KAZHDAN-LUSZTIG IMMANANTS AND PRODUCTS OF MATRIX MINORS

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ABSTRACT. We define a family of polynomials of the form $\sum f(\sigma)x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ in terms of the Kazhdan-Lusztig basis $\{C'_w(1)\,|\,w \in S_n\}$ for the symmetric group algebra $\mathbb{C}[S_n]$. Using this family, we obtain nonnegativity properties of polynomials of the form $\sum c_{I,I'}\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$. In particular, we show that the application of certain of these polynomials to Jacobi-Trudi matrices yields symmetric functions which are nonnegative linear combinations of Schur functions.

RÉSUMÉ. Nous definissons une famille de polynômes $\sum f(\sigma)x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ en terme de la base $\{C'_w(q)\,|\,w \in S_n\}$ de Kazhdan-Lusztig pour l'algébra $\mathbb{C}[S_n]$. En utilisant cette famille, nous obtenons quelques propriétées des polynômes totalement non négatifs de la forme $\sum c_{I,I'}\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$. En particulier, nous démontrons que l'application des certains de ces polynômes aux matrices de Jacobi-Trudi rapporte des fonctions symmétriques qui sont des combinaisons linéaires non négatives des fonctions de Schur.

1. INTRODUCTION

Since its introduction in [14], the Kazhdan-Lusztig basis $\{C'_w(q)\,|\,w \in S_n\}$ of the Hecke algebra $H_n(q)$ has found many applications related to algebraic geometry, combinatorics, and Lie theory. One such application, due to Haiman [12], clarifies three nonnegativity properties of certain polynomials which arise in the representation theory of $H_n(q)$. Years later, two of these nonnegativity properties were observed in a family of polynomials which arise in the study of inequalities satisfied by minors of totally nonnegative matrices [4, 18]. Building upon the arguments of Haiman [12], we will show that this family posesses the third nonnegativity property as well.

The nonnegativity properties are as follows. Let $x = (x_{ij})$ be a generic square matrix. For each pair (I, I') of subsets of $[n] = \{1, \dots, n\}$, define $\Delta_{I,I'}(x)$ to be the (I, I') minor of x, i.e., the determinant of the submatrix of x corresponding to rows I and columns I'. A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative. A polynomial $p(x) = p(x_{1,1}, \dots, x_{n,n})$ in n^2 variables is called

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totally nonnegative if for every TNN matrix A, the number

$$p(A) \underset{def}{=} p(a_{1,1}, \ldots, a_{n,n})$$

is nonnegative. Much current work in total nonnegativity is motivated by problems in quantum Lie theory. (See e.g. [7, 15, 28].) The strong connection between total nonnegativity and Jacobi-Trudi matrices leads to more nonnegativity properties. (See [17, 20] for information on Jacobi-Trudi matrices, and [8] for connections to total nonnegativity.) We will call the polynomial $p(x)$ *Schur nonnegative* (SNN) if for every $n \times n$ Jacobi-Trudi matrix A, the symmetric function $p(A)$ is equal to a nonnegative linear combination of Schur functions. We will also call such a symmetric function Schur nonnegative. Much current work in Schur nonnegativity is motivated by problems concerning the cohomology ring of the Grassmannian variety. (See e.g. [2, 6].) In analogy to Schur nonnegativity, we will call $p(x)$ *monomial nonnegative* (MNN) if for every $n \times n$ Jacobi-Trudi matrix A, $p(A)$ is equal to a nonnegative linear combination of monomial symmetric functions. We will also call such a symmetric function monomial nonnegative. Since each Schur function is itself monomial nonnegative, any SNN polynomial must also be MNN.

Some nontrivial classes of polynomials possessing the TNN, SNN and MNN properties are contained in the complex span of the monomials $\{x_{1,w(1)} \cdots x_{n,w(n)} \,|\, w \in S_n\}$. We will call such polynomials *immanants*. In particular, for every function $f : S_n \to \mathbb{C}$ we define the *f-immanant* (as in [21, Sec. 3]) by

$$\mathrm{Imm}_f(x) \underset{\mathrm{def}}{=} \sum_{w \in S_n} f(w) x_{1,w(1)} \cdots x_{n,w(n)}.$$

Some familiar immanants are those of the form $\mathrm{Imm}_{\chi^\lambda}(x)$, where χ^λ is an irreducible character of S_n. Goulden and Jackson conjectured [10] and Greene proved [11] these immanants to be MNN. Stembridge then conjectured [26] these immanants to be TNN and SNN, and he [23] and Haiman [12] proved these two conjectures. (See [12, 13, 22, 23, 24] for related conjectures and results.) Other immanants of the form

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \tag{1.1}$$

characterize the inequalities satisfied by products of two minors of TNN matrices. (Equivalently, these characterize the inequalities satisfied by products of two entries of the exterior algebra representation of TNN elements of $GL_n(\mathbb{C})$.) Fallat, Gekhtman and Johnson characterized [4] the TNN immanants of the form (1.1), in the principal minor case ($I = I'$, etc.) A characterization of the general case followed in [16, 18], as did a proof that all such TNN immanants are MNN.

In Section 2 we define more immanants in terms of the Kazhdan-Lusztig basis of $\mathbb{C}[S_n]$. We then use the Schur nonnegativity of these Kazhdan-Lusztig immanants in Section 3 to prove the Schur nonnegativity of all TNN immanants of the form

(1.1). More properties of the Kazhdan-Lusztig immanants and open problems follow in Section 4.

2. Kazhdan-Lusztig immanants

Let q be a formal parameter and define the *Hecke algebra* $H_n(q)$ to be the $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra generated by elements $T_{s_1}, \ldots, T_{s_{n-1}}$, subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned}$$

For each permutation w we define the Hecke algebra element T_w by

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}.$$

where $s_{i_1} \cdots s_{i_\ell}$ is any reduced expression for w. Specializing at $q = 1$ gives the symmetric group algebra $\mathbb{C}[S_n]$.

The elements $\{C'_v(q) \mid v \in S_n\}$ of the Kazhdan-Lusztig basis of $H_n(q)$ have the form

$$C'_v(q) = \sum_{u \leq v} P_{u,v}(q) q^{-\ell(v)/2} T_u, \tag{2.1}$$

where the comparison of permutations is in the Bruhat order, and

$$\{P_{u,v}(q) \mid u, v \in S_n\}$$

are certain polynomials in q, known as the *Kazhdan-Lusztig polynomials* [14]. Solving the equations (2.1) for T_v, we have

$$T_v = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0v,w_0u}(q) q^{\ell(u)/2} C'_u(q),$$

where w_0 is the longest permutation in S_n.

For each permutation v in S_n define the function $f_v : S_n \to \mathbb{C}$ by

$$f_v(w) = (-1)^{\ell(w)-\ell(v)} P_{w_0w,w_0v}(1).$$

Extending these functions linearly to $\mathbb{C}[S_n]$, we see that they are dual to the Kazhdan-Lusztig basis in the sense that

$$f_v(C'_w(1)) = \delta_{v,w}.$$

We will denote the f_v-immanant by

$$\mathrm{Imm}_v(x) \underset{\mathrm{def}}{=} \sum_{w \geq v} f_v(w) x_{1,w(1)} \cdots x_{n,w(n)},$$

and will call these immanants the *Kazhdan-Lusztig immanants*. In the case that v is the identity permutation, we obtain the determinant.

Results in [12, 23] imply that the Kazhdan-Lusztig immanants are TNN and SNN. To give brief proofs, we shall consider the following elements of $H_n(q)$. Given indices $1 \leq i \leq j \leq n$, define $z_{[i,j]}$ to be the element of $H_n(q)$ which is the sum of elements T_w corresponding to permutations w in the subgroup of S_n generated by $s_i, \ldots, s_{j-1}$.

Proposition 2.1. *Let z be an element of $H_n(q)$ of the form*

$$z = z_{[i_1,j_1]} \cdots z_{[i_r,j_r]}. \tag{2.2}$$

Then we have

$$z = \sum_{w \in S_n} p_{z,w}(q) C'_w(q),$$

where the expressions $p_{z,w}(q)$ are Laurent polynomials in $q^{1/2}$ with nonnegative coefficients. In particular, an element of the form (2.2) in $\mathbb{C}[S_n]$ is equal to a nonnegative linear combination of the Kazhdan-Lusztig basis elements $\{C'_w(1) \,|\, w \in S_n\}$.

Proof. Let $s_{[i,j]}$ be the longest permutation in the subgroup generated by $s_i, \ldots s_{j-1}$. By [12, Prop. 3.1], we have

$$z_{[i,j]} = q^{\ell(w)/2} C'_{s_{[i,j]}}(q).$$

A result of Springer [19] implies that for every pair (u, v) of permutations in S_n, we have

$$C'_u(q) C'_v(q) = \sum_{w \in S_n} f^w_{u,v}(q) C'_w(q),$$

where the expressions $f^w_{u,v}(q)$ are Laurent polynomials in $q^{1/2}$ with nonnegative coefficients. (See also [12, Appendix].) $\square$

Proposition 2.2. *For each permutation w in S_n, the Kazhdan-Lusztig immanant* $\mathrm{Imm}_w(x)$ *is totally nonnegative.*

Proof. For any complex matrix A and any function $f : S_n \to \mathbb{C}$ we have

$$\mathrm{Imm}_f(A) = \sum_z c_z f(z),$$

where the sum is over elements z of $\mathbb{C}[S_n]$ of the form (2.2), and the coefficients c_z depend on A. If A is a totally nonnegative matrix, then these coefficients are real and nonnegative. (See, e.g., [16, Lem. 2.5], [23, Thm. 2.1].)

Let A be a TNN matrix. By Propositiion 2.1 we have

$$\begin{aligned}\mathrm{Imm}_w(A) &= \sum_z c_z f_w(z) \\ &= \sum_z c_z \sum_v p_{z,v}(1) f_w(C'_v(1)) \\ &= \sum_z c_z p_{z,w}(1) \\ &\geq 0.\end{aligned}$$

□

The following easy consequence of [12, Thm. 1.5] implies the Schur nonnegativity of the Kazhdan-Lusztig immanants. Following [12], we define a *generalized Jacobi-Trudi matrix* to be a finite matrix whose i, j entry is the homogeneous symmetric function $h_{\mu_i - \nu_i}$, where $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ are weakly decreasing nonnegative sequences, and by convention $h_m = 0$ if m is negative. Thus each generalized Jacobi-Trudi matrix is constructed from an ordinary Jacobi-Trudi matrix by repeating some rows and/or columns.

Proposition 2.3. *For each permutation w in S_n, and each $n \times n$ generalized Jacobi-Trudi matrix A, the symmetric function* $\mathrm{Imm}_w(A)$ *is Schur nonnegative.*

Proof. By [12, Thm. 1.5], we have

$$\sum_{v \in S_n} a_{1,v(1)} \cdots a_{n,v(n)} v = \sum_u g_{v,u}(A) C'_u(1),$$

where $g_{v,u}(A)$ is a Schur nonnegative symmetric function which depends upon A. Applying the function f_w to both sides of this equations, we have

$$\begin{aligned}\mathrm{Imm}_w(A) &= \sum_u g_{w,u}(A) f_w(C'_u(1)) \\ &= g_{w,w}(A).\end{aligned}$$

□

3. Main Results

Studying inequalities satisfied by products of principal minors of TNN matrices, Fallat, Gekhtman and Johnson [4, Thm. 4.6] characterized all TNN immanants of the form

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x),$$

(where $\overline{I} = [n] \smallsetminus I$, $\overline{J} = [n] \smallsetminus J$) and more generally, all TNN polynomials of the form

$$\Delta_{J,J}(x)\Delta_{L,L}(x) - \Delta_{I,I}(x)\Delta_{K,K}(x).$$

This result was generalized in [18, Thm. 3.2] as follows.

Proposition 3.1. *Let I, J, K, L be subsets of $[n]$ and let I', J', K', L' be subsets of $[n']$, and define the subsets I'', J'', K'', L'' of $[n+n']$ by*

$$\begin{aligned} I'' &= I \cup \{n+n'+1-i \,|\, i \in K'\}, \\ K'' &= K \cup \{n+n'+1-i \,|\, i \in I'\}, \\ J'' &= J \cup \{n+n'+1-i \,|\, i \in L'\}, \\ L'' &= L \cup \{n+n'+1-i \,|\, i \in J'\}. \end{aligned} \tag{3.1}$$

Then the polynomial

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) \tag{3.2}$$

is totally nonnegative if and only if the sets $I, \dots, L$, $I', \dots, L'$ satisfy

$$\begin{aligned} I \cup K = J \cup L, \qquad I' \cup K' = J' \cup L', \\ I \cap K = J \cap L, \qquad I' \cap K' = J' \cap L', \end{aligned} \tag{3.3}$$

and for each subinterval B of $[n+n']$ the sets $I'', \dots, K''$ satisfy

$$\max\{|B \cap J''|, |B \cap L''|\} \leq \max\{|B \cap I''|, |B \cap K''|\}. \tag{3.4}$$

The proof in [18] shows that these polynomials are MNN as well. (See [16, Cor. 6.1].) Two combinatorial alternatives to the system of inequalities (3.4) are given in [16, Thms. 5.2, 5.4]. The second of these proves the total nonnegativity of the polynomials (3.2) by relating them to TNN immanants defined in terms of the Temperley-Lieb algebra.

Given a formal parameter ξ, we define the *Temperley-Lieb algebra* $TL_n(\xi)$ to be the $\mathbb{C}[\xi]$-algebra generated by elements $t_1, \dots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \xi t_i, && \text{for } i = 1, \dots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

The rank of $TL_n(\xi)$ as a $\mathbb{C}[\xi]$-module is well-known to be $\frac{1}{n+1}\binom{2n}{n}$, and a natural basis is given by the elements of the form $t_{i_1} \cdots t_{i_\ell}$, where $i_1 \cdots i_\ell$ is a reduced word for a 321-avoiding permutation in S_n. We shall call these elements the standard basis elements of $TL_n(\xi)$, or simply *the basis elements* of $TL_n(\xi)$.

The Temperley-Lieb algebra may be realized as a quotient of the Hecke algebra by

$$H_n(q)/(z_{[1,3]}) \cong TL_n(q^{1/2} + q^{-1/2}),$$

where the element $z_{[1,3]}$ of $H_n(q)$ is defined as before Proposition 2.1. We will let θ be the homomorphism

$$H_n(q) \to TL_n(q^{1/2} + q^{-1/2})$$
$$q^{-1/2}(T_{s_i} + 1) \mapsto t_i.$$

(See e.g. [5], [9, Sec. 2.1, Sec. 2.11], [27, Sec. 7].)

Immanants called *Temperley-Lieb immanants* in [16] were defined in terms of the homomorphism θ, specialized at $q = 1$. For each basis element τ of $TL_n(2)$, let $f_\tau : S_n \to \mathbb{R}$ be the function defined by

$$f_\tau(v) = \text{ coefficient of } \tau \text{ in } \theta(T_v),$$

and let

$$\mathrm{Imm}_\tau(x) = \sum_{w \in S_n} f_\tau(w) x_{1,w(1)} \cdots x_{n,w(n)}$$

be the corresponding immanant. By [16, Thm. 3.1], the Temperley-Lieb immanants are TNN. Furthermore, the following result shows that the Temperley-Lieb immanants are Kazhdan-Lusztig immanants. To prove this, we define for each 321-avoiding permutation w in S_n an element $D_w(q)$ of $H_n(q)$ as follows. For any reduced word $i_1 \cdots i_\ell$ for w, define

$$D_w(q) \underset{\mathrm{def}}{=} q^{-1/\ell}(T_{s_{i_1}} + 1) \cdots (T_{s_{i_\ell}} + 1).$$

(This element does not depend upon the particular reduced word.) The element $D_w(q)$ satisfies

$$\theta(D_w(q)) = t_{i_1} \cdots t_{i_\ell},$$

and it follows that the set

$$\{\theta(D_w(q)) \,|\, w \text{ a 321-avoiding permutation } \}$$

is equal to the standard basis of $TL_n(q^{1/2} + q^{-1/2})$.

Proposition 3.2. *Let w be any 321-avoiding permutation in S_n, and define $\tau = \theta(D_w(1))$. Then the Temperley-Lieb immanant $\mathrm{Imm}_\tau(x)$ is equal to the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$.*

Proof. Let v be any permutation in S_n. Then we have

$$v = \sum_{u \le v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) C'_u(1).$$

The coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in $\mathrm{Imm}_\tau(x)$ is equal to $f_\tau(v)$, which is the coefficient of τ in

$$\theta(v) = \sum_{u \le v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) \theta(C'_u(1)). \tag{3.5}$$

A result of Fan and Green [5, Thm. 3.8.2] implies that we have

$$\theta(C'_w(q)) = \begin{cases} \theta(D_w(q)) & \text{if } w \text{ is 321-avoiding,} \\ 0 & \text{otherwise.} \end{cases}$$

(See also [3, Thm. 4].) We may therefore assume that each permutation u appearing in (3.5) is 321-avoiding, and we may rewrite the sum as

$$\theta(v) = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0v,w_0u}(1)\theta(D_u(1)).$$

The coefficient of $\tau = \theta(D_w(1))$ in this expression is $(-1)^{\ell(v)-\ell(w)} P_{w_0v,w_0u}(1)$. But this is precisely the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in $\mathrm{Imm}_w(x)$. □

Thus the Temperley-Lieb immanants are precisely the Kazhdan-Lusztig immanants corresponding to 321-avoiding permutations. From the Schur nonnegativity of the Kazhdan-Lusztig immanants, it then follows that all TNN polynomials of the form (3.2) are SNN.

Theorem 3.3. *Let I, J, K, L be subsets of $[n]$, let I', J', K', L' be subsets of $[n']$, and suppose that these satisfy the conditions of Proposition 3.1. Then the polynomial*

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) \tag{3.6}$$

is Schur nonnegative.

Proof. Define $r = |I|+|K|$, and let $k_1 \leq \cdots \leq k_r$ be the nondecreasing rearrangement of the elements of I and K, including repeated elements. Define $k'_1, \ldots, k'_r$ analogously, and let y be the $r \times r$ matrix whose i, j entry is the variable x_{k_i,k'_j}. Thus y is the matrix obtained from x by duplicating rows whose indices belong to $I \cap K$ and columns whose indices belong to $I' \cap K'$.

By Proposition 3.1, the polynomial (3.6) is TNN, and by [16, Prop. 5.3, Thm. 5.4] we have

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{\tau} \mathrm{Imm}_\tau(y),$$

where the sum is over a subset of basis elements of $TL_r(2)$. By Proposition 3.2 this is a sum of Kazhdan-Lustig immanants,

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{w} \mathrm{Imm}_w(y), \tag{3.7}$$

for an appropriate set of 321-avoiding permutations w in S_r.

Now let A be an arbitrary $n \times n'$ Jacobi-Trudi matrix, and let B be the generalized Jacobi-Trudi matrix whose i, j entry is a_{k_i,k'_j}. Then the evaluation of the left-hand side of (3.7) at $x = A$ is equal to the evaluation of the right-hand side at $y = B$.

By Proposition 2.3, the resulting symmetric function on the right-hand side is SNN. Thus the polynomial $\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$ is SNN. □

Theorem 3.3 provides new machinery for proving that certain symmetric functions of the form $s_{\alpha/\kappa}s_{\beta/\lambda} - s_{\gamma/\mu}s_{\delta/\nu}$ are SNN. For example, the combinatorial test in [16, Thm. 4.2] makes it easy to see that for

$$J = \{ i \in [n] \,|\, i \text{ odd } \}$$

and for any subsets I, I' of $[n]$, the immanant

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$$

is SNN. Choosing $n = 6$ and $I = \{1, 3, 4\}$, $I' = \{1, 2, 4\}$, we may apply this immanant,

$$\Delta_{135,135}(x)\Delta_{246,246}(x) - \Delta_{134,124}(x)\Delta_{256,356}(x)$$

to the Jacobi-Trudi matrix indexed by the skew shape 766655/22211,

$$\begin{bmatrix} h_5 & h_6 & h_7 & h_9 & h_{10} & h_{12} \\ h_3 & h_4 & h_5 & h_7 & h_8 & h_{10} \\ h_2 & h_3 & h_4 & h_6 & h_7 & h_9 \\ h_1 & h_2 & h_3 & h_5 & h_6 & h_8 \\ 0 & 1 & h_1 & h_3 & h_4 & h_6 \\ 0 & 0 & 1 & h_2 & h_3 & h_5 \end{bmatrix},$$

to see the Schur nonnegativity of the symmetric function

$$s_{864/32}s_{875/42} - s_{755/22}s_{855/31}.$$

4. Open Questions

The Littlewood-Richardson coefficients, defined by

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu,$$

and the inequalities satisfied by these coefficients have have interesting interpretations in algebraic geometry and representation theory. (See, e.g., [1, 6, 25].) A basic open question about these inequalities may be stated as follows.

Question 4.1. For what conditions on partitions α, β, γ, δ is the symmetric function $s_\alpha s_\beta - s_\gamma s_\delta$ Schur nonnegative? Equivalently, what conditions on these four partitions imply that $c^\nu_{\alpha,\beta} \geq c^\nu_{\gamma,\delta}$ for all ν?

Some conjectured sufficient conditions are given by Fomin, Fulton, Li and Poon [6, Conj. 2.8, Conj. 5.1]. Generalizing the second of these conjectures, Bergeron, Biagioli and Rosas [2, Conj. 2.9] have conjectured sufficient conditions for Schur nonnegativity of symmetric functions of the form

$$s_{\alpha/\kappa}s_{\beta/\lambda} - s_{\gamma/\mu}s_{\delta/\nu}. \tag{4.1}$$

It would be interesting to determine which of the conjectured sufficient conditions can be derived from Theorem 3.3. On the other hand it would be interesting to find symmetric functions of the form (4.1) for which Schur nonnegativity follows from Theorem 3.3, but not from the conjectured sufficient conditions.

The fact that Theorem 3.3 may be applied to *generalized* Jacobi-Trudi matrices highlights an important difference between the determinant and other Kazhdan-Lusztig immanants. Specifically, Kazhdan-Lusztig immanants do not vanish on a matrix having a pair of equal rows. It was shown in [16, Prop. 3.14] that Temperley-Lieb immanants vanish on matrices having *three* equal rows. This fact generalizes nicely to arbitrary Kazhdan-Lusztig immanants.

Proposition 4.1. *Let w be a permutation in S_n and suppose that the one-line notation $w(1)\cdots w(n)$ contains no decreasing subsequence of length k. Then* $\mathrm{Imm}_w(x)$ *vanishes on any $n \times n$ matrix having k equal rows or columns.*

Proof. Omitted. □

It would be interesting to generalize other determinantal formulas and identities to Kazhdan-Lusztig immanants.

Some work on immanants related to representations of S_n has led to the study of certain elements of $\mathbb{C}[S_n]$ associated to total nonnegativity. Following Stembridge [23], we define the *cone of total nonnegativity* to be the smallest cone in $\mathbb{C}[S_n]$ containing the set

$$\{\sum_w a_{1,w(1)} \cdots a_{n,w(n)} w \,|\, A \text{ TNN } \}.$$

We shall denote this cone by $\mathcal{C}_{TNN}$. (We omit the number n from this notation, although the cone obviously depends upon n.) Dual to $\mathcal{C}_{TNN}$ is the cone of TNN immanants, which we shall denote by $\check{\mathcal{C}}_{TNN}$,

$$\check{\mathcal{C}}_{TNN} = \{\mathrm{Imm}_f(x) \,|\, f(z) \geq 0 \text{ for all } z \in \mathcal{C}_{TNN}\}.$$

No simple description of the extremal rays of these cones is known. However, Stembridge showed [23, Thm. 2.1] that $\mathcal{C}_{TNN}$ is contained in the cone whose extremal rays are elements of $\mathbb{C}[S_n]$ of the form (2.2). Furthermore, Stembridge showed that this containment is proper for $n \geq 4$. We shall denote this third cone by $\mathcal{C}_{INT}$.

Define C_{KL} to be the cone whose extremal rays are the Kazhdan-Lusztig basis elements $\{C'_w(1) \,|\, w \in S_n\}$. It is not difficult to show that $\mathcal{C}_{INT}$ is contained in $\mathcal{C}_{KL}$

and that this containment is proper for $n \geq 4$. Thus we have the proper containment of the dual cones

$$\check{\mathcal{C}}_{KL} \subset \check{\mathcal{C}}_{INT} \subset \check{\mathcal{C}}_{TNN}.$$

For small n, many of the Kazhdan-Lusztig immanants seem to be extremal rays in $\check{\mathcal{C}}_{TNN}$. In the case $n = 4$ we have the following.

Proposition 4.2. *Let a totally nonnegative immanant* $\mathrm{Imm}_f(x)$ *in* V_4 *have coordinates* $\{d_w \,|\, w \in S_4\}$ *with respect to the basis of Kazhdan-Lusztig immanants,*

$$\mathrm{Imm}_f(x) = \sum_{w \in S_4} d_w \mathrm{Imm}_w(x).$$

Then d_w *can be negative only for* $w \in \{3412, 4231\}$.

Proof. Omitted. □

Question 4.2. To restate the previous proposition for arbitrary n, would it suffice to say that d_w can be negative only when the Schubert variety Γ_w in $\mathcal{F}_n$ is not smooth? (i.e. when w avoids the patterns $3412, 4231$?)

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References

[1] H. Barcelo and A. Ram. Combinatorial representation theory. In *New Perspectives in Algebraic Combinatorics (Berkeley, CA 1996-97)* (G. C. Rota, ed.), vol. 38 of *Math. Sci. Res. Inst. Publ.*. Cambridge University Press, Cambridge, 1999 pp. 23–90.
[2] F. Bergeron, R. Biagioli, and M. Rosas. On a conjecture concerning Littlewood-Richardson coefficients, 2003. In progress.
[3] S. C. Billey and G. Warrington. Kazhdan-Lusztig polynomials for 321-hexagon avoiding permutations. *J. Algebraic Combin.*, **13** (2001) pp. 111–136.
[4] S. M. Fallat, M. I. Gekhtman, and C. R. Johnson. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30** (2003) pp. 442–470.
[5] K. Fan and R. M. Green. Monomials and Temperley-Lieb algebras. *J. Algebra*, **190** (1997) pp. 498–517.
[6] S. Fomin, W. Fulton, C. Li, and Y. Poon. Eigenvalues, singular values, and Littlewood-Richardson coefficients, 2003. Preprint math.AG/03013078 on ArXiv.
[7] S. Fomin and A. Zelevinsky. Total positivity: Tests and parametrizations. *Math. Intelligencer*, **22** (2000) pp. 23–33.
[8] I. Gessel and G. Viennot. Determinants and plane partitions, 1989. Preprint.
[9] F. M. Goodman, P. de la Harpe, and V. Jones. *Coxeter Graphs and Towers of Algebras*. Springer–Verlag, New York, 1989.

[10] I. P. Goulden and D. M. Jackson. Immanants of combinatorial matrices. *J. Algebra*, **148** (1992) pp. 305–324.
[11] C. Greene. Proof of a conjecture on immanants of the Jacobi-Trudi matrix. *Linear Algebra Appl.*, **171** (1992) pp. 65–79.
[12] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6** (1993) pp. 569–595.
[13] P. Heyfron. Immanant dominance orderings for hook partitions. *Linear and Multilinear Algebra*, **24** (1988) pp. 65–78.
[14] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Inv. Math.*, **53** (1979) pp. 165–184.
[15] G. Lusztig. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston, 1994 pp. 531–568.
[16] B. Rhoades and M. Skandera. Temperley-Lieb immanants, 2004. Submitted.
[17] B. Sagan. *The Symmetric Group*. Springer, New York, 2001.
[18] M. Skandera. Inequalities in products of minors of totally nonnegative matrices. *J. Alg. Combin.*, **20** (2004).
[19] T. A. Springer. Quelques aplications de la cohomologie d'intersection. In *Séminaire Bourbaki, Vol. 1981/1982*, vol. 92 of *Astérisque*. Soc. Math. France, Paris, 1982 pp. 249–273.
[20] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge, 1999.
[21] R. Stanley. Positivity problems and conjectures. In *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.). American Mathematical Society, Providence, RI, 2000 pp. 295–319.
[22] R. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.
[23] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.
[24] J. Stembridge. Some conjectures for immanants. *Can. J. Math.*, **44** (1992) pp. 1079–1099.
[25] J. Stembridge. Multiplicity-free products of Schur functions. *Ann. Comb.*, **5** (2001) pp. 113–121.
[26] J. Stembridge, 2004. Personal communication.
[27] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, **219** (1995) pp. 539–565.
[28] A. Zelevinsky. From Littlewood-Richardson coefficients to cluster algebras in three lectures. In *Symmetric Functions 2001: Surveys of Developments and Perspectives* (S. Fomin, ed.), vol. 74 of *NATO Science Series II: Mathematics, Physics, and Chemistry*. Kluwer, Dordrecht, 2002 pp. 253–273.