TABLEAUX ON PERIODIC SKEW DIAGRAMS AND IRREDUCIBLE REPRESENTATIONS OF DOUBLE AFFINE HECKE ALGEBRA OF TYPE A

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ABSTRACT. The irreducible representations of the symmetric group S_n are parameterized by combinatorial objects called *Young diagrams,* or shapes. A given irreducible representation has a basis indexed by *Young tableaux* of that shape. In fact, this basis consists of weight vectors (simultaneous eigenvectors) for a commutative subalgebra $\mathbb{F}[\mathfrak{X}]$ of the group algebra $\mathbb{F}S_n$.

The double affine Hecke algebra (DAHA) is a deformation of the group algebra of the *affine* symmetric group and it also contains a commutative subalgebra $\mathbb{F}[\mathfrak{X}]$.

Not every irreducible representation of the DAHA has a basis of weight vectors (and in fact it is quite difficult to parameterize all of its irreducible representations), but if we restrict our attention to those that do, these irreducible representations are parameterized by "affine shapes" and have a basis (of $\mathfrak{X}$-weight vectors) indexed by the "affine tableaux" of that shape. In this talk, we will construct these irreducible representations.

Introduction.

We introduce and study an affine analogue of skew Young diagrams and tableaux on them. The double affine Hecke algebra of type A acts on the space spanned by standard tableaux on each diagram. We show that the modules obtained this way are irreducible, and they exhaust all irreducible modules of a certain class over the double affine Hecke algebra. In particular, the classification of irreducible modules of this class, announced by Cherednik, is recovered.

As is well-known, Young diagrams consisting of n boxes parameterize isomorphism classes of finite dimensional irreducible representations of the symmetric group $\mathfrak{S}_n$, and moreover the structure of each irreducible representation is described in terms of tableaux on the corresponding Young diagram; namely, a basis of the representation is labeled by standard tableaux, on which the action of $\mathfrak{S}_n$ generators is explicitly described. This combinatorial description due to A. Young has played an essential role in the study of the representation theory of the symmetric group (or the affine Hecke algebra), and its generalization for the (degenerate) affine Hecke algebra $H_n(q)$ of GL_n has been given in [Ch1, Ra1, Ra2], where skew Young diagrams appear on combinatorial side.

The purpose of this paper is to introduce an "affine analogue" of skew Young diagrams and tableaux, which give a parameterization and a combinatorial description of a family of irreducible representations of the double affine Hecke algebra $\ddot{H}_n(q)$ of GL_n over a field $\mathbb{F}$, where $q \in \mathbb{F}$ is a parameter of the algebra.

The double affine Hecke algebra was introduced by I. Cherednik [Ch2, Ch3] and has since been used by him and by several authors to obtain important results about diagonal coinvariants, Macdonald polynomials, and certain Macdonald identities.

In this paper, we focus on the case where q is not a root of 1, and we consider representations of $\ddot{H}_n(q)$ that are $\mathfrak{X}$-semisimple; namely, we consider representations which have basis of simultaneous eigenvectors with respect to all elements in the commutative subalgebra $\mathbb{F}[\mathfrak{X}] = \mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \xi^{\pm 1}]$ of $\ddot{H}_n(q)$. (In [Ra1, Ra2], such representations for affine Hecke algebras are referred to as "calibrated.")

On combinatorial side, we introduce *periodic skew diagrams* as skew Young diagrams consisting of infinitely many boxes satisfying certain periodicity conditions. We define a tableau on a periodic skew diagram as a bijection from the diagram to $\mathbb{Z}$ which satisfies the condition reflecting the periodicity of the diagram.

Periodic skew diagrams are natural generalization of skew Young diagrams and have appeared in [Ch4] (or implicitly in [AST]), but the notion of tableaux on them seems new.

To connect the combinatorics with the representation theory of the double affine Hecke algebra $\ddot{H}_n(q)$, we construct, for each periodic skew diagram, an $\ddot{H}_n(q)$-module that has a basis of $\mathbb{F}[\mathfrak{X}]$-weight vectors labeled by standard tableaux on the diagram by giving the explicit action of the $\ddot{H}_n(q)$ generators.

Such modules are $\mathfrak{X}$-semisimple by definition. We show that they are irreducible, and that our construction gives a one-to-one correspondence between the set of periodic skew diagrams and the set of isomorphism classes of irreducible representations of the double affine Hecke algebra that are $\mathfrak{X}$-semisimple.

The classification results here recover those of Cherednik's in [Ch4] (see also [Ch5]), but in this paper we provide a detailed proof based on purely combinatorial arguments concerning standard tableaux on periodic skew diagrams.

Note that the corresponding results for the degenerate double affine Hecke algebra of GL_n easily follow from a parallel argument.

1. The affine root system and Weyl group

1.1. **The affine root system.** Let $n \in \mathbb{Z}_{\geq 2}$. Let $\tilde{\mathfrak{h}}$ be an $(n+2)$-dimensional vector space over $\mathbb{Q}$ with the basis $\{\epsilon_1^\vee, \epsilon_2^\vee, \ldots, \epsilon_n^\vee, c, d\}$:

$$\tilde{\mathfrak{h}} = \left(\oplus_{i=1}^n \mathbb{Q}\epsilon_i^\vee\right) \oplus \mathbb{Q}c \oplus \mathbb{Q}d.$$

Introduce the non-degenerate symmetric bilinear form $(\ |\ )$ on $\tilde{\mathfrak{h}}$ by

$$(\epsilon_i^\vee|\epsilon_j^\vee) = \delta_{ij}, \quad (\epsilon_i^\vee|c) = (\epsilon_i^\vee|d) = 0, \quad (c|d) = 1, \quad (c|c) = (d|d) = 0.$$

Put $\mathfrak{h} = \oplus_{i=1}^n \mathbb{Q}\epsilon_i^\vee$ and $\dot{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{Q}c$. Let $\tilde{\mathfrak{h}}^* = (\oplus_{i=1}^n \mathbb{Q}\epsilon_i) \oplus \mathbb{Q}c^* \oplus \mathbb{Q}\delta$ be the dual space of $\tilde{\mathfrak{h}}$, where ϵ_i, c^* and δ are the dual vectors of $\epsilon_i^\vee$, c and d respectively. We identify the dual space $\dot{\mathfrak{h}}^*$ of $\dot{\mathfrak{h}}$ as a subspace of $\tilde{\mathfrak{h}}^*$ via the identification $\dot{\mathfrak{h}}^* = \tilde{\mathfrak{h}}^*/\mathbb{Q}\delta \cong \mathfrak{h}^* \oplus \mathbb{Q}c^*$.

The natural pairing is denoted by $\langle\,|\,\rangle : \tilde{\mathfrak{h}}^* \times \tilde{\mathfrak{h}} \to \mathbb{Q}$. There exists an isomorphism $\tilde{\mathfrak{h}}^* \to \tilde{\mathfrak{h}}$ such that $\epsilon_i \mapsto \epsilon_i^\vee$, $\delta \mapsto c$ and $c^* \mapsto d$. We denote by $\zeta^\vee \in \tilde{\mathfrak{h}}$ the image of $\zeta \in \tilde{\mathfrak{h}}^*$ under this isomorphism. Introduce the bilinear form $(\,|\,)$ on $\tilde{\mathfrak{h}}^*$ through this isomorphism. Note that

$$(\zeta \mid \eta) = \langle \zeta \mid \eta^\vee \rangle = (\zeta^\vee \mid \eta^\vee), \quad (\zeta, \eta \in \tilde{\mathfrak{h}}^*).$$

Put $\alpha_{ij} = \epsilon_i - \epsilon_j$ $(1 \leq i \neq j \leq n)$ and $\alpha_i = \alpha_{ii+1}$ $(1 \leq i \leq n-1)$. Then

$$R = \{\alpha_{ij} \mid i, j \in [1, n],\ i \neq j\}, \ R^+ = \{\alpha_{ij} \mid i, j \in [1, n],\ i < j\}, \ \Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$$

give the system of roots, positive roots and simple roots of type A_{n-1} respectively.

Put $\alpha_0 = -\alpha_{1n} + \delta$, and define the set $\dot{R}$ of (real) roots, $\dot{R}^+$ of positive roots and $\dot{\Pi}$ of simple roots of type $A^{(1)}_{n-1}$ by

$$\begin{aligned}
&\dot{R} = \{\alpha + k\delta \,|\, \alpha \in R,\ k \in \mathbb{Z}\}\,,\\
&\dot{R}^+ = \{\alpha + k\delta \mid \alpha \in R^+,\ k \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + k\delta \mid \alpha \in R^+,\ k \in \mathbb{Z}_{\geq 1}\}\,,\\
&\dot{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}\,.
\end{aligned}$$

1.2. **Affine Weyl group.**

Definition 1.1. For $n \in \mathbb{Z}_{\geq 2}$, the *extended affine Weyl group* $\dot{W}_n$ of $\mathfrak{gl}_n$ is the group defined by the following generators and relations:

$$\begin{aligned}
&\textit{generators}\ : && s_0, s_1, \dots, s_{n-1}, \pi^{\pm 1}.\\
&\textit{relations for } n \geq 3\ : && s_i^2 = 1\ (i \in [0, n-1]),\\
& && s_i s_j\ s_i = s_j s_i s_j\ (i - j \equiv \pm 1 \bmod n),\\
& && s_i s_j = s_j s_i\ (i - j \not\equiv \pm 1 \bmod n),\\
& && \pi s_i = s_{i+1}\pi,\ (i \in [0, n-2]), \quad \pi s_{n-1} = s_0 \pi,\\
& && \pi\pi^{-1} = \pi^{-1}\pi = 1.\\
&\textit{relations for } n = 2\ : && s_0^2 = s_1^2 = 1,\\
& && \pi s_0 = s_1 \pi, \quad \pi s_1 = s_0 \pi, \quad \pi\pi^{-1} = \pi^{-1}\pi = 1.
\end{aligned}$$

The subgroup W_n of $\dot{W}_n$ generated by the elements $s_1, s_2, \dots, s_{n-1}$ is called the *Weyl group* of $\mathfrak{gl}_n$. The group W_n is isomorphic to the symmetric group of degree n.

In the following, we fix $n \in \mathbb{Z}_{\geq 2}$ and denote $\dot{W} = \dot{W}_n$ and $W = W_n$.

Put

$$P = \oplus_{i=1}^n \mathbb{Z}\epsilon_i.$$

Put $\tau_{\epsilon_1} = \pi s_{n-1} \cdots s_2 s_1$ and $\tau_{\epsilon_i} = \pi^{i-1}\tau_{\epsilon_1}\pi^{-i+1}$ $(i \in [2, n])$. Then there exists a group embedding $P \to \dot{W}$ such that $\epsilon_i \mapsto \tau_{\epsilon_i}$. By τ_η we denote the element in $\dot{W}$ corresponding to $\eta \in P$. It is well-known that the group $\dot{W}$ is isomorphic to the semidirect product $P \rtimes W$ with the relation $w\tau_\eta w^{-1} = \tau_{w(\eta)}$.

The group $\dot{W}$ acts on $\tilde{\mathfrak{h}}$ by

$$\begin{aligned}
&s_i(h) = h - \langle\alpha_i|h\rangle\alpha_i^\vee \quad \text{for } i \in [1, n-1],\ h \in \tilde{\mathfrak{h}},\\
&\tau_{\epsilon_i}(h) = h + \langle\delta|h\rangle\epsilon_i^\vee - \left(\langle\epsilon_i|h\rangle + \frac{1}{2}\langle\delta|h\rangle\right)c \quad \text{for } i \in [1, n],\ h \in \tilde{\mathfrak{h}}.
\end{aligned}$$

The dual action on $\tilde{\mathfrak{h}}^*$ is given by

$$\begin{aligned}
&s_i(\zeta) = \zeta - (\alpha_i|\zeta)\alpha_i \ \ \text{for } i \in [1, n-1],\ \zeta \in \tilde{\mathfrak{h}}^*,\\
&\tau_{\epsilon_i}(\zeta) = \zeta + (\delta|\zeta)\epsilon_i - \left((\epsilon_i|\zeta) + \frac{1}{2}(\delta|\zeta)\right)\delta \quad \text{for } i \in [1, n],\ h \in \tilde{\mathfrak{h}}^*.
\end{aligned}$$

With respect to these actions, the inner products on $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}^*$ are $\dot{W}$-invariant. Note that the set $\dot{R}$ of roots is preserved by the dual action of $\dot{W}$ on $\tilde{\mathfrak{h}}^*$. For $\alpha \in \dot{R}$, there exists $i \in [0, n-1]$ and $w \in \dot{W}$ such that $w(\alpha_i) = \alpha$. We set $s_\alpha = w s_i w^{-1}$. Then s_α is independent of the choice of i and w, and we have

$$s_\alpha(h) = h - \langle\alpha \mid h\rangle\alpha^\vee$$

for $h \in \tilde{\mathfrak{h}}$. The element s_α is called the reflection corresponding to α. Note that $s_{\alpha_i} = s_i$. For $w \in \dot{W}$, set

$$R(w) = \dot{R}^+ \cap w^{-1}\dot{R}^-,$$

where $\dot{R}^- = \dot{R}\backslash\dot{R}^+$. The length $l(w)$ of $w \in \dot{W}$ is defined as the number $\sharp R(w)$ of elements in $R(w)$. For $w \in \dot{W}$, an expression $w = \pi^k s_{j_1} s_{j_2} \cdots s_{j_m}$ is called a reduced expression if $m = l(w)$. It can be seen that

$$R(w) = \{s_{j_m} \cdots s_{j_2}(\alpha_{j_1}), s_{j_m} \cdots s_{j_3}(\alpha_{j_2}), \ldots, \alpha_{j_m}\} \tag{1.1}$$

if $w = \pi^k s_{j_1} s_{j_2} \cdots s_{j_m}$ is a reduced expression.

Define the Bruhat order $\preceq$ in $\dot{W}$ by

$$x \preceq w \Leftrightarrow x \text{ is equal to a subexpression of a reduced expression of } w.$$

Let I be a subset of $[0, n-1]$. Put

$$\dot{\Pi}_I = \{\alpha_i \mid i \in I\} \subseteq \dot{\Pi}, \quad \dot{W}_I = \langle s_i \mid i \in I\rangle \subseteq \dot{W}, \quad \dot{R}_I^+ = \{\alpha \in \dot{R}^+ \mid s_\alpha \in \dot{W}_I\}.$$

The subgroup $\dot{W}_I$ is called the parabolic subgroup corresponding to $\dot{\Pi}_I$. Define

$$\dot{W}^I = \left\{w \in \dot{W} \mid R(w) \cap \dot{R}_I^+ = \emptyset\right\}.$$

1.3. **Notation.** For any integer i, we introduce the following notation:

$$\epsilon_i = \epsilon_{\underline{i}} - k\delta \in \tilde{\mathfrak{h}}^*, \quad \epsilon_i^\vee = \epsilon_{\underline{i}}^\vee - kc \in \tilde{\mathfrak{h}}, \tag{1.2}$$

where $i = \underline{i} + kn$ with $\underline{i} \in [1, n]$ and $k \in \mathbb{Z}$.

Put $\alpha_{ij} = \epsilon_i - \epsilon_j$ and $\alpha_{ij}^\vee = \epsilon_i^\vee - \epsilon_j^\vee$ for any $i, j \in \mathbb{Z}$. Noting that $\epsilon_0 - \epsilon_1 = \delta + \epsilon_n - \epsilon_1 = \alpha_0$, we reset $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$ for any $i \in \mathbb{Z}$.

Define the action of $\dot{W}$ on the set $\mathbb{Z}$ of integers by

$$\begin{aligned} s_i(j) &= j+1 \quad \text{for } j \equiv i \bmod n, & s_i(j) &= j \quad \text{for } j \not\equiv i, i+1 \bmod n,\\ s_i(j) &= j-1 \quad \text{for } j \equiv i+1 \bmod n, & \pi(j) &= j+1 \quad \text{for all } j. \end{aligned}$$

It is easy to see that the action of τ_{ϵ_i} $(i \in [1, n])$ is given by

$$\tau_{\epsilon_i}(j) = j + n \quad \text{for } j \equiv i \bmod n, \qquad \tau_{\epsilon_i}(j) = j \quad \text{for } j \not\equiv i \bmod n,$$

and that the following formula holds for any $w \in \dot{W}$:

$$w(j+n) = w(j) + n \quad \text{for all } j.$$

Lemma 1.2. *Let $w \in \dot{W}$.*

(i) $w(\epsilon_j) = \epsilon_{w(j)}$ *and* $w(\epsilon_j^\vee) = \epsilon_{w(j)}^\vee$ *for any* $j \in \mathbb{Z}$.

(ii) $w(\alpha_{ij}) = \alpha_{w(i)w(j)}$ *and* $w(\alpha_{ij}^\vee) = \alpha_{w(i)w(j)}^\vee$ *for any* $i, j \in \mathbb{Z}$.

2. Periodic skew diagrams and tableaux on them

Throughout this paper, we let $\mathbb{F}$ denote a field whose characteristic is not equal to 2.

2.1. **Periodic skew diagrams.** For $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, put

$$\widehat{\mathcal{P}}^{+}_{m,\ell} = \{\mu \in \mathbb{Z}^m \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \text{ and } \ell \geq \mu_1 - \mu_m\}, \tag{2.1}$$

where μ_i denotes the i-th component of μ, i.e., $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. Fix $n \in \mathbb{Z}_{\geq 2}$ and introduce the following subsets of $\mathbb{Z}^m \times \mathbb{Z}^m$:

$$\widehat{\mathcal{J}}^{n}_{m,\ell} = \left\{ (\lambda, \mu) \in \widehat{\mathcal{P}}^{+}_{m,\ell} \times \widehat{\mathcal{P}}^{+}_{m,\ell} \,\middle|\, \lambda_i \geq \mu_i \ (i \in [1,m]),\ \sum_{i=1}^{m} (\lambda_i - \mu_i) = n \right\},$$

$$\widehat{\mathcal{J}}^{*n}_{m,\ell} = \left\{ (\lambda, \mu) \in \widehat{\mathcal{P}}^{+}_{m,\ell} \times \widehat{\mathcal{P}}^{+}_{m,\ell} \,\middle|\, \lambda_i > \mu_i \ (i \in [1,m]),\ \sum_{i=1}^{m} (\lambda_i - \mu_i) = n \right\}.$$

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^{n}_{m,\ell}$, define the subsets λ/μ and $\widehat{\lambda/\mu}_{(m,-\ell)}$ of $\mathbb{Z}^2$ by

$$\lambda/\mu = \left\{ (a,b) \in \mathbb{Z}^2 \mid a \in [1,m],\ b \in [\mu_a + 1, \lambda_a] \right\},$$

$$\widehat{\lambda/\mu}_{(m,-\ell)} = \left\{ (a + km, b - k\ell) \in \mathbb{Z}^2 \mid (a,b) \in \lambda/\mu,\ k \in \mathbb{Z} \right\}.$$

Let $\lambda/\mu[k] = \lambda/\mu + k(m, -\ell)$. Obviously we have

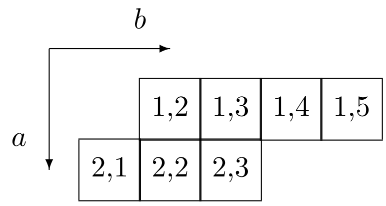
$$\widehat{\lambda/\mu}_{(m,-\ell)} = \bigsqcup_{k \in \mathbb{Z}} \lambda/\mu[k] = \bigsqcup_{k \in \mathbb{Z}} (\lambda/\mu + k(m, -\ell)).$$

The set λ/μ is the skew diagram (or skew Young diagram) associated with (λ, μ).
We call the set $\widehat{\lambda/\mu}_{(m,-\ell)}$ the *periodic skew diagram* associated with (λ, μ).
We will denote $\widehat{\lambda/\mu}_{(m,-\ell)}$ just by $\widehat{\lambda/\mu}$ when m and ℓ are fixed.

Example 2.1. Let $n = 7$, $m = 2$ and $\ell = 3$. Put $\lambda = (5,3)$, $\mu = (1,0)$. Then $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and we have

$$\lambda/\mu = \{(1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3)\}.$$

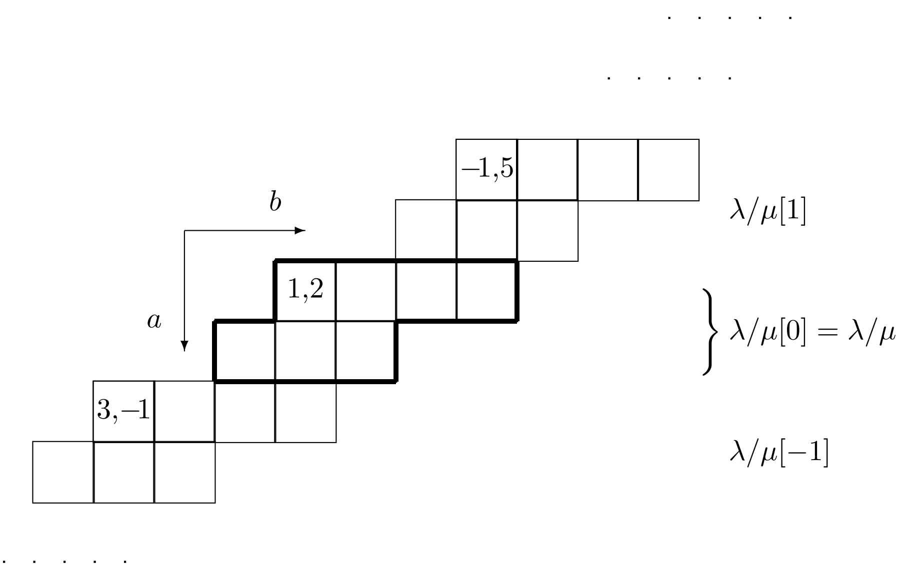
The set λ/μ is expressed by the following picture (usually, the coordinate in the boxes are omitted):



The periodic skew diagram

$$\widehat{\lambda/\mu}_{(2,-3)} = \bigsqcup_{k \in \mathbb{Z}} \lambda/\mu[k] = \bigsqcup_{k \in \mathbb{Z}} (\lambda/\mu + k(2, -3))$$

is expressed by the following picture:



2.2. **Tableaux on periodic skew diagram.** Fix $n \in \mathbb{Z}_{\geq 2}$. Recall that a bijection from a skew Young diagram λ/μ of degree n to the set $[1, n]$ is called a tableau on λ/μ.

Definition 2.2. Given $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, $\gamma = (m, -\ell)$, a bijection $T : \widehat{\lambda/\mu} \to \mathbb{Z}$ is said to be a γ-*tableau* on $\widehat{\lambda/\mu}$ if T satisfies

$$T(u + \gamma) = T(u) + n \quad \text{for all } u \in \widehat{\lambda/\mu}. \tag{2.2}$$

Let

$$\mathrm{Tab}(\widehat{\lambda/\mu}) = \mathrm{Tab}_{(m,-\ell)}(\widehat{\lambda/\mu})$$

denote the set of all γ-tableaux on $\widehat{\lambda/\mu}$.

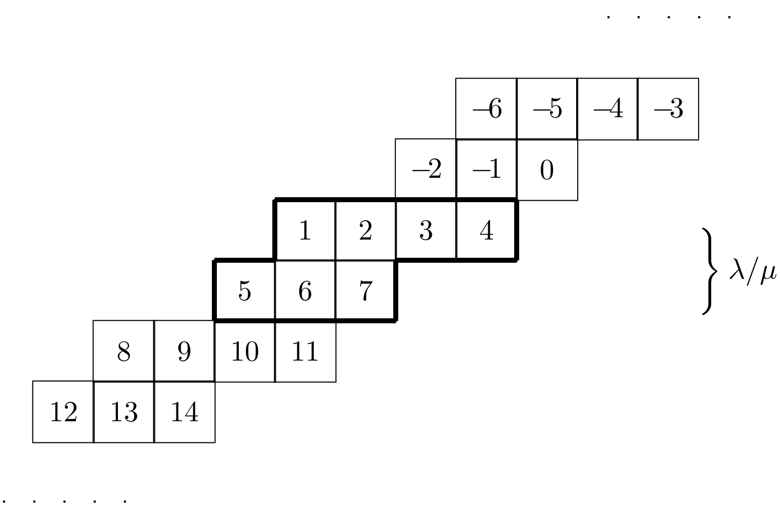
Remark 2.3. A tableau on $\widehat{\lambda/\mu}$ is determined uniquely from the values on a fundamental domain of $\widehat{\lambda/\mu}$ with respect to the action of $\mathbb{Z}\gamma$. It also holds that any bijection from a fundamental domain of $\mathbb{Z}\gamma$ to the set $[1, n]$ uniquely extends to a tableau on $\widehat{\lambda/\mu}$.

There exists a unique tableau $T_0^{\widehat{\lambda/\mu}} = T_0$ on $\widehat{\lambda/\mu}$ such that

$$T_0(i, \mu_i + j) = \sum_{k=1}^{i-1} (\lambda_k - \mu_k) + j \quad \text{for } i \in [1, m],\ j \in [1, \lambda_i - \mu_i]. \tag{2.3}$$

We call T_0 the *row reading tableau* on $\widehat{\lambda/\mu}$.

Example 2.4. Let $n = 7$, $m = 2$, $\ell = 3$ and $\lambda = (5, 3)$, $\mu = (1, 0)$. The tableau T_0 on $\widehat{\lambda/\mu}$ given above is expressed as follows:



Proposition 2.5. *Let $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. The group $\dot{W}$ acts on the set* $\mathrm{Tab}(\widehat{\lambda/\mu})$ *by*

$$(wT)(u) = w(T(u)) \tag{2.4}$$

for $w \in \dot{W}$, $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ and $u \in \widehat{\lambda/\mu}$.

For each $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, define the map $\psi_T : \dot{W} \to \mathrm{Tab}(\widehat{\lambda/\mu})$ by $\psi_T(w) = wT$ $(w \in \dot{W})$.

Proposition 2.6. *Let $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. For any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, the correspondence ψ_T is a bijection.*

Lemma 2.7. $T^{-1}(w^{-1}(i)) = (wT)^{-1}(i)$ *for any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, $w \in \dot{W}$ and $i \in \mathbb{Z}$.*

2.3. **Content and weight.** Let C denote the map from $\mathbb{Z}^2$ to $\mathbb{Z}$ given by $C(a,b) = b - a$ for $(a,b) \in \mathbb{Z}^2$.

For a tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, define the map $C_T^{\widehat{\lambda/\mu}} : \mathbb{Z} \to \mathbb{Z}$ by

$$C_T^{\widehat{\lambda/\mu}}(i) = C(T^{-1}(i)) \quad (i \in \mathbb{Z}),$$

and call $C_T^{\widehat{\lambda/\mu}}$ the *content* of T. We simply denote $C_T^{\widehat{\lambda/\mu}}$ by C_T when (λ,μ) is fixed.

Lemma 2.8. *Let $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$. Then*
(i) $C_T(i+n) = C_T(i) - (\ell + m)$ *for all $i \in \mathbb{Z}$.*
(ii) $C_{wT}(i) = C_T(w^{-1}(i))$ *for all $w \in \dot{W}$ and $i \in \mathbb{Z}$.*

For $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, we define $\zeta_T \in \dot{\mathfrak{h}}^*$ by

$$\zeta_T = \sum_{i=1}^{n} C_T(i)\epsilon_i + (\ell+m)c^*.$$

Then ζ_T belongs to the lattice $\dot{P} \overset{def}{=} P \oplus \mathbb{Z}c^* = (\oplus_{i=1}^n \mathbb{Z}\epsilon_i) \oplus \mathbb{Z}c^*$. Note that the action of $\dot{W}$ on $\dot{\mathfrak{h}}^*$ preserves $\dot{P}$. Lemma 2.8 immediately implies the following:

Lemma 2.9. *Let* $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$. *Then*
(i) $\langle \zeta_T \mid \epsilon_i^\vee \rangle = C_T(i)$ *for all* $i \in \mathbb{Z}$.
(ii) $w(\zeta_T) = \zeta_{wT}$ *for all* $w \in \dot{W}$.

2.4. **The affine Weyl group and row increasing tableaux.** Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$.

Definition 2.10. A tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ is said to be *row increasing* (resp. *column increasing*) if

$$(a,b), (a,b+1) \in \widehat{\lambda/\mu} \ \Rightarrow\ T(a,b) < T(a,b+1).$$
$$(\text{resp. } (a,b), (a+1,b) \in \widehat{\lambda/\mu} \ \Rightarrow\ T(a,b) < T(a+1,b).)$$

A tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ which is row increasing and column increasing is called *a standard tableau* (or *a row-column increasing tableau*).

Denote by $\mathrm{Tab}^{\mathrm{R}}(\widehat{\lambda/\mu})$ (resp. $\mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$) the set of all row increasing (resp. standard) tableaux on $\widehat{\lambda/\mu}$.

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, put

$$I_{\lambda,\mu} = [1, n-1] \setminus \{n_1, n_2, \ldots, n_{m-1}\},$$

where $n_i = \sum_{j=1}^{i} (\lambda_j - \mu_j)$ for $i \in [1, m-1]$.

We write $\dot{R}^+_{\lambda-\mu} = \dot{R}^+_{I_{\lambda,\mu}}$, $\dot{W}_{\lambda-\mu} = \dot{W}_{I_{\lambda,\mu}}$ and $\dot{W}^{\lambda-\mu} = \dot{W}^{I_{\lambda,\mu}}$.

Note that $\dot{R}^+_{\lambda-\mu} \subseteq R^+$ and $\dot{W}_{\lambda-\mu} = W_{\lambda_1-\mu_1} \times W_{\lambda_2-\mu_2} \times \cdots \times W_{\lambda_m-\mu_m} \subseteq W$.

Recall that the correspondence $\psi_T : \dot{W} \to \mathrm{Tab}(\widehat{\lambda/\mu})$ given by $w \mapsto wT$ is bijective (Proposition 2.6) for any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$.

Proposition 2.11. *Let* $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. *Then*

$$\psi_{T_0}^{-1}(\mathrm{Tab}^{\mathrm{R}}(\widehat{\lambda/\mu})) = \dot{W}^{\lambda-\mu},$$

or equivalently, $\mathrm{Tab}^{\mathrm{R}}(\widehat{\lambda/\mu}) = \dot{W}^{\lambda-\mu} T_0 = \{wT_0 \mid w \in \dot{W}^{\lambda-\mu}\}$.

2.5. **The set of standard tableaux.** The next lemma follows easily:

Lemma 2.12. *Let* $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ *and* $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. *If* $(a,b) \in \widehat{\lambda/\mu}$ *and* $(a+1, b+1) \in \widehat{\lambda/\mu}$, *then* $T(a+1, b+1) - T(a,b) > 1$.

Proposition 2.13. *Let* $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ *and* $T, S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. *If* $C_T = C_S$ *then* $T = S$.

For $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$, put

$$\dot{Z}_T^{\widehat{\lambda/\mu}} = \left\{ w \in \dot{W} \,\middle|\, \langle \zeta_T \mid \alpha^\vee \rangle \notin \{-1, 1\} \text{ for all } \alpha \in R(w) \right\}. \tag{2.5}$$

Theorem 2.14. *Let* $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ *and* $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. *Then*

$$\psi_T^{-1}(\mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})) = \dot{Z}_T^{\widehat{\lambda/\mu}},$$

or equivalently, $\mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}) = \dot{Z}_T^{\widehat{\lambda/\mu}} T$.

For $m \in \mathbb{Z}_{\geq 1}$, define an automorphism ω_m of $\mathbb{Z}^m$ by

$$\omega_m \cdot \lambda = (\lambda_m + \ell + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_{m-1} + 1), \tag{2.6}$$

for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}^m$. Let $\langle \omega_m \rangle$ denote the free group generated by ω_m, and let $\langle \omega_m \rangle$ act on $\mathbb{Z}^m \times \mathbb{Z}^m$ by $\omega_m \cdot (\lambda, \mu) = (\omega_m \cdot \lambda, \omega_m \cdot \mu)$ for $(\lambda, \mu) \in \mathbb{Z}^m \times \mathbb{Z}^m$. Note that $\langle \omega_m \rangle$ preserves the subsets $\widehat{\mathcal{J}}^n_{m,\ell}$ and $\widehat{\mathcal{J}}^{*n}_{m,\ell}$ of $\mathbb{Z}^m \times \mathbb{Z}^m$.

Proposition 2.15. *Let $m, m' \in [1, n]$ and $\ell, \ell' \in \mathbb{Z}_{\geq 0}$. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and $(\eta, \nu) \in \widehat{\mathcal{J}}^{*n}_{m',\ell'}$. The following are equivalent*:

(a) $C_T^{\widehat{\lambda/\mu}} = C_S^{\widehat{\eta/\nu}}$ *for some* $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$ *and* $S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\eta/\nu})$,

(b) $m = m'$, $\ell = \ell'$ *and* $\widehat{\lambda/\mu} = \widehat{\eta/\nu} + (r, r)$ *for some* $r \in \mathbb{Z}$.

(c) $m = m'$, $\ell = \ell'$ *and* $(\eta, \nu) = \omega_m^r \cdot (\lambda, \mu)$ *for some* $r \in \mathbb{Z}$.

3. Representations of the double affine Hecke algebra

Let $\mathbb{F}$ denote a field whose characteristic is not equal to 2.

3.1. Double affine Hecke algebra of type A.

Let $q \in \mathbb{F}$.

The double affine Hecke algebra was introduced by Cherednik [Ch2, Ch3].

Definition 3.1. Let $n \in \mathbb{Z}_{\geq 2}$.

(i) The *double affine Hecke algebra* $\ddot{H}_n(q)$ of GL_n is the unital associative algebra over $\mathbb{F}$ defined by the following generators and relations:

$$\begin{array}{ll}
\textit{generators}: & t_0, t_1, \dots, t_{n-1}, \pi^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, \xi^{\pm 1}. \\
\textit{relations for } n \geq 3: & (t_i - q)(t_i + 1) = 0 \ (i \in [0, n-1]), \\
& t_i t_j t_i = t_j t_i t_j \ (j \equiv i \pm 1 \bmod n), \quad t_i t_j = t_j t_i \ (j \not\equiv i \pm 1 \bmod n), \\
& \pi \pi^{-1} = \pi^{-1} \pi = 1, \\
& \pi t_i \pi^{-1} = t_{i+1} \ (i \in [0, n-2]), \ \pi t_{n-1} \pi^{-1} = t_0, \\
& x_i x_i^{-1} = x_i^{-1} x_i = 1 \ (i \in [1, n]), \quad x_i x_j = x_j x_i \ (i, j \in [1, n]), \\
& t_i x_i t_i = q x_{i+1} \ (i \in [1, n-1]), \ t_0 x_n t_0 = \xi^{-1} q x_1 \\
& t_i x_j = x_j t_i \ (j \not\equiv i, i+1 \bmod n), \\
& \pi x_i \pi^{-1} = x_{i+1} \ (i \in [1, n-1]), \ \pi x_n \pi^{-1} = \xi^{-1} x_1, \\
& \xi \xi^{-1} = \xi^{-1} \xi = 1, \quad \xi^{\pm 1} h = h \xi^{\pm 1} \ (h \in \ddot{H}_n(q)). \\
\textit{relations for } n = 2: & (t_i - q)(t_i + 1) = 0 \ (i \in [0, 1]), \\
& \pi \pi^{-1} = \pi^{-1} \pi = 1, \quad \pi t_0 \pi^{-1} = t_1, \ \pi t_1 \pi^{-1} = t_0, \\
& x_i x_i^{-1} = x_i^{-1} x_i = 1 \ (i \in [1, 2]), \quad x_1 x_2 = x_2 x_1, \\
& t_1 x_1 t_1 = q x_2, \quad t_0 x_2 t_0 = \xi^{-1} q x_1 \\
& \pi x_1 \pi^{-1} = x_2, \ \pi x_2 \pi^{-1} = \xi^{-1} x_1, \\
& \xi \xi^{-1} = \xi^{-1} \xi = 1, \quad \xi^{\pm 1} h = h \xi^{\pm 1} \ (h \in \ddot{H}_2(q)).
\end{array}$$

(ii) Define the *affine Hecke algebra* $\dot{H}_n(q)$ of GL_n as the subalgebra of $\ddot{H}_n(q)$ generated by $\{t_0, t_1, \ldots, t_{n-1}, \pi^{\pm 1}\}$.

Remark 3.2. It is known that the subalgebra of $\ddot{H}_n(q)$ generated by

$$\{t_1, t_2, \ldots, t_{n-1}, x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\}$$

is also isomorphic to $\dot{H}_n(q)$.

For $\nu = \sum_{i=1}^n \nu_i \epsilon_i + \nu_c c^* \in \dot{P}$, put

$$x^\nu = x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n} \xi^{\nu_c}.$$

Let $\mathfrak{X}$ denote the commutative group $\{x^\nu \mid \nu \in \dot{P}\} \subseteq \ddot{H}_n(q)$. The group algebra $\mathbb{F}[\mathfrak{X}] = \mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \xi^{\pm 1}]$ is a commutative subalgebra of $\ddot{H}_n(q)$.

For $w \in \dot{W}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$, put

$$t_w = \pi^r t_{i_1} t_{i_2} \cdots t_{i_k}.$$

Then t_w does not depend on the choice of the reduced expression, and $\{t_w\}_{w \in \dot{W}}$ forms a basis of the affine Hecke algebra $\dot{H}_n(q) \subset \ddot{H}_n(q)$.

It is easy to see that $\{t_w x^\nu\}_{w \in \dot{W}, \nu \in \dot{P}}$ and $\{x^\nu t_w\}_{w \in \dot{W}, \nu \in \dot{P}}$ respectively form bases of $\ddot{H}_n(q)$.

Let $\mathfrak{X}^*$ denote the set of characters of $\mathfrak{X}$:

$$\mathfrak{X}^* = \mathrm{Hom}_{group}(\mathfrak{X}, \mathrm{GL}_1(\mathbb{F})).$$

Consider the correspondence $\dot{P} \to \mathfrak{X}^*$ which maps $\zeta \in \dot{P}$ to the character $q^\zeta \in \mathfrak{X}^*$ defined by

$$q^\zeta(x_i) = q^{\langle \zeta | \epsilon_i^\vee \rangle} \ (i \in [1, n]), \quad q^\zeta(\xi) = q^{\langle \zeta | c \rangle},$$

or equivalently, defined by $q^\zeta(x^\nu) = q^{\langle \zeta | \nu^\vee \rangle}$ $(\nu \in \dot{P})$. Through this correspondence, $\dot{P}$ is identified with the subset

$$\{\chi \in \mathfrak{X}^* \mid \chi(x^\nu) \in q^{\mathbb{Z}} \ (\forall \nu \in \dot{P})\}$$

of $\mathfrak{X}^*$, where $q^{\mathbb{Z}} = \{q^r \mid r \in \mathbb{Z}\}$.

For an $\ddot{H}_n(q)$-module M and $\zeta \in \dot{P}$, define the weight space M_ζ and the generalized weight space M_ζ^{gen} of weight ζ with respect to the action of $\mathbb{F}[\mathfrak{X}]$ by

$$M_\zeta = \left\{ v \in M \,\middle|\, (x^\nu - q^{\langle \zeta | \nu^\vee \rangle}) v = 0 \ \text{ for any } \nu \in \dot{P} \right\},$$
$$M_\zeta^{\mathrm{gen}} = \bigcup_{k \geq 1} \left\{ v \in M \,\middle|\, (x^\nu - q^{\langle \zeta | \nu^\vee \rangle})^k v = 0 \text{ for any } \nu \in \dot{P} \right\}.$$

For an $\ddot{H}_n(q)$-module M, an element $\zeta \in \dot{P}$ is called a weight of M if $M_\zeta \neq 0$, and an element $v \in M_\zeta$ (resp. M_ζ^{gen}) is called a weight vector (resp. generalized weight vector) of weight ζ.

For $\zeta \in \dot{P}$, put

$$\dot{\mathcal{Z}}_\zeta = \{w \in \dot{W} \mid \langle \zeta \mid \alpha^\vee \rangle \notin \{-1, 1\} \text{ for all } \alpha \in R(w)\}. \tag{3.1}$$

Note that $\dot{\mathcal{Z}}_{\zeta_T} = \dot{Z}_T^{\widehat{\lambda/\mu}}$ for $(\lambda, \mu) \in \widehat{\mathcal{J}}_{m,\ell}^n$ and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

3.2. **$\mathfrak{X}$-semisimple modules.** Fix $n \in \mathbb{Z}_{\geq 2}$. Let $q \in \mathbb{F}$ and suppose that q is not a root of 1.

Fix $\kappa \in \mathbb{Z}$ and put $P_\kappa = P + \kappa c^* = \{\zeta \in \dot{P} \mid \langle \zeta \mid c \rangle = \kappa\}$.

Definition 3.3. Define $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$ as the set consisting of those $\ddot{H}_n(q)$-modules M which are finitely generated and admit a decomposition

$$M = \bigoplus_{\zeta \in P_\kappa} M_\zeta$$

with $\dim M_\zeta < \infty$ for all $\zeta \in P_\kappa$.

A module in $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$ is also called $\mathfrak{X}$-semisimple. We remark that the structure of all irreducible $\mathfrak{X}$-semisimple modules, without requiring the eigenvalues of the x_k to live in $\{q^i \mid i \in \mathbb{Z}\}$, is easily obtained once we understand the modules in $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$.

3.3. **Representations associated with periodic skew diagrams.** In the rest of this paper, we always assume that q is not a root of 1.

Let $n \in \mathbb{Z}_{\geq 2}$, $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$.

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, set

$$\ddot{V}(\lambda, \mu) = \bigoplus_{T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})} \mathbb{F} v_T. \tag{3.2}$$

Define linear operators $\tilde{x}_i$ $(i \in [1, n])$, $\tilde{\pi}$ and $\tilde{t}_i$ $(i \in [0, n-1])$ on $\ddot{V}(\lambda, \mu)$ by

$$\tilde{x}_i v_T = q^{C_T(i)} v_T, \tag{3.3}$$

$$\tilde{\pi} v_T = v_{\pi T}, \tag{3.4}$$

$$\tilde{t}_i v_T = \begin{cases} \frac{1-q^{1+\tau_i}}{1-q^{\tau_i}} v_{s_i T} - \frac{1-q}{1-q^{\tau_i}} v_T & \text{if } s_i T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}), \\ -\frac{1-q}{1-q^{\tau_i}} v_T & \text{if } s_i T \notin \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}), \end{cases} \tag{3.5}$$

where

$$\tau_i = C_T(i) - C_T(i+1) = \langle \zeta_T \mid \alpha_i^\vee \rangle \quad (i \in [0, n-1]).$$

The following lemma is easy and ensures that the operator $\tilde{t}_i$ is well-defined:

Lemma 3.4. *$C_T(i) - C_T(i+1) \neq 0$ for any $i \in [0, n-1]$ and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.*

Theorem 3.5. *Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. There exists an algebra homomorphism $\theta_{\lambda,\mu} : \ddot{H}_n(q) \to \mathrm{End}_{\mathbb{F}}(\ddot{V}(\lambda, \mu))$ such that*

$$\theta_{\lambda,\mu}(t_i) = \tilde{t}_i \ (i \in [0, n-1]),\ \theta_{\lambda,\mu}(\pi) = \tilde{\pi},$$
$$\theta_{\lambda,\mu}(x_i) = \tilde{x}_i \ (i \in [1, n]), \quad \theta_{\lambda,\mu}(\xi) = q^{\ell+m}.$$

Theorem 3.6. *Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$.*

(i) $\ddot{V}(\lambda, \mu) = \bigoplus_{T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})} \ddot{V}(\lambda, \mu)_{\zeta_T}$, *and* $\ddot{V}(\lambda, \mu)_{\zeta_T} = \mathbb{F} v_T$ *for all* $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

(ii) *The $\ddot{H}_n(q)$-module $\ddot{V}(\lambda, \mu)$ is irreducible.*

3.4. **Classification of $\mathfrak{X}$-semisimple modules.** Fix $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Let $q \in \mathbb{F}$ and suppose that q is not a root of 1.

Theorem 3.7. *Let $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_{\kappa}(\ddot{H}_n(q))$. Then there exist $m \in [1, \kappa]$ and $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\kappa-m}$ such that $L \cong \ddot{V}(\lambda, \mu)$.*

Theorem 3.8. *Let $m, m' \in \mathbb{Z}_{\geq 1}$ and $\ell, \ell' \in \mathbb{Z}_{\geq 0}$. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and $(\eta, \nu) \in \widehat{\mathcal{J}}^{*n}_{m',\ell'}$. Then the following are equivalent*:
(a) $\ddot{V}(\lambda, \mu) \cong \ddot{V}(\eta, \nu)$.
(b) $m = m'$, $\ell = \ell'$ *and* $\widehat{\lambda/\mu} = \widehat{\eta/\nu} + (r, r)$ *for some* $r \in \mathbb{Z}$.
(c) $m = m'$, $\ell = \ell'$ *and* $(\eta, \nu) = \omega^r_m \cdot (\lambda, \mu)$ *for some* $r \in \mathbb{Z}$.

Remark 3.9. Combining Theorem 3.7 and Theorem 3.8, the classification we obtain agrees with that announced in [Ch4], where he also considers general q and ξ.

An alternative approach to prove these results is to use the result in [Va, Su], where the classification of irreducible modules over $\ddot{H}_n(q)$ of a more general class is obtained. Actually, it is easy to see that the $\ddot{H}_n(q)$-module $\ddot{V}(\lambda, \mu)$ coincides with the unique simple quotient $\ddot{L}(\lambda, \mu)$ of the induced module $\ddot{M}(\lambda, \mu)$ with the notation in [Su].

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