DECREASING SUBSEQUENCES IN PERMUTATIONS AND WILF EQUIVALENCE FOR INVOLUTIONS

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Abstract. In a recent paper, Backelin, West and Xin describe a map ϕ^* that recursively replaces all occurrences of the pattern $k \cdots 21$ in a permutation σ by occurrences of the pattern $(k-1) \cdots 21k$. The resulting permutation $\phi^*(\sigma)$ contains no decreasing subsequence of length k. We prove that, rather unexpectedly, the map ϕ^* commutes with taking the inverse of a permutation.

In the BWX paper, the definition of ϕ^* is actually extended to full rook placements on a Ferrers board (the permutations correspond to square boards), and the construction of the map ϕ^* is the key step in proving the following result. Let T be a set of patterns starting with the prefix $12 \cdots k$. Let T' be the set of patterns obtained by replacing this prefix by $k \cdots 21$ in every pattern of T. Then for all n, the number of permutations of the symmetric group $\mathcal{S}_n$ that avoid T equals the number of permutations of $\mathcal{S}_n$ that avoid T'.

Our commutation result, generalized to Ferrers boards, implies that the number of *involutions* of $\mathcal{S}_n$ that avoid T is equal to the number of involutions of $\mathcal{S}_n$ avoiding T', as recently conjectured by Jaggard.

Version française. Dans un article récent, Backelin, West et Xin ont défini une transformation ϕ^* qui détruit récursivement toutes les sous-suites décroissantes de longueur k d'une permutation (k est fixé). Cette transformation s'obtient en itérant une transformation élémentaire ϕ qui détruit *une* sous-suite décroissante de longueur k.

Ces deux transformations peuvent être étendues à des objets plus généraux que les permutations : des placements de tours sur des diagrammes de Ferrers. Le trio BWX s'est servi de ϕ^* pour démontrer le résultat suivant. Soit T un ensemble de motifs commençant tous par le préfixe $12 \cdots k$. Soit T' l'ensemble de motifs obtenu en remplaçant ce préfixe par $k \cdots 21$ dans chacun des motifs de T. Alors pour tout n, le nombre de permutations de $\mathcal{S}_n$ qui évitent T est égal au nombre de permutations de $\mathcal{S}_n$ qui évitent T'.

Le résultat principal de notre travail est que, très curieusement, la transformation itérée ϕ^* commute avec l'inversion des permutations (alors que ça n'est pas le cas de la transformation élémentaire ϕ). Plus généralement, elle commute avec la symétrie diagonale des placements de tours.

Un corollaire est l'analogue du résultat de BWX pour les involutions : avec des notations évidentes, $I_n(T) = I_n(T')$, où T et T' sont définis comme ci-dessus. Ce résultat avait été conjecturé par Jaggard.

1. Introduction

Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation of length n. Let $\tau = \tau_1 \cdots \tau_k$ be another permutation. An *occurrence* of τ in π is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$ of π that is order-isomorphic to τ. For instance, 246 is an occurrence of $\tau = 123$ in $\pi = 251436$. We say that π *avoids* τ if π contains no occurrence of τ. For instance, the above

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permutation π avoids 1234. The set of permutations of length n is denoted by $\mathcal{S}_n$, and $\mathcal{S}_n(\tau)$ denotes the set of τ-avoiding permutations of length n.

The idea of systematically studying pattern avoidance in permutations appeared in the mid-eighties [20]. The main problem in this field is to determine $S_n(\tau)$, the cardinality of $\mathcal{S}_n(\tau)$, for any given pattern τ. This question has subsequently been generalized and refined in various ways (see for instance [1, 4, 8, 17], and [16] for a recent survey). However, relatively little is known about the original question. The case of patterns of length 4 is not yet completed, since the pattern 1324 still remains unsolved. See [5, 9, 22, 21, 25] for other patterns of length 4.

For length 5 and beyond, all the solved cases follow from three important generic results. The first one, due to Gessel [9, 10], gives the generating function of the numbers $S_n(12\cdots k)$. The second one, due to Stankova and West [23], states that $S_n(231\tau) = S_n(312\tau)$ for any pattern τ on $\{4, 5, \ldots, k\}$. The third one, due to Backelin, West and Xin [3], shows that $S_n(12\cdots k\tau) = S_n(k\cdots 21\tau)$ for any pattern τ on the set $\{k+1, k+2, \ldots, \ell\}$. In the present paper an analogous result is established for pattern-avoiding *involutions*. We denote by $\mathcal{I}_n(\tau)$ the set of involutions avoiding τ, and by $I_n(\tau)$ its cardinality.

The systematic study of pattern avoiding involutions was also initiated in [20], continued in [9, 11] for increasing patterns, and then by Guibert in his thesis [12]. Guibert discovered experimentally that, for a surprisingly large number of patterns τ of length 4, $I_n(\tau)$ is the nth Motzkin number:

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(k+1)!(n-2k)!}.$$

This was already known for $\tau = 1234$ (see [18]), and consequently for $\tau = 4321$, thanks to the properties of the Schensted correspondence [19]. Guibert explained all the other instances of the Motzkin numbers, except for two of them: 2143 and 3214. However, he was able to describe a two-label generating tree for the class $\mathcal{I}_n(2143)$. Several years later, the Motzkin result for the pattern 2143 was at last derived from this tree: first in a bijective way [13], then using generating functions [6]. No simple generating tree could be described for involutions avoiding 3214, and it was only in 2003 that Jaggard [15] gave a proof of this final conjecture, inspired by [2]. More generally, he proved that for $k = 2$ or 3, $I_n(12\cdots k\tau) = I_n(k\cdots 21\tau)$ for all τ. He conjectured that this holds for all k, which we prove here.

We derive this from another result, which may be more interesting than its implication in terms of forbidden patterns. This result deals with a transformation ϕ^* that was defined in [3] to prove that $S_n(12\cdots k\tau) = S_n(k\cdots 21\tau)$. This transformation acts not only on permutations, but on more general objects called *full rook placements on a Ferrers shape* (see Section 2 for precise definitions). The map ϕ^* may, at first sight, appear as an *ad hoc* construction, but we prove that it has a remarkable, and far from obvious, property: it commutes with the inversion of a permutation, and more generally with the corresponding diagonal reflection of a full rook placement. (By the inversion of a permutation π we mean the map that sends π, seen as a bijection, to its inverse.)

The map ϕ^* is defined by iterating a transformation ϕ, which chooses a certain occurrence of the pattern $k\cdots 21$ and replaces it by an occurrence of $(k-1)\cdots 21k$. The map ϕ itself does *not* commute with the inversion of permutations, and our proof of the commutation theorem is actually quite complicated.

This strongly suggests that we need a better description of the map ϕ^*, on which the commutation theorem would become obvious. By analogy, let us recall what

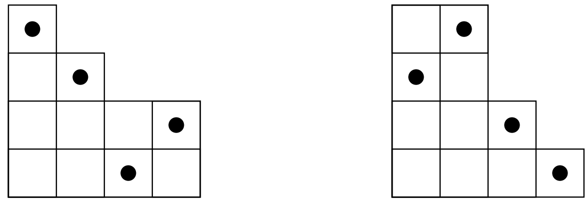


FIGURE 1. A full rook placement on a Ferrers board, and its inverse.

happened for the Schensted correspondence: the fact that the inversion of permutations exchanges the two tableaux only became completely clear with Viennot's description of the correspondence [24].

Actually, since the Schensted correspondence has nice properties regarding the monotone subsequences of permutations, and provides one of the best proofs of the identity $I_n(12\cdots k) = I_n(k\cdots 21)$, we suspect that the map ϕ^* might be related to this correspondence, or to an extension of it to rook placements.

2. Wilf equivalence for involutions

One of the main implications of this paper is the following.

Theorem 1. *Let $k \geq 1$. Let T be a set of patterns, each starting with the prefix $12\cdots k$. Let T' be the set of patterns obtained by replacing this prefix by $k\cdots 21$ in every pattern of T. Then, for all $n \geq 0$, the number of involutions of $\mathcal{S}_n$ that avoid T equals the number of involutions of $\mathcal{S}_n$ that avoid T'.*

In particular, the involutions avoiding $12\cdots k\tau$ and the involutions avoiding $k\cdots 21\tau$ are equinumerous, for any permutation τ of $\{k+1, k+2, \ldots, \ell\}$.

This theorem was proved by Jaggard for $k = 2$ and $k = 3$ [15]. It is the analogue, for involutions, of a result recently proved by Backelin, West and Xin for permutations [3]. Thus it is not very suprising that we follow their approach. This approach requires looking at pattern avoidance for slightly more general objects than permutations, namely, *full rook placements on a Ferrers board.*

Let λ be an integer partition, which we represent as a Ferrers board (Figure 1). A full rook placement on λ, or a *placement* for short, is a distribution of dots on this board, such that every row and column contains exactly one dot. This implies that the board has as many rows as columns.

Each cell of the board will be denoted by its coordinates: in the first placement of Figure 1, there is a dot in the cell $(1, 4)$. If the placement has n dots, we associate with it a permutation π of $\mathcal{S}_n$, defined by $\pi(i) = j$ if there is a dot in the cell (i, j). The permutation corresponding to the first placement of Figure 1 is $\pi = 4312$. This induces a bijection between placements on the $n \times n$ square and permutations of $\mathcal{S}_n$.

The *inverse* of a placement p on the board λ is the placement p' obtained by reflecting p and λ with respect to the main diagonal; it is thus a placement on the *conjugate* of λ, usually denoted by λ'. This terminology is of course an extension to placements of the classical terminology for permutations.

Definition 2. Let p be a placement on the board λ, and let π be the corresponding permutation. Let τ be a permutation of $\mathcal{S}_k$. We say that p *contains* τ if there exists in π an occurrence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ of τ such that the corresponding dots are contained in a *rectangular* sub-board of λ. In other words, the cell with coordinates $(i_k, \max_j \pi_{i_j})$ must belong to λ.

The placement of Figure 1 contains the pattern 12, but avoids the pattern 21, even though the associated permutation $\pi = 4312$ contains several occurrences of 21. We denote by $\mathcal{S}_\lambda(\tau)$ the set of placements on λ that avoid τ. If λ is self-conjugate, we denote by $\mathcal{I}_\lambda(\tau)$ the set of *symmetric* (that is, self-inverse) placements on λ that avoid τ. We denote by $S_\lambda(\tau)$ and $I_\lambda(\tau)$ the cardinalities of these sets.

In [2, 3, 23], it was shown that the notion of pattern avoidance in placements is well suited to deal with prefix exchanges in patterns. This was adapted by Jaggard [15] to involutions:

Proposition 3. *Let α and β be two involutions of $\mathcal{S}_k$. Let T_α be a set of patterns, each beginning with α. Let T_β be obtained by replacing, in each pattern of T_α, the prefix α by β. If, for every self-conjugate shape λ, $I_\lambda(\alpha) = I_\lambda(\beta)$, then $I_\lambda(T_\alpha) = I_\lambda(T_\beta)$ for every self-conjugate shape.*

Hence Theorem 1 will be proved if we can prove that $I_\lambda(12\cdots k) = I_\lambda(k\cdots 21)$ for any self-conjugate shape λ. A simple induction on k, combined with Proposition 3, shows that it is actually enough to prove the following:

Theorem 4. *Let λ be a self-conjugate shape. Then $I_\lambda(k\cdots 21) = I_\lambda((k-1)\cdots 21k)$.*

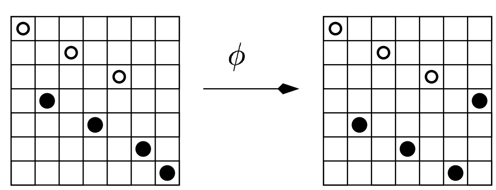
A similar result was proved in [3] for general (asymmetric) placements: for every shape λ, one has $S_\lambda(k\cdots 21) = S_\lambda((k-1)\cdots 21k)$. The proof relies on the description of a recursive bijection between the sets $\mathcal{S}_\lambda(k\cdots 21)$ and $\mathcal{S}_\lambda((k-1)\cdots 21k)$. What we prove here is that this complicated bijection actually *commutes with the inversion of a placement*, and this implies Theorem 4.

But let us first describe (and slightly generalize) the transformation defined by Backelin, West and Xin [3]. This transformation depends on k, which from now on is supposed to be fixed. Since Theorem 4 is trivial for $k = 1$, we assume $k \geq 2$.

Definition 5 (The transformation ϕ). Let p be a placement containing $k\cdots 21$, and let π be the associated permutation. To each occurrence of $k\cdots 21$ in p, there corresponds a decreasing subsequence of length k in π. The *$\mathcal{A}$-sequence* of p, denoted by $\mathcal{A}(p)$, is the smallest of these subsequences for the lexicographic order.

The corresponding dots in p form an occurrence of $k\cdots 21$. Rearrange these dots cyclically so as to form an occurrence of $(k-1)\cdots 21k$. The resulting placement is defined to be $\phi(p)$.

If p avoids $k\cdots 21$, we simply define $\phi(p) := p$. The transformation ϕ is also called the *$\mathcal{A}$-shift*.



$7\ \underline{4}\ 6\ \underline{3}\ 5\ \underline{2}\ \underline{1}$ $\qquad$ $7\ 3\ 6\ 2\ 5\ 1\ 4$

FIGURE 2. The $\mathcal{A}$-shift on the permutation 7 4 6 3 5 2 1, when $k = 4$.

An example is provided by Figure 2 (the letters of the $\mathcal{A}$-sequence are underlined, and the corresponding dots are black). It is easy to see that the $\mathcal{A}$-shift decreases the inversion number of the permutation associated with the placement (details will be given in the proof of Corollary 11). This implies that after finitely many iterations of

ϕ, there will be no more decreasing subsequences of length k in the placement. We denote by ϕ^* the iterated transformation, that recursively transforms *every* pattern $k\cdots 21$ into $(k-1)\cdots 21k$. For instance, with the permutation $\pi = 7\ 4\ 6\ 3\ 5\ 2\ 1$ of Figure 2 and $k=4$, we find

$$\pi = 7\ \underline{4}\ 6\ \underline{3}\ 5\ \underline{2}\ \underline{1} \longrightarrow \underline{7}\ \underline{3}\ 6\ \underline{2}\ 5\ \underline{1}\ 4 \longrightarrow 3\ 2\ 6\ 1\ 5\ 7\ 4 = \phi^*(\pi).$$

The main property of ϕ^* that was proved and used in [3] is the following:

Theorem 6 (The BWX bijection). *For every shape λ, the transformation ϕ^* induces a bijection from $\mathcal{S}_\lambda((k-1)\cdots 21k)$ to $\mathcal{S}_\lambda(k\cdots 21)$.*

The key to our paper is the following rather unexpected theorem.

Theorem 7 (Global commutation). *The transformation ϕ^* commutes with the inversion of a placement.*

For instance, with π as above, we have

$$\pi^{-1} = 7\ \underline{6}\ \underline{4}\ \underline{2}\ 5\ 3\ \underline{1} \longrightarrow \underline{7}\ \underline{4}\ \underline{2}\ \underline{1}\ 5\ 3\ 6 \longrightarrow 4\ 2\ 1\ 7\ 5\ 3\ 6 = \phi^*(\pi^{-1})$$

and we observe that

$$\phi^*(\pi^{-1}) = (\phi^*(\pi))^{-1}.$$

Note, however, that $\phi(\pi^{-1}) \neq (\phi(\pi))^{-1}$. Indeed, $\phi(\pi^{-1}) = 7\ 4\ 2\ 1\ 5\ 3\ 6$ while $(\phi(\pi))^{-1} = 6\ 4\ 2\ 7\ 5\ 3\ 1$, so that the elementary transformation ϕ, that is, the $\mathcal{A}$-shift, does not commute with the inversion.

Theorems 6 and 7 together imply that ϕ^* induces a bijection from $\mathcal{I}_\lambda((k-1)\cdots 21k)$ to $\mathcal{I}_\lambda(k\cdots 21)$, for every self-conjugate shape λ. This proves Theorem 4, and hence Theorem 1. The rest of the paper is devoted to proving Theorem 7, which we call the theorem of *global commutation.* By this, we mean that the inversion commutes with the global tranformation ϕ^* (but not with the elementary transformation ϕ).

Remarks

1. At first sight, our definition of the $\mathcal{A}$-sequence (Definition 5), does not seem to coincide with the definition given in [3]. Let $a_k\cdots a_2a_1$ denote the $\mathcal{A}$-sequence of the placement p, with $a_k > \cdots > a_1$. We identify this sequence with the corresponding set of dots in p. The dot a_k is the lowest dot that is the leftmost point in an occurrence of $k\cdots 21$ in p. Then a_{k-1} is the lowest dot such that a_ka_{k-1} is the beginning of an occurrence of $k\cdots 21$ in p, and so on.

However, in [3], the dot a_k is chosen as above, but then each of the next dots $a'_{k-1},\ldots,a'_1$ is chosen to be as far *left* as possible, and not as *low* as possible. Let us prove that the two procedures give the same sequence of dots. Assume not, and let $a_j \neq a'_j$ be the first (leftmost) point where the two sequences differ. By definition, a_j is lower than a'_j, and to the right of it. But then the sequence $a_{k-1}\cdots a_{j+1}a'_ja_j\cdots a_2a_1$ is an occurrence of the pattern $k\cdots 21$ in p, which is smaller than $a_k\cdots a_2a_1$ for the lexicographic order, a contradiction.

The fact that the $\mathcal{A}$-sequence can be defined in two different ways will be used very often in the paper.

2. At this stage, we have reduced the proof of Theorem 1 to the proof of the global commutation theorem, Theorem 7.

3. From local commutation to global commutation

In order to prove that ϕ^* commutes with the inversion of placements, it would naturally be tempting to prove that ϕ itself commutes with the inversion. However, this is not the case, as shown above. Given a placement p and its inverse p', we thus want to know how the placements $\phi(p)$ and $\phi(p')'$ differ.

Definition 8. For any shape λ and any placement p on λ, we define $\psi(p)$ by

$$\psi(p) := \phi(p')'.$$

Thus $\psi(p)$ is also a placement on λ.

Note that $\psi^m(p) = (\phi^m(p'))'$, so that the theorem of global commutation, Theorem 7, can be restated as $\psi^* = \phi^*$.

Combining the above definition of ψ with Definition 5 gives an alternative description of ψ.

Lemma 9 (The transformation ψ). *Let p be a placement containing $k \cdots 21$. Let $b_1, b_2, \ldots, b_k$ be defined recursively as follows: For all j, b_j is the leftmost dot such that $b_j \cdots b_2 b_1$ ends an occurrence of $k \cdots 21$ in p. We call $b_k \cdots b_2 b_1$ the $\mathcal{B}$-*sequence *of p, and denote it by $\mathcal{B}(p)$.*

Rearrange the k dots of the $\mathcal{B}$-sequence cyclically so as to form an occurrence of $(k-1) \cdots 21k$: the resulting placement is $\psi(p)$.

*If p avoids $k \cdots 21$, then $\psi(p) = p$. The transformation ψ is also called the $\mathcal{B}$-*shift.

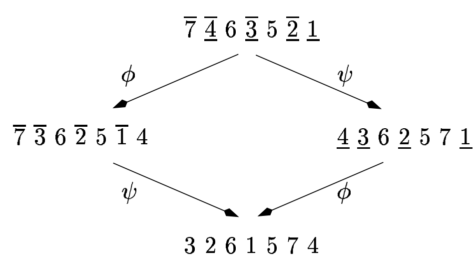
According to the first remark that concludes Section 2, we can alternatively define b_j, for $j \geq 2$, as the *lowest* dot such that $b_j \cdots b_2 b_1$ ends an occurrence of $k \cdots 21$ in p.

We have seen that, in general, ϕ does not commute with the inversion. That is, $\phi(p) \neq \psi(p)$ in general. The above lemma tells us that $\phi(p) = \psi(p)$ if and only if the $\mathcal{A}$-sequence and the $\mathcal{B}$-sequence of p coincide. If they do *not* coincide, then we still have the following remarkable property, whose proof is deferred to the very end of the paper.

Theorem 10 (Local commutation). *Let p be a placement for which the $\mathcal{A}$- and $\mathcal{B}$-sequences do not coincide. Then $\phi(p)$ and $\psi(p)$ still contain the pattern $k \cdots 21$, and*

$$\phi(\psi(p)) = \psi(\phi(p)).$$

For instance, for the permutation of Figure 2 and $k = 4$, we have the following commutative diagram, in which the underlined (resp. overlined) letters correspond to the $\mathcal{A}$-sequence (resp. $\mathcal{B}$-sequence):



A classical argument, which is sometimes stated in terms of *locally confluent* and *globally confluent rewriting systems* (see [14] and references therein), will show that Theorem 10 implies $\psi^* = \phi^*$, and actually the more general following corollary.

Corollary 11. *Let p be a placement. Any iterated application of the transformations ϕ and ψ yields ultimately the same placement, namely $\phi^*(p)$. Moreover, all the minimal sequences of transformations that yield $\phi^*(p)$ have the same length.*

Before we prove this corollary, let us illustrate it. We think of the set of permutations of length n as the set of vertices of an oriented graph, the edges of which are given by the maps ϕ and ψ. Figure 3 shows a connected component of this graph. The dotted edges represent ϕ while the plain edges represent ψ. The dashed edges correspond to the cases where ϕ and ψ coincide. We see that all the paths that start at a given point converge to the same point.

Proof. For any placement p, define the inversion number of p as the inversion number of the associated permutation π (that is, the number of pairs (i,j) such that $i<j$ and $\pi_i > \pi_j$). Assume p contains at least one occurrence of $k\cdots 21$, and let $i_1 < \cdots < i_k$ be the positions (abscissae) of the elements of the $\mathcal{A}$-sequence of p. A careful examination of the inversions of p and $\phi(p)$ shows that

$$\operatorname{inv}(p) - \operatorname{inv}(\phi(p)) = k-1+2\sum_{m=1}^{k-1} \operatorname{Card}\{i : i_m < i < i_{m+1} \text{ and } \pi_{i_1} > \pi_i > \pi_{i_{m+1}}\}.$$

In particular, $\operatorname{inv}(\phi(p)) < \operatorname{inv}(p)$. By symmetry, together with the fact that $\operatorname{inv}(\pi^{-1}) = \operatorname{inv}(\pi)$, it follows that $\operatorname{inv}(\psi(p)) < \operatorname{inv}(p)$ too.

We encode the compositions of the maps ϕ and ψ by words on the alphabet $\{\phi,\psi\}$. For instance, if u is the word $\phi\psi^2$, then $u(p) = \phi\psi^2(p)$. Let us prove, by induction on $\operatorname{inv}(p)$, the following two statements:
1. If u and v are two words such that $u(p)$ and $v(p)$ avoid $k\cdots 21$, then $u(p) = v(p)$.
2. Moreover, if u and v are minimal for this property (that is, for any non-trivial factorization $u = u_0u_1$, the placement $u_1(p)$ still contains an occurrence of $k\cdots 21$ — and similarly for v), then u and v have the same length.

If the first property holds for p, then $u(p) = v(p) = \phi^*(p)$. If the second property holds, we denote by $L(p)$ the length of any minimal word u such that $u(p)$ avoids $k\cdots 21$.

If π is the identity, then the two results are obvious. They remain obvious, with $L(p) = 0$, if p does not contain any occurrence of $k\cdots 21$.

Now assume p contains such an occurrence, and $u(p)$ and $v(p)$ avoid $k\cdots 21$. By assumption, neither u nor v is the empty word. Let f (resp. g) be the rightmost letter of u (resp. v), that is, the *first* transformation that is applied to p in the evaluation of $u(p)$ (resp. $v(p)$). Write $u = u'f$ and $v = v'g$.

If $f(p) = g(p)$, let q be the placement $f(p)$. Given that $\operatorname{inv}(q) < \operatorname{inv}(p)$, and that the placements $u(p) = u'(q)$ and $v(p) = v'(q)$ avoid $k\cdots 21$, both statements follow by induction.

If $f(p) \neq g(p)$, we may assume, without loss of generality, that $f = \phi$ and $g = \psi$. Let $q_1 = \phi(p)$, $q_2 = \psi(p)$ and $q = \phi(\psi(p) = \psi(\phi(p))$ (Theorem 10). The induction

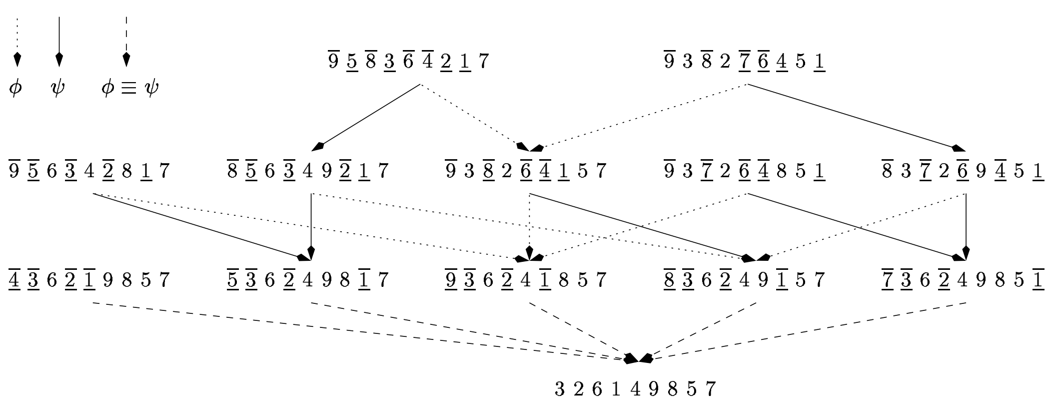
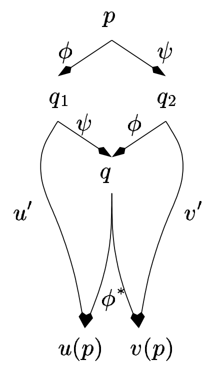


FIGURE 3. The action of ϕ and ψ on a part of $\mathcal{S}_9$, for $k=4$.

hypothesis, applied to q_1, gives $u'(q_1) = \phi^*(\psi(q_1)) = \phi^*(q)$, that is, $u(p) = \phi^*(q)$ (see the figure below). Similarly, $v'(q_2) = \phi^*(q_2) = \phi^*(q)$, that is, $v(p) = \phi^*(q)$. This proves the first statement. If u and v are minimal for p, then so are u' and v' for q_1 and q_2 respectively. By the first statement of Theorem 10, q_1 and q_2 still contain the pattern $k \cdots 21$, so $L(q) = L(q_1) - 1 = L(q_2) - 1$, and the words u' and v' have the same length. Consequently, u and v have the same length too. □



4. The local commutation theorem

We have reduced the proof of Theorem 1 to the proof of the local commutation theorem, Theorem 10. The rest of the paper is devoted to this proof, which turns out to be unexpectedly complicated. There is no question that one needs to find a more illuminating description of ϕ^*, or of $\phi \circ \psi$, which makes Theorems 7 and 10 clear.

In the full version of the paper, available on the arXiv [7], this theorem is first proved for permutations, and then extended to placements. In this extended abstract, we simply study one big example, and use it to describe the main steps of the proof (for permutations). This example is illustrated in Figure 4.

Example. Let π be the following permutation of length 21:

$$\pi = 17\ 21\ 20\ 16\ 19\ 18\ 13\ 15\ 11\ 14\ 12\ 8\ 10\ 9\ 7\ 4\ 2\ 6\ 5\ 3\ 1.$$

1. Let $k = 12$. The $\mathcal{A}$-sequence of π is

$$\mathcal{A}(\pi) = 17\ 16/15\ 14\ 12\ 10\ 9\ 7/6\ 5\ 3\ 1,$$

while its $\mathcal{B}$-sequence is

$$\mathcal{B}(\pi) = 21\ 20\ 19\ 18/15\ 14\ 12\ 10\ 9\ 7/4\ 2.$$

Observe that the intersection of $\mathcal{A}(\pi)$ and $\mathcal{B}(\pi)$ (delimited by '/') consists of the letters 15 14 12 10 9 7, and that they are consecutive both in $\mathcal{A}(\pi)$ and $\mathcal{B}(\pi)$. Also, $\mathcal{B}$ contains more letters than $\mathcal{A}$ before this intersection, while $\mathcal{A}$ contains more letters than $\mathcal{B}$ after the intersection. In the full version of the paper, we prove that this is always true.

2. Let us now apply the $\mathcal{B}$-shift to π. One finds:

$$\psi(\pi) = 17\ 20\ 19\ 16\ 18\ 15\ 13\ 14\ 11\ 12\ 10\ 8\ 9\ 7\ 4\ 2\ 21\ 6\ 5\ 3\ 1.$$

The new $\mathcal{A}$-sequence is now $\mathcal{A}(\psi(\pi)) = 17\ 16/15\ 13\ 11\ 10\ 8\ 7/6\ 5\ 3\ 1$. Observe that all the letters of $\mathcal{A}(\pi)$ that were before or after the intersection with $\mathcal{B}(\pi)$ are still in the new $\mathcal{A}$-sequence, as well as the first letter of the intersection. We prove that this is always true. In this example, the last letter of the intersection is still in the new $\mathcal{A}$-sequence, but this is not true in general.

By symmetry with respect to the main diagonal, after the $\mathcal{A}$-shift, the letters of $\mathcal{B}$ that were before or after the intersection are in the new $\mathcal{B}$-sequence, as well as the first letter of $\mathcal{A}$ following the intersection. This can be checked on our example:

$$\phi(\pi) = 16\ 21\ 20\ 15\ 19\ 18\ 13\ 14\ 11\ 12\ 10\ 8\ 9\ 7\ 6\ 4\ 2\ 5\ 3\ 1\ 17,$$

and the new $\mathcal{B}$-sequence is $\mathcal{B}(\phi(\pi)) = 21\ 20\ 19\ 18/13\ 11\ 10\ 8\ 7\ 6\ /4\ 2$.

3. Let $a_i = b_j$ denote the first (leftmost) point in $\mathcal{A}(\pi) \cap \mathcal{B}(\pi)$, and let $a_d = b_e$ be the last point in this intersection. We have seen that after the $\mathcal{B}$-shift, the new $\mathcal{A}$-sequence begins with $a_k \cdots a_i = 17\ 16\ 15$, and ends with $a_{d-1} \cdots a_1 = 6\ 5\ 3\ 1$. The letters in the center of the new $\mathcal{A}$-sequence, that is, the letters replacing $a_{i-1} \cdots a_d$, are $x_{i-1} \cdots x_d = 13\ 11\ 10\ 8\ 7$. Similarly, after the $\mathcal{A}$-shift, the new $\mathcal{B}$-sequence begins with $b_k \cdots b_{j+1} = 21\ 20\ 19\ 18$, and ends with $a_{d-1} b_{e-1} \cdots b_1 = 6\ 4\ 2$. The central letters are again $x_{i-1} \cdots x_d = 13\ 11\ 10\ 8\ 7$! (See Figure 4). This is not a coincidence; we prove in [7] that this always holds.

4. We finally combine all these properties to describe explicitly how the maps $\phi \circ \psi$ and $\psi \circ \phi$ act on a permutation π, and conclude that they yield the same permutation if the $\mathcal{A}$- and $\mathcal{B}$-sequences of π do not coincide.

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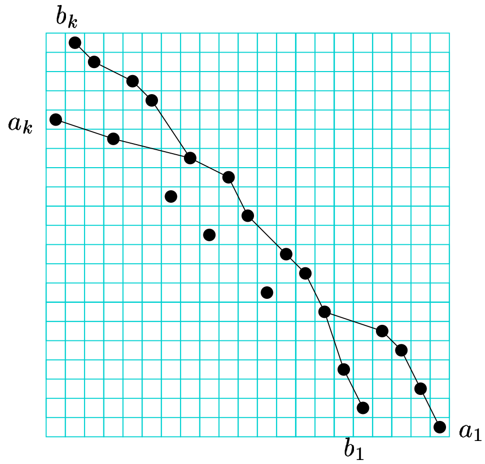
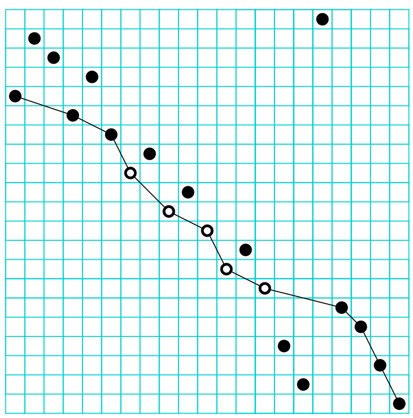
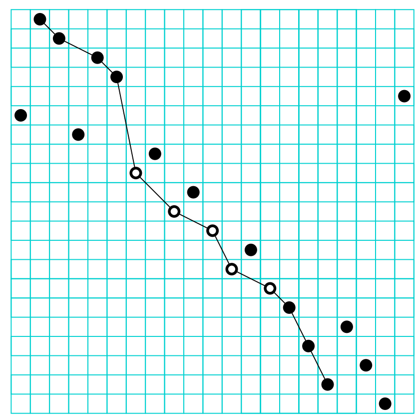




The $\mathcal{A}$-sequence after the $\mathcal{B}$-shift (ψ)



The $\mathcal{B}$-sequence after the $\mathcal{A}$-shift (ϕ)

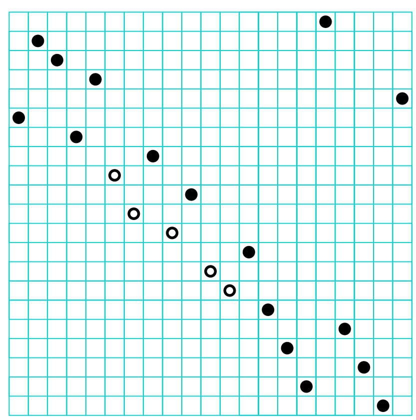


FIGURE 4. Top: A permutation π, with its $\mathcal{A}$- and $\mathcal{B}$-sequences shown. Left: After the $\mathcal{B}$-shift. Right: After the $\mathcal{A}$-shift. Bottom: After the composition of ϕ and ψ.

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