nu}^{\lambda,d} \geq 0$, we conclude from (3.1) that $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ is Schur-positive. On the other hand, we observe that $s_{\lambda/d/\mu}(x)$ may not be Schur-positive. For example, when $k = n-k = 2$ and $\lambda/d/\mu = (1,0)/1/(1,0)$,

$$s_{\lambda/d/\mu} = m_{22} + 2m_{211} + 4m_{1111} = s_{22} + s_{211} - s_{1111}.$$

In the next section, we answer the following question:

Question 3.3. *For what cylindric skew shapes C is $s_C(x)$ Schur-positive?*

4. Schur-positivity

We saw in Example 2.2 that the skew shape λ/μ can be regarded as a cylindric skew shape C when λ/μ fits inside a box of height k and width $n-k$. In this case, we then know that s_C is Schur-positive. The following theorem, which is the main result of this section, states that these are the only

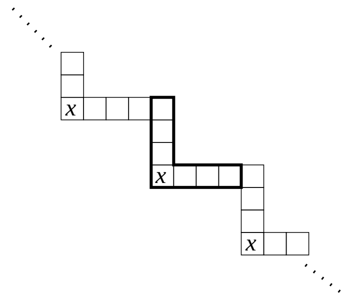


FIGURE 4. $H_{4,3}$

Schur-positive cylindric skew Schur functions. To state the theorem, we need to define what it means for cylindric skew shapes to be isomorphic. Recall that every cylindric skew shape can be viewed as a subposet of $\mathfrak{C}_{vu}$, for some positive numbers v and u. For cylindric skew shapes to be isomorphic, we certainly need the corresponding subposets to be isomorphic, but we also need to preserve certain "strictness" relations in the columns. Therefore, we will say that cylindric skew shapes C_1 and C_2 are isomorphic if there exists a poset isomorphism $f : C_1 \to C_2$ such that, for $x, y \in C_1$, x and y are in the same column of C_1 if and only if $f(x)$ and $f(y)$ are in the same column of C_2.

Theorem 4.1. *Let C be a cylindric skew shape. Then $s_C(x)$ is Schur-positive if and only if C is isomorphic to a skew shape.*

In other words, s_C is never Schur-positive except in the trivial case of C being a skew shape.

Let us say as a few words about the proof of this result. The first of two main steps is to prove it to be true for cylindric ribbons. The second step involves using the outer coproduct for the ring of quasisymmetric functions, as defined in [9, 10], to deduce the result for general cylindric skew shapes.

Cylindric ribbons are also interesting in their own right, and they will be our next topic of discussion. We begin with a special class.

Example 4.2. A cylindric ribbon is said to be a *cylindric hook* if it has a unique minimal element (when viewed as a subposet of $\mathfrak{C}_{k,n-k}$). See Figure 4 for an example. We see that, unlike hooks in the classical case, cylindric hooks have just one maximal element. Also note that $\mathfrak{C}_{k,n-k}$ has just one cylindric hook as a subposet, up to isomorphism. We denote this cylindric hook by $H_{k,n-k}$. A cylindric hooks is the simplest example of a cylindric skew shape that is not toric. It follows that $s_{H_{k,n-k}}(x_1, \ldots, x_k) = 0$. This is also evident in the following result which shows that the Schur expansion of $s_{H_{k,n-k}}(x)$ is a nice alternating sum of Schur functions of hooks.

Lemma 4.3. *With all functions in the variables $x = (x_1, x_2, \ldots)$, we have*

$$s_{H_{k,n-k}} = s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})} + \cdots + (-1)^{n-k-2} s_{(2,1^{n-2})} + (-1)^{n-k-1} s_{(1^n)}.$$

Using this result, we can now describe the Schur expansions of general cylindric ribbons.

Proposition 4.4. *Let C by a cylindric ribbon which is a subposet of the cylinder $\mathfrak{C}_{k,n-k}$. Then*

$$s_C(x) = \left(\sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu(x) \right) + s_{H_{k,n-k}}(x),$$

with c_ν a non-negative integer for all $\nu \subseteq k \times (n-k)$.

Let C be a cylindric skew shape that is not isomorphic to a skew shape. We know from Theorem 3.2 that s_C in k variables is Schur-positive. On the other hand, by Theorem 4.1, s_C in an infinite number of variables is not Schur-positive. We conclude this section with a discussion of the minimum number of variables in which s_C fails to be Schur-positive.

If C is a cylindric ribbon, we deduce from Proposition 4.4 and Lemma 4.3 that s_C remains Schur-positive in $k+1$ variables but always fails to be Schur-positive in $k+2$ variables. By looking at coproducts, we can use this fact to say something about a general cylindric skew shape C. Let $C[-1]$ denote the cylindric skew shape that results when we remove the cylindric ribbon that runs all the way along the top of C.

Proposition 4.5. *Let C be a cylindric skew shape that is not isomorphic to a skew shape and that is a subposet of $\mathfrak{C}_{k,n-k}$. If m denotes the maximum number of elements in a column of $C[-1]$, then s_C is not Schur-positive in $m+k+2$ variables.*

We do not claim, and it is not true, that $m+k+2$ is the best possible value. In other words, it can be the case that s_C is not Schur-positive in some number of variables that is less than $m+k+2$. For toric shapes, it is clear that $m \leq k-1$, and so we get the following result.

Corollary 4.6. *Let C be a toric shape that is not isomorphic to a skew shape and that is a subposet of $\mathfrak{C}_{k,n-k}$. Then s_C is not Schur-positive in $2k+1$ variables.*

5. From cylindric skew shapes to skew shapes

So far, we have not discussed any tools for dealing with cylindric skew Schur functions. The subject of this section is a rule for expressing any cylindric skew Schur function as a signed sum of skew Schur functions. Our rule is based on a result of Gessel and Krattenthaler from [5], with our reformulation modelled on a result from [2]. We begin with an exposition of these two results, starting with the latter.

By saying that a partition τ is obtained from λ by adding d n-ribbons, we mean that there is a sequence of partitions

$$\lambda = \nu_0 \subseteq \nu_1 \subseteq \cdots \subseteq \nu_d = \tau \tag{5.1}$$

such that ν_i/ν_{i-1} is an n-ribbon for $i = 1, \ldots, d$. We say that the *width* of a ribbon is its number of non-empty columns. If $\tau_1 \leq n-k$, then we define

$$\varepsilon(\tau/\lambda) = (-1)^{\sum_{i=1}^{d}(n-k-\mathrm{width}(\nu_i/\nu_{i-1}))}.$$

It can be shown that $\varepsilon(\tau/\lambda)$ is independent of the choice of the sequence in (5.1).

The result of interest from [2] is the following:

Theorem 5.1. *Suppose we have $\lambda, \mu, \nu \subseteq k \times (n-k)$ with $|\mu| + |\nu| = |\lambda| + dn$ for some $d \geq 0$. Then the Gromov-Witten invariant $C_{\mu\nu}^{\lambda,d}$ can be expressed in terms of Littlewood-Richardson coefficients as*

$$C_{\mu\nu}^{\lambda,d} = \sum_{\tau} \varepsilon(\tau/\lambda) c_{\mu\nu}^{\tau}, \tag{5.2}$$

where the sum is over all τ with $\tau_1 \leq n-k$ that can be obtained from λ by adding d n-ribbons.

Formulas for $C_{\mu\nu}^{\lambda,d}$ similar to (5.2) have appeared in different contexts in [4, 6, 7, 16]. Multiplying both sides of (5.2) by $s_\nu(x_1, \ldots, x_k)$, summing over all $\nu \subseteq k \times (n-k)$, and applying Theorem 3.2, we get:

Corollary 5.2. *For any cylindric skew shape $\lambda/d/\mu$ with $\lambda, \mu \subseteq k \times (n-k)$, we have*

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\tau} \varepsilon(\tau/\lambda) s_{\tau/\mu}(x_1, \ldots, x_k), \tag{5.3}$$

where the sum is over all τ with $\tau_1 \leq n-k$ that can be obtained from λ by adding d n-ribbons.

From our point of view, the obvious disadvantage of Corollary 5.2 is that it only gives certain terms in the expansion of $s_{\lambda/d/\mu}(x)$. For example, for cylindric shapes that are not toric, both sides of (5.3) will be zero. Gessel and Krattenthaler's setting does not have this limitation. To apply their result to get an expression for $s_{\lambda/d/\mu}(x)$, we first have some work to do. Their basic result, Proposition 1, is stated in terms of lattice paths. In their Section 9, they show how to apply Proposition 1 to obtain expressions for Schur functions. Mimicking their approach, we first obtain an expression for $s_{\lambda/d/\mu}$ in terms of the elementary symmetric functions. Recall from page 5 that, for a given $\lambda/d/\mu$, Λ is the

r	$\Lambda' + rn$	resulting partition	sign
$(0,0,0,0)$	$(7,5,4,4)$	$(7,5,4,4)$	$+$
$(-1,0,0,1)$	$(0,5,4,11)$	$(8,5,4,3)$	$-$
$(-1,0,1,0)$	$(0,5,11,4)$	$(9,5,3,3)$	$+$
$(-1,1,0,0)$	$(0,12,4,4)$	$(11,3,3,3)$	$-$
$(0,-1,0,1)$	$(7,-2,4,11)$	$(8,8,4,0)$	$+$
$(0,-1,1,0)$	$(7,-2,11,4)$	$(9,8,3,0)$	$-$
$(1,-1,0,0)$	$(14,-2,4,4)$	$(14,3,3,0)$	$+$
$(-1,-1,1,1)$	$(0,-2,11,11)$	$(9,9,2,0)$	$+$
$(-1,-1,0,2)$	$(0,-2,4,18)$	$(15,3,2,0)$	$-$
$(-1,-1,2,0)$	$(0,-2,18,4)$	$(16,2,2,0)$	$+$

TABLE 1. Applying Theorem 5.3

unique partition satisfying $\Lambda[-d] = \lambda$. In this case, we also write $\lambda[d] = \Lambda$ and we see that Λ is obtained from λ by adding d n-ribbons, each starting in $\lambda's$ rightmost column (column $n-k$) and ending in column 1. We get that

$$s_{\lambda/d/\mu}(x) = \sum_{\substack{r_1+\cdots+r_{n-k}=0 \\ r_i\in\mathbb{Z}}} \det\left(e_{r_s n+\Lambda'_s-\mu'_t-s+t}(x)\right)_{s,t=1}^{n-k}, \tag{5.4}$$

where, as usual, we set $e_0 = 1$ and $e_i = 0$ for $i < 0$. We wish to simplify this expression using the dual Jacobi-Trudi identity and, to do so, we must introduce the modification rule

$$s_{(\tau_1,\ldots,\tau_i,\tau_{i+1},\ldots,\tau_{n-k})'/\mu}(x) = -s_{(\tau_1,\ldots,\tau_{i-1},\tau_{i+1}-1,\tau_i+1,\tau_{i+2},\ldots,\tau_{n-k})'/\mu}(x), \tag{5.5}$$

which allows us to interpret $s_{\tau'/\mu}$ when $\tau = (\tau_1,\ldots,\tau_{n-k})$ is not necessarily a partition containing μ. For example,

$$s_{(7,-2,11,4)'/(2,1)} = -s_{(7,10,-1,4)'/(2,1)} = s_{(7,10,3,0)'/(2,1)} = -s_{(9,8,3,0)'/(2,1)}.$$

On the other hand,

$$s_{(-7,5,4,18)'/(2,1)} = 0,$$

since no number of applications of the rule (5.5) will result in a skew Schur function. We can further simplify (5.4) by letting $\Lambda' + rn$ denote the integer sequence $(\Lambda'_1 + r_1 n, \ldots, \Lambda'_{n-k} + r_{n-k} n)$.

Putting this all together, (5.4) becomes

Theorem 5.3. [5] *For any cylindric shape $\lambda/d/\mu$ that is a subposet of $\mathfrak{C}_{k,n-k}$, we have*

$$s_{\lambda/d/\mu}(x) = \sum_{\substack{r_1+\cdots+r_{n-k}=0 \\ r_i\in\mathbb{Z}}} s_{(\Lambda'+rn)'/\mu}(x) \tag{5.6}$$

where $\Lambda = \lambda[d]$ and the right-hand side is interpreted in terms of the modification rule (5.5).

Example 5.4. Consider $\lambda/d/\mu = (3,3)/2/(2,1)$ as depicted in Figure 3. We see that $n = 7$, $n-k = 4$, $\Lambda' = (7,5,4,4)$ and $\mu = (2,1,0,0)$. The values of $r = (r_1,\ldots,r_{n-k})$ that make $s_{(\Lambda'+rn)'/\mu} \neq 0$ are listed in the first column of Table 1. We conclude that

$$\begin{aligned} s_{(3,3)/2/(2,1)}(x) &= s_{(7,5,4,4)'/(2,1)}(x) - s_{(8,5,4,3)'/(2,1)}(x) + s_{(9,5,3,3)'/(2,1)}(x) \\ &\quad - s_{(11,3,3,3)'/(2,1)}(x) + s_{(8,8,4,0)'/(2,1)}(x) - s_{(9,8,3,0)'(2,1)}(x) \\ &\quad + s_{(14,3,3,0)'/(2,1)}(x) + s_{(9,9,2,0)'/(2,1)}(x) - s_{(15,3,2,0)'/(2,1)}(x) \\ &\quad + s_{(16,2,2,0)'/(2,1)}(x). \end{aligned}$$

Using Theorem 5.3, we can actually show that Corollary 5.2 extends to the case of infinitely many variables. This is the main result of this section.

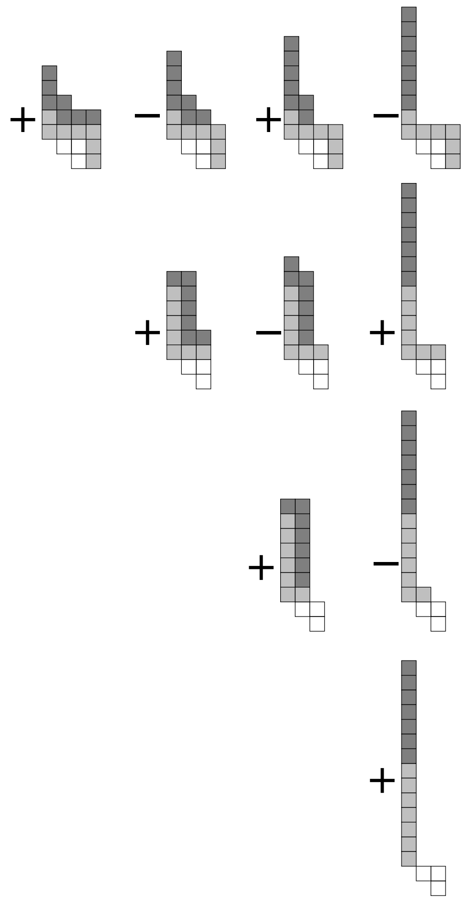


FIGURE 5.

Theorem 5.5. *For any cylindric skew shape $\lambda/d/\mu$ that is a subposet of $\mathfrak{C}_{k,n-k}$, we have*

$$s_{\lambda/d/\mu}(x) = \sum_{\tau} \varepsilon(\tau/\lambda) s_{\tau/\mu}(x), \tag{5.7}$$

where the sum is over all τ with $\tau_1 \leq n-k$ that can be obtained from λ by adding d n-ribbons.

Note 5.6. While $\lambda \subseteq k \times (n-k)$ by definition, we do not require that $l(\mu) \leq k$, unlike in Theorem 5.1 and Corollary 5.2.

It does not seem that the proof of Theorem 5.1 from [2] can be easily modified to prove Theorem 5.5, and our proof of Theorem 5.5 is somewhat technical.

Example 5.7. Again, consider $\lambda/d/\mu = (3,3)/2/(2,1)$ as depicted in Figure 3. Figure 5 shows the set of all possible $\varepsilon(\tau/\lambda)\tau/\mu$ with $\tau_1 \leq n-k$ such that τ can be obtained from $(3,3)$ by adding 2 7-ribbons. The positioning of the partitions in the figure is supposed to be helpful, as it is determined by the rightmost column of the added ribbons. There can be more than one way to add ribbons to λ and get a particular τ, but this does not affect our expression for $s_{\lambda/d/\mu}$.

We see that we get the same result as in Example 5.4. While the result obtained from Theorem 5.3 is more compact to write, we find the graphical description of $s_{\lambda/d/\mu}$ in Theorem 5.5 preferable, especially from the point of view of intuition. We make much use of Theorem 5.5 in proving the results of the next section.

Remark 5.8. Because the expression of a cylindric skew shape C in the form $\lambda/d/\mu$ is not unique, Theorem 5.5 can be used to give a host of identities among skew Schur functions. For example, consider

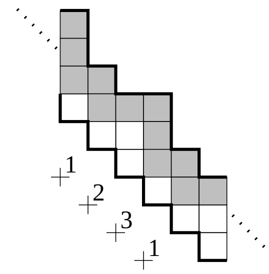


FIGURE 6.

the cylindric skew shape C shown in Figure 6 with $k = n - k = 3$. By choosing the origins labelled 1, 2 and 3 respectively, we see that C can be written as $(3,3,1)/1/(2,1)$, $(3,2,2)/1/(2,1)$ or $(1)/2/(2,1)$. It follows that

$$\begin{aligned} s_C(x) &= s_{333211/21} - s_{3322111/21} + s_{331111111/21} \\ &= s_{33331/21} - s_{32221111/21} + s_{322111111/21} \\ &= s_{33322/21} - s_{3222211/21} + s_{3211111111/21} + s_{2222221/21} - s_{22111111111/21}. \end{aligned}$$

6. Cylindric Schur-positivity

Before presenting the conjecture which is the main subject of this section, we begin with a relevant application of Theorem 5.5.

In the same way that Schur functions are those skew Schur functions $s_{\lambda/\mu}(x)$ with $\mu = \emptyset$, we will say that *cylindric Schur functions* are those cylindric skew Schur functions $s_{\lambda/d/\mu}(x)$ with $\mu = \emptyset$. While the Schur functions are known to be a basis for the symmetric functions, we have the following result for the cylindric Schur functions.

Proposition 6.1. *For a given $k, n-k$, the cylindric Schur functions of the form $s_{\lambda/d/\emptyset}(x)$, with $\lambda/d/\emptyset$ a subposet of $\mathfrak{C}_{k,n-k}$, are linearly independent.*

We might next ask if every cylindric skew Schur function $s_{\lambda/d/\mu}(x)$ with $\lambda/d/\mu$ a subposet of $\mathfrak{C}_{k,n-k}$ can be expressed as a linear combination of cylindric Schur functions of the form $s_{\nu/e/\emptyset}(x)$, where each $\nu/e/\emptyset$ is also a subposet of $\mathfrak{C}_{k,n-k}$. As we shall see, an affirmative answer to this question would also imply Conjecture 6.3 below.

Definition 6.2. Suppose $\lambda/d/\mu$ is a cylindric skew shape that is a subposet of $\mathfrak{C}_{k,n-k}$. We say that $s_{\lambda/d/\mu}(x)$ in the variables $x = (x_1, x_2, \ldots)$ is *cylindric Schur-positive* if it can be expressed as a linear combination of cylindric Schur functions $s_{\nu/e/\emptyset}(x)$ with positive coefficients, where each such $\nu/e/\emptyset$ is also a subposet of $\mathfrak{C}_{k,n-k}$.

As an analogue of the fact that every skew Schur function is Schur-positive, we propose the following conjecture.

Conjecture 6.3. *Every cylindric skew Schur function is cylindric Schur-positive.*

Proposition 4.4 implies this conjecture is true for cylindric ribbons. The rest of this section will be devoted to other evidence in favour of the conjecture.

It follows from (3.2) that we can split $s_{\lambda/d/\mu}(x)$ into two sums as follows:

$$s_{\lambda/d/\mu}(x) = \sum_{\nu \subseteq k \times (n-k)} a_\nu s_\nu(x) + \sum_{\substack{\nu : \nu_1 \leq n-k \\ l(\nu) > k}} b_\nu s_\nu(x). \tag{6.1}$$

When $\nu \subseteq k \times (n-k)$, we know that $s_\nu(x)$ is a cylindric Schur function. Furthermore, we know from Theorem 3.2 that $a_\nu \geq 0$ for all $\nu \subseteq k \times (n-k)$. Therefore, the first sum is cylindric Schur-positive.

Now consider the second sum, which we denote by $B(\lambda/d/\mu, x)$. We know that $s_{\lambda/d/\mu}(x)$ is cylindric Schur-positive when $d = 0$. Therefore, we can assume by induction that $s_{\lambda/(d-1)/\mu}(x)$ is cylindric Schur-positive:

$$s_{\lambda/(d-1)/\mu}(x) = \sum_{\substack{\nu,e \\ \nu\subseteq k\times(n-k)}} c_{\nu,e} s_{\nu/e/\emptyset}(x), \tag{6.2}$$

where $c_{\nu,e} \geq 0$ for all ν, e, and e is a always non-negative integer. (For $s_{\nu/e/\emptyset}(x) \neq 0$, we require that $ne = |\lambda| - |\mu| + n(d-1) - |\nu|$.) We conjecture, in fact, that $B(\lambda/d/\mu, x)$ can be expressed exactly in terms of $s_{\lambda/(d-1)/\mu}(x)$ as:

$$B(\lambda/d/\mu, x) = \sum_{\substack{\nu,e \\ \nu\subseteq k\times(n-k)}} c_{\nu,e} s_{\nu/e+1/\emptyset}(x).$$

Plugging this into (6.1), we get

$$s_{\lambda/d/\mu}(x) = \sum_{\nu\subseteq k\times(n-k)} a_\nu s_\nu(x) + \sum_{\substack{\nu,e \\ \nu\subseteq k\times(n-k)}} c_{\nu,e} s_{\nu/e+1/\emptyset}(x), \tag{6.3}$$

where $a_\nu, c_{\nu,e} \geq 0$ for all ν, e. This expression is a strong refinement of Conjecture 6.3 as it gives much information about the form of the cylindric Schur-positive expansion of $s_{\lambda/d/\mu}(x)$. Using [3, 15], we have verified (6.3) for all $\lambda/d/\mu$ with $k, n-k, d \leq 5$.

As promised, we can also reformulate Conjecture 6.3 into a seemingly easier statement.

Theorem 6.4. *Conjecture 6.3 holds if and only if every cylindric skew Schur function $s_{\lambda/d/\mu}(x)$ with $\lambda/d/\mu$ a subposet of $\mathfrak{C}_{k,n-k}$ can be expressed as a linear combination of cylindric Schur functions $s_{\nu/e/\emptyset}(x)$, where each $\nu/e/\emptyset$ is also a subposet of $\mathfrak{C}_{k,n-k}$.*

In other words, to prove Conjecture 6.3, we don't have to show that the coefficients are positive.

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