STABLE GROTHENDIECK POLYNOMIALS AND K-THEORETIC FACTOR SEQUENCES

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ABSTRACT. We give a nonrecursive combinatorial formula for the expansion of a stable Grothendieck polynomial in the basis of stable Grothendieck polynomials for partitions. The proof is based on a generalization of the Edelman-Greene insertion algorithm. This result is applied to prove a number of formulas and properties for K-theoretic quiver polynomials and Grothendieck polynomials. In particular we formulate and prove a K-theoretic analogue of Buch and Fulton's factor sequence formula for the cohomological quiver polynomials.

1. INTRODUCTION

1.1. **Stable Grothendieck polynomials.** For each permutation w there is a symmetric power series $G_w = G_w(x_1, x_2, \dots)$ called the stable Grothendieck polynomial for w. These power series were defined by Fomin and Kirillov [13, 12] as a limit of the ordinary Grothendieck polynomials of Lascoux and Schützenberger [18]. We recall this definition in Section 2. The term of lowest degree in G_w is the Stanley function (or stable Schubert polynomial) F_w. The *Stanley coefficients* in the Schur expansion of a Stanley function are interesting combinatorial invariants which generalize the Littlewood-Richardson coefficients. It was proved by Edelman and Greene [10] and Lascoux and Schützenberger [19] that Stanley coefficients are nonnegative. Various combinatorial rules have been given for these coefficients [11, 15, 23].

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, the *Grassmannian permutation* w_λ for λ is uniquely defined by the requirement that $w_\lambda(i) = i + \lambda_{k+1-i}$ for $1 \leq i \leq k$ and $w_\lambda(i) < w_\lambda(i+1)$ for $i \neq k$. The power series $G_\lambda := G_{w_\lambda}$ play a role in combinatorial K-theory similar to the role of Schur functions in cohomology. Buch has shown [3] that any stable Grothendieck polynomial G_w can be written as a finite linear combination

$$G_w = \sum_\lambda c_{w,\lambda} G_\lambda \qquad (1)$$

of stable Grothendieck polynomials indexed by partitions, using integer coefficients $c_{w,\lambda}$ that generalize the Stanley coefficients [2]. Lascoux gave a recursive formula for stable Grothendieck polynomials which confirms a conjecture that these coefficients have signs that alternate with degree, i.e. $(-1)^{|\lambda|-\ell(w)} c_{w,\lambda} \geq 0$ [17]. The central result of this paper is a new formula for the coefficients $c_{w,\lambda}$ which generalizes Fomin and Greene's rule [11] for Stanley coefficients.

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To state our formula, we need the 0-*Hecke monoid*, which is the quotient of the free monoid of all finite words in the alphabet $\{1, 2, \dots\}$ by the relations

$$pp \equiv p \qquad \text{for all } p \tag{2}$$

$$pqp \equiv qpq \qquad \text{for all } p, q \tag{3}$$

$$pq \equiv qp \qquad \text{for } |p-q| \geq 2. \tag{4}$$

There is a bijection between the 0-Hecke monoid and the infinite symmetric group $S_\infty = \bigcup_{n \geq 1} S_n$. Given any word a there is a unique permutation $w \in S_\infty$ such that $a \equiv b$ for some (or equivalently every) reduced word b of w. In this case we write $w(a) = w$ and say that a is a *Hecke word* for w. Notice that the reduced words for w are precisely the Hecke words for w that are of minimum length. The *Hecke product* $u \cdot v$ of two permutations $u, v \in S_\infty$ is the element $w(ab) \in S_\infty$ where a and b are words such that $w(a) = u$ and $w(b) = v$.

We use the English notation for partitions and tableaux. A *decreasing tableau*[1] is a Young tableau whose rows decrease strictly from left to right, and whose columns decrease strictly from top to bottom. The (row reading) *word* of a tableau is obtained by reading the rows of the tableau from left to right, starting with the bottom row, followed by the next-to-bottom row, etc. We shall identify a tableau with its word.

Theorem 1. *For any permutation w we have*

$$G_w = \sum_\lambda c_{w,\lambda} G_\lambda$$

where $c_{w,\lambda}$ is equal to $(-1)^{|\lambda| - \ell(w)}$ times the number of decreasing tableaux T of shape λ such that $w(T) = w$.

Example 2. Let $w = 31524$. The decreasing tableaux that are Hecke words for w are:

$$\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}$$

So $G_w = G_{22} + G_{31} - G_{32}$.

When the permutation w is 321-avoiding, Theorem 1 also generalizes Buch's rule for the coefficients $c_{w,\lambda}$ in terms of *set-valued* tableaux [3], in the sense that there is an explicit bijection between the relevant decreasing and set-valued tableaux. As a consequence, we obtain a new proof of the set-valued Littlewood-Richardson rule for K-theoretic Schubert structure constants on Grassmannians, as well as an alternative rule based on decreasing tableaux.

1.2. **Hecke insertion.** Fomin and Kirillov proved that the monomial coefficients of (stable) Grothendieck polynomials are counted by combinatorial objects called resolved wiring diagrams (also known as FK-graphs, pipe dreams, or nonreduced RC-graphs) [13, 12]. This formula was used in [3] to express the monomial coefficients of stable Grothendieck polynomials for partitions in terms of set-valued tableaux. We prove Theorem 1 by exhibiting an explicit bijection between the set of FK-graphs for a permutation w and the set of pairs (T, U) where T is a decreasing tableau with $w(T) = w$ and U is a set-valued tableau of the same shape as T. This

[1] The use of decreasing tableaux rather than increasing, is merely for convenience in the definition of a K-theoretic factor sequence.

bijection, called *Hecke insertion*, is the technical core of our paper. It is a subtle extension of the Edelman-Greene insertion algorithm from the set of reduced words to the set of all (Hecke) words.

Hecke insertion allows us to define a product of decreasing tableaux $(T_1, T_2) \mapsto T_1 \cdot T_2$ (see section 3.5). This product is used in the definition of K-theoretic factor sequences in the next section.

1.3. **Quiver coefficients.** Our main application of Theorem 1 concerns quiver coefficients. A sequence of vector bundle morphisms $E_0 \to E_1 \to \cdots \to E_n$ over a variety X together with a set of *rank conditions* $r = \{r_{ij}\}$ for $0 \le i \le j \le n$ define a *quiver variety* $\Omega_r \subset X$ of points where each composition of bundle maps $E_i \to E_j$ has rank at most r_{ij}. We demand that the rank conditions can *occur*, which is equivalent to the requirement that $r_{ii} = e_i := \operatorname{rank}(E_i)$ for all i, $0 \le r_{ij} \le \min(r_{i,j-1}, r_{i+1,j})$ for $0 \le i < j \le n$, and $r_{ij} + r_{i-1,j+1} \ge r_{i-1,j} + r_{i,j+1}$ for $0 < i \le j < n$. We also demand that the bundle maps are generic, so that the quiver variety Ω_r obtains its expected codimension $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. Buch and Fulton proved a formula for the cohomology class of Ω_r [5] which was later generalized to K-theory by Buch [2]. The K-theory version states that the Grothendieck class of Ω_r is given by

$$[\mathcal{O}_{\Omega_r}] = \sum_{\mu} c_\mu(r)\, G_{\mu_1}(E_1 - E_0) G_{\mu_2}(E_2 - E_1) \cdots G_{\mu_n}(E_n - E_{n-1})\,, \tag{5}$$

where the sum is over sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions μ_i such that $\sum |\mu_i| \ge d(r)$ and each partition μ_i can be contained in the rectangle $e_i \times e_{i-1}$ with e_i rows and e_{i-1} columns. The coefficients $c_\mu(r)$ in this formula are integers called *quiver coefficients*. When $\sum |\mu_i| = d(r)$, the coefficient $c_\mu(r)$ also appears in the cohomology formula from [5] and is called a *cohomological quiver coefficient*. It was conjectured that cohomological quiver coefficients are nonnegative, while K-theoretic quiver coefficients have signs that alternate with degree, i.e. $(-1)^{\sum |\mu_i| - d(r)} c_\mu(r) \ge 0$. The conjecture for cohomological quiver coefficients was proved by Knutson, Miller, and Shimozono [16], after which the general case was proved by Buch [4] and Miller [21]. Both of the latter proofs are based on the *ratio formula* from [16], which expresses the Grothendieck class of a quiver variety as a ratio of two double Grothendieck polynomials.

A more precise conjecture for cohomological quiver coefficients was posed in [5], which asserts that any such coefficient $c_\mu(r)$ counts the number of *factor sequences* of tableaux with shapes given by the sequence of partitions μ. A factor sequence is a sequence of semistandard Young tableaux that can be obtained by performing a series of plactic factorizations and multiplications of chosen tableaux arranged in a *tableau diagram*. For a specific choice of tableau diagram, this more precise conjecture was also proved by Knutson, Miller and Shimozono [16]. It appears, however, that the original definition of factor sequences from [5] has no natural generalization to K-theory.

In this paper, we prove that K-theoretic quiver coefficients are counted by a new type of factor sequence. These sequences are constructed from a tableau diagram of decreasing tableaux using the same algorithm that defines the original factor sequences, except that the plactic product is replaced with a product $(U, T) \mapsto U \cdot T$ of decreasing tableaux which respects Hecke words (see section 3.5).

For each $0 \leq i < j \leq n$ let R_{ij} be a rectangle with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns. Let U_{ij} be the unique decreasing tableau of shape R_{ij} such that the lower left box contains the number $r_{i,j-1}$, and each box is one larger than the box below it and one smaller than the box to the left of it. For example, if $r_{i,j-1} = 6$, $r_{i+1,j} = 5$, and $r_{ij} = 2$ then

$$U_{ij} = \begin{array}{|c|c|c|c|}\hline 8 & 7 & 6 & 5 \\ \hline 7 & 6 & 5 & 4 \\ \hline 6 & 5 & 4 & 3 \\ \hline \end{array}\,.$$

These tableaux U_{ij} can be arranged in a triangular tableau diagram as in [5, §4]. We define a *K-theoretic factor sequence* for the rank conditions r by induction on n. If $n = 1$ then the only factor sequence is the sequence (U_{01}) consisting of the only tableau in the tableau diagram. If $n \geq 2$ then the numbers $\overline{r} = \{\overline{r}_{ij} : 0 \leq i \leq j \leq n-1\}$ defined by $\overline{r}_{ij} = r_{i,j+1}$ form a valid set of rank conditions corresponding to a sequence of $n-1$ bundle maps. In this case, a factor sequence for r is any sequence of the form $(U_{01} \cdot A_1, \ldots, B_{i-1} \cdot U_{i-1,i} \cdot A_i, \cdots, B_{n-1} \cdot U_{n-1,n})$, for a choice of decreasing tableaux A_i and B_i such that $(A_1 \cdot B_1, \ldots, A_{n-1} \cdot B_{n-1})$ is a factor sequence for $\overline{r}$.

Theorem 3. *The K-theoretic quiver coefficient $c_\mu(r)$ is equal to $(-1)^{\sum |\mu_i| - d(r)}$ times the number of K-theoretic factor sequences $(T_1, \ldots, T_n)$ for the rank conditions r, such that T_i has shape μ_i for each i.*

Using results about Demazure characters it was proved in [16] that cohomological quiver coefficients are special cases of the Stanley coefficients associated to the *Zelevinsky permutation* $z(r)$ [24, 16]. We recall the definition of this permutation in Section 4. In this paper we prove more generally that the K-theoretic quiver coefficients are special cases of the coefficients $c_{z(r),\lambda}$ in the expansion (1) of the stable Grothendieck polynomial for $z(r)$. This result also sharpens the fact from [4, 8] that quiver coefficients are special cases of the decomposition coefficients of Grothendieck polynomials studied in [6] (see section 1.4.1). Given a sequence of partitions $\mu = (\mu_1, \ldots, \mu_n)$ such that μ_i is contained in the rectangle $e_i \times e_{i-1}$, let $\lambda(\mu)$ be the partition obtained by concatenating the partitions $(e_0 + \cdots + e_{n-2-i})^{e_i} + \mu_{n-i}$ for $i = 0, \ldots, n-1$. Our proof of the following identity is based on a bijection between the K-theoretic factor sequences for r and the decreasing tableaux representing $z(r)$.

Theorem 4. *For any set of rank conditions r and sequence of partitions μ we have $c_\mu(r) = c_{z(r),\lambda(\mu)}$.*

Central to the proof of the nonnegativity of cohomological quiver coefficients given in [16] is the *stable component formula*, which writes the cohomology class of a quiver variety as a sum of products of Stanley functions. This sum is over all *lace diagrams* representing the rank conditions r, which have the smallest possible number of crossings. The K-theoretic version of the component formula from [4, 21] states that

$$[\mathcal{O}_{\Omega_r}] = \sum_{(w_1,\ldots,w_n)} (-1)^{\sum \ell(w_i) - d(r)} G_{w_1}(E_1 - E_0) \cdots G_{w_n}(E_n - E_{n-1}) \tag{6}$$

where the sum is over a generalization of minimal lace diagrams, which was named *KMS-factorizations* in [4]. We recall this definition in Section 4. The K-theoretic factor sequences also have the following characterization.

Theorem 5. *A sequence of decreasing tableaux $(T_1, \dots, T_n)$ is a K-theoretic factor sequence for the rank conditions r if and only if $(w(T_1), \dots, w(T_n))$ is a KMS-factorization for r.*

We will use the statement of Theorem 5 as our definition of K-theoretic factor sequences. When this definition is used, Theorem 3 is an immediate consequence of Theorem 1 combined with the K-theoretic stable component formula (6). The above inductive construction of factor sequences is then derived from a similar construction of KMS-factorizations proved in [4].

1.4. **Other applications.** We list other applications for the methods presented in this extended abstract that are not developed here but which will appear in the full version of this paper.

1.4.1. *Decomposition coefficients of Grothendieck polynomials.* Fulton's *universal Schubert polynomials* from [14] describe certain quiver varieties associated to a sequence of vector bundles $E_1 \to \cdots \to E_{n-1} \to E_n \to F_n \to F_{n-1} \to \cdots \to F_1$ over X, such that $\mathrm{rank}(E_i) = \mathrm{rank}(F_i) = i$ for each i. Given a permutation $w \in S_{n+1}$, we let $\Omega_w \subset X$ be the degeneracy locus of points where the rank of each composed map $E_q \to F_p$ is at most equal to the number of integers $1 \leq i \leq p$ such that $w(i) \leq q$. The quiver formula (5) can be applied to give a formula

$$[\mathcal{O}_{\Omega_w}] = \sum_{\mu} c^{(n)}_{w,\mu} G_{\mu_1}(E_2 - E_1) \cdots G_{\mu_n}(F_n - E_n) \cdots G_{\mu_{2n-1}}(F_1 - F_2) \tag{7}$$

for the Grothendieck class of Ω_w, where the coefficients $c^{(n)}_{w,\mu}$ are special cases of quiver coefficients. It was shown in [2] that the coefficients $c_{w,\lambda}$ of the expansion (1) of the stable Grothendieck polynomial for w can be obtained as the specializations $c^{(n)}_{w,(\emptyset^{n-1},\lambda,\emptyset^{n-1})}$, where $\emptyset^{n-1}$ denotes a sequence of $n-1$ empty partitions. More generally, it was proved in [6, Thm. 4] that the coefficients $c^{(n)}_{w,\lambda}$ can be used to expand a double Grothendieck polynomial as a linear combination of products of stable Grothendieck polynomials applied to disjoint intervals of variables. In [6], the formula (7) was also used to prove that

$$[\mathcal{O}_{\Omega_w}] = \sum (-1)^{\ell(u_1 \cdots u_{2n-1} w)} G_{u_1}(E_2 - E_1) \cdots G_{u_n}(F_n - E_n) \cdots G_{u_{2n-1}}(F_1 - F_2)$$

where this sum is over all sequences of permutations $(u_1, \dots, u_{2n-1})$ such that $u_i \in S_{\min(i,2n-i)+1}$ and w is equal to the Hecke product $u_1 \cdot u_2 \cdots u_{2n-1}$. Combining this with Theorem 1, we obtain the following generalization of [7, Thm. 1].

Theorem 6. *The coefficient $c^{(n)}_{w,\mu}$ of (7) is equal to $(-1)^{\sum |\mu_i| - \ell(w)}$ times the number of sequences $(T_1, \dots, T_{2n-1})$ of decreasing tableaux of shapes $(\mu_1, \dots, \mu_{2n-1})$, such that the entries of T_i are at most $\min(i, 2n-i)$ and $w(T_1 T_2 \cdots T_n) = w$.*

1.4.2. *Expansion of Grothendieck polynomials.* Theorem 1 may be refined to give an expansion of Grothendieck polynomials. The cohomological analogue is the combinatorial rule [20, 22, 23] for the expansion of a Schubert polynomial as a positive sum of Demazure characters [9]. Taking a suitable limit, the Schubert polynomial becomes a Stanley function and each Demazure character becomes a Schur function. Using divided difference operators, one may introduce a new basis of $\mathbb{Z}[x_1, x_2, \dots]$ called Grothendieck-Demazure polynomials. We have a conjectural expansion of a Grothendieck polynomial as an alternating sum of Grothendieck-Demazure polynomials. In the limit this expansion becomes that in Theorem 1.

2. Grothendieck polynomials

2.1. **Definition.** Lascoux and Schützenberger's original definition of Grothendieck polynomials was based on divided difference operators [19]. In this paper we will use Fomin and Kirillov's construction of these polynomials [12], in notation that generalizes Billey, Jockusch, and Stanley's formula for Schubert polynomials [1].

Define a *compatible pair* to be a pair (a, i) of words $a = a_1 a_2 \cdots a_p$ and $i = i_1 i_2 \cdots i_p$, such that $i_1 \leq i_2 \leq \cdots \leq i_p$, and so that $i_j < i_{j+1}$ whenever $a_j \leq a_{j+1}$. For $w \in S_\infty$, the *stable Grothendieck polynomial* for w is given by [12]

$$G_w = \sum_{(a,i)} (-1)^{\ell(i)-\ell(w)} x^i \tag{8}$$

where the sum is over all compatible pairs (a, i) such that $w(a) = w$. Here $\ell(i)$ is the common length of a and i, and $x^i = x_{i_1} x_{i_2} \cdots x_{i_{\ell(i)}}$. The ordinary *Grothendieck polynomial* $\mathfrak{G}_w$ is given by the same sum (8), but only including the compatible pairs (a, i) for which $a_j \geq i_j$ for each j. The *Schubert polynomial* for w is equal to the lowest term of $\mathfrak{G}_w$, while the *Stanley function* F_w is the lowest term of G_w.

We also require Buch's formula [3] for the stable Grothendieck polynomial G_λ. A *set-valued tableau* of shape λ is a filling of the boxes of λ with finite nonempty sets of positive integers, such that these sets are weakly increasing along rows and strictly increasing down columns. In other words, all integers in a box must be smaller than or equal to the integers in the box to the right of it, and strictly smaller than the integers in the box below it. For a set-valued tableau S, let x^S denote the monomial where the exponent of x_i is equal to the number of boxes containing the integer i, and let $|S|$ be the degree of this monomial. We have [3]

$$G_\lambda = \sum_S (-1)^{|S|-|\lambda|} x^S \tag{9}$$

where S runs over all set-valued tableaux of shape λ.

2.2. **The required bijection.** For any permutation $w \in S_n$, it follows from (8) and the symmetry of stable Grothendieck polynomials that $G_w = G_{w_0 w^{-1} w_0}$, where $w_0 = w_0^{(n)}$ is the longest permutation in S_n. Theorem 1 is therefore equivalent to the following statement. Define an *increasing tableau* to be a Young tableau with strictly increasing rows and columns.

Theorem 7. *The coefficient $c_{w,\lambda}$ is equal to $(-1)^{|\lambda|-\ell(w)}$ times the number of increasing tableaux T of shape λ such that $w(T) = w^{-1}$.*

In light of (8) and (9), to prove this theorem, it suffices to establish a bijection $(a, i) \mapsto (T, U)$ between all compatible pairs (a, i) such that $w(a) = w$, and all pairs of tableaux (T, U) of the same shape, such that T is increasing with $w(T) = w^{-1}$ and U is set-valued. In addition, this bijection must satisfy that $x^U = x^i$.

3. Hecke Insertion

Let M_1 be the set of pairs (Y, x) where Y is an increasing tableau and x is a letter. Let M_2 be the set of triples (Z, r, α) where Z is an increasing tableau, Z has a corner cell (r, c) in the r-th row, and $\alpha \in \{0, 1\}$. *Hecke insertion* defines a bijection

$$\begin{aligned} \Phi : M_1 &\to M_2 \\ (Y, x) &\mapsto (Z, r, \alpha) \end{aligned} \tag{10}$$

such that either

(1) $\alpha = 0$ and $\mathrm{shape}(Z) = \mathrm{shape}(Y)$.
(2) $\alpha = 1$ and $\mathrm{shape}(Z) = \mathrm{shape}(Y) \bigsqcup \{(r, c)\}$.

This bijection defines the Hecke insertion of x into the increasing tableau Y, resulting in the increasing tableau Z, ending at the corner cell (r, c) of Z. Unlike ordinary Schensted insertion, it is possible for a Hecke insertion not to add a cell to the tableau: a new cell is created if and only if $\alpha = 1$.

3.1. **Hecke (Row) Insertion.** Let $(Y, x') \in M_1$. Inductively for $i \geq 1$, suppose that the first $i - 1$ rows of Y have been suitably modified, and that the number x is being inserted into the i-th row. Let y be the smallest entry in the i-th row such that $y > x$. If no such element exists, set $y = \infty$ and let z be the last entry in the i-th row. If the i-th row is empty then set $z = 0$.

(H1) If $y = \infty$ and $z = x$: The insertion terminates. Let (r, c) be the corner cell in the same column as z and $\alpha = 0$.
(H2) If $y = \infty$ and $z < x$: The insertion terminates. (a) If adjoining x to the end of the i-th row, results in an increasing tableau, then do so, with $r = i$ and $\alpha = 1$. (b) If not (and this can only happen if $i > 1$), let (r, c) be the corner cell in the same column as z and $\alpha = 0$.
(H3) If $y < \infty$, replace y by x if the result is an increasing tableau and otherwise leave the row unchanged. Continue by inserting y into the $(i + 1)$-th row.

Let Z be the resulting increasing tableau. This algorithm produces a triple $\Phi(Y, x') = (Z, r, \alpha) \in M_2$. Write $Z = (Y \xleftarrow{H} x')$.

Remark 8. A increasing tableau is produced in cases (H2) and (H3) unless x would be placed just to the right of, or just below, another x.

Example 9.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \xleftarrow{H} 3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

3 is inserted into the first row, which contains 3. So 5 is inserted into the second row, whose largest value is $z = 5$. This is case (H1). Then $\alpha = 0$ and $r = 3$, since $(3, 2)$ is the cell at the bottom of the column of z.

Example 10.

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \xleftarrow{H} 2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

2 is inserted into the first row, which contains 2. 4 is inserted into the second row, displacing the 5. The 5 is inserted into the third row, where it comes to rest. This is case (H2a). Then $\alpha = 1$ and $r = 3$.

Example 11.

$$\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & 4 & \\ 5 & & \end{array} \overset{H}{\longleftarrow} 2 = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & 4 & \\ 5 & & \end{array}$$

2 is inserted into the first row, which contains a 2. 4 is inserted into the second row, which has largest entry $z = 3$. 4 can't come to rest at the cell $(2,3)$ since that is just below the 4 in cell $(1,3)$. Case (H2b) holds. Then $\alpha = 0$ and $r = 3$ because $(3,2)$ is the cell at the bottom of the column of z.

Example 12.

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{array} \overset{H}{\longleftarrow} 1 = \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 5 & \end{array}$$

1 is inserted into the first row, which already contains a 1. So 3 is inserted into the second row. It would have replaced 4, but this replacement would place a 3 directly below another 3, violating the increasing tableau condition. So the second row is unchanged and 4 is inserted into the third row. Similarly 4 cannot replace 5. So 5 is inserted into the fourth row, where it comes to rest in the cell $(4,1)$ with $\alpha = 1$.

3.2. **Reverse Hecke insertion.** The inverse map $\Psi : M_2 \to M_1$ is defined as follows. Let $(Z, r, \alpha) \in M_2$, (r, c) the corner cell in the r-th row of Z, and $y = Z_{r,c}$. If $\alpha = 1$ then remove y. In any case, reverse insert y up into the previous row.

Whenever a value y is reverse inserted into a row, let x be the largest entry in the row such that $x < y$. If replacing x by y yields an increasing tableau then do so; otherwise leave the row unchanged. In any case, reverse insert x into the previous row.

Eventually a value x' reverse inserted out of the first row, leaving behind an increasing tableau Y. Call x' the output value. Define $\Psi(Z, r, \alpha) = (Y, x')$.

Remark 13. Note that the only obstructions for replacing x by y, are when the entry below or to the right of x already contains y.

Example 14. Let us apply reverse Hecke insertion to the tableau computed in Example 12 at the cell $(4,1)$ with $\alpha = 1$. Since $\alpha = 1$ the entry 5 in cell $(4,1)$ is removed. Then 5 is reverse inserted into the third row. Since 5 is already in the third row, the third row remains unchanged and 3 is reverse inserted into the second row. 3 cannot replace 2 because this would place a 3 directly atop a 3, creating a vertical violation of the increasing tableau condition. The second row is unchanged and 2 is reverse inserted into the first row. 2 cannot replace 1 for the same reason. The first row is unchanged and 1 is the output value. This recovers the initial tableau of Example 12.

Proposition 15. *The maps Φ and Ψ are mutually inverse bijections.*

3.3. **Properties of Hecke insertion.** Hecke insertion respects Hecke words.

Lemma 16. *Suppose reverse Hecke insertion of the tableau T at some corner cell results in the tableau T' and the output value x. Then $w(T) = w(T'x)$.*

Hecke insertion also satisfies the following Pieri property.

Lemma 17. *Suppose we first reverse Hecke insert starting from one corner C_1 of T, and then reverse Hecke insert from a corner C_2 of the modification of T. Then the first output value is strictly smaller than the second output value if and only if C_1 is strictly lower than C_2.*

3.4. **Proof of Theorem 7 via column Hecke Robinson-Schensted.** In this section we give the bijection that was sought in section 2.2. We may define Hecke column insertion by switching the roles of rows and columns in Hecke row insertion. Write $\Phi' : M_1 \to M_2$ for this bijection.

Let (a, i) be as in section 2.2 with $a = a_1 a_2 \cdots a_p$ and $i = i_1 i_2 \cdots i_p$. We start with the empty tableau pair $(T_0, U_0) = (\varnothing, \varnothing)$. If (T_{j-1}, U_{j-1}) has been defined for some $j \geq 1$, let $(T_j, s_j, \alpha_j) = \Phi'(T_{j-1}, a_j)$. Let U_j be obtained from U_{j-1} by adjoining a new cell to the end of the s_j-th row containing the singleton set $\{i_j\}$ if $\alpha_j = 1$. Otherwise U_j is obtained from U_{j-1} by putting i_j into the existing set in the corner cell in row s_j. Define $(T, U) = (T_p, U_p)$. The map $(a, i) \mapsto (T, U)$ has the desired properties. U is a set-valued tableau by Lemma 17 and $x^i = x^U$ by definition. The fact that $w(T) = w^{-1}$ follows from Lemma 16 combined with the fact that the reversal of a word gives a bijection between the Hecke words for w and those for w^{-1}. This proves Theorems 7 and 1.

3.5. **Product of decreasing tableaux.** For use with factor sequences, we define the product of the decreasing tableaux T_1 and T_2. Consider the variant of Hecke insertion in which larger numbers bump smaller numbers. In other words, we reverse the order of the positive integers in the algorithm of Section 3.1. Let $T_1 \cdot T_2$ be the decreasing tableau obtained by inserting the word of T_2 into T_1 using this variant of Hecke insertion. More precisely, if $a_1 a_2 \cdots a_p$ is the word of T_2 then $T_1 \cdot T_2 = (((T_1 \xleftarrow{H} a_1) \xleftarrow{H} a_2) \cdots) \xleftarrow{H} a_p$. This product has the following properties.

Lemma 18. *(1) For decreasing tableaux T_1, T_2 we have $w(T_1 \cdot T_2) = w(T_1) \cdot w(T_2)$.*

(2) Suppose a decreasing tableau T is cut along a vertical line into T_{left} and T_{right}. Then $T = T_{left} \cdot T_{right}$.

(3) Suppose T is cut along a horizontal line into tableaux T_{bottom} and T_{top}. Then $T = T_{bottom} \cdot T_{top}$.

Our applications to factor sequences require that the product of decreasing tableaux satisfies the properties of this lemma. When the concatenation of the words of T_1 and T_2 is a reduced word of a permutation, then these conditions imply that $T_1 \cdot T_2$ agrees with the Coxeter-Knuth product, but no such uniqueness statement holds in general. The product $T_1 \cdot T_2$ also fails to be associative.

4. Quiver varieties

Let $r = \{r_{ij}\}$ be a set of rank conditions for $0 \leq i, j \leq n$, and set $N = e_0 + \cdots + e_n$ where $e_i = r_{ii}$. A result of Zelevinsky shows that when the base variety X is a product of matrix spaces, the quiver variety $\Omega_r \subset X$ is identical to a dense open subset of a Schubert variety [24]. The *Zelevinsky permutation* corresponding to this Schubert variety was used in [16] to prove the ratio formula for quiver varieties.

With the notation from [4], the Zelevinsky permutation can be constructed as a product of permutations as follows (see [16, Prop. 1.6] for a different construction). Extend the rank conditions $r = \{r_{ij}\}$ by setting $r_{ij} = e_j + \cdots + e_i$ for $0 \leq j < i \leq n$.

Then define decreasing tableaux U_{ij} as in the introduction, but for all $0 \le i < n$ and $0 < j \le n$. The corresponding permutation $W_{ij} = w(U_{ij})$ is given by

$$W_{ij}(p) = \begin{cases} p + r_{i,j-1} - r_{ij} & \text{if } r_{ij} < p \le r_{i+1,j} \\ p - r_{i+1,j} + r_{ij} & \text{if } r_{i+1,j} < p \le r_{i+1,j} + r_{i,j-1} - r_{ij} \\ p & \text{otherwise.} \end{cases}$$

The Zelevinsky permutation can now be defined by $z(r) = \prod_{j=1}^{n} \prod_{i=0}^{n-1} W_{ij}$.

For each $1 \le j \le n-1$ we set $\delta_j = W_{jj} W_{j+1,j} \cdots W_{n-1,j} \in S_N$. A *KMS-factorization* for the rank conditions r is any sequence $(w_1, \dots, w_n)$ of permutations with $w_i \in S_{e_{i-1}+e_i}$, such that the Zelevinsky permutation $z(r)$ is equal to the Hecke product

$$w_1 \cdot \delta_1 \cdot w_2 \cdot \delta_2 \cdots \delta_{n-1} \cdot w_n \,.$$

These sequences of permutations generalize the notion of a minimal lace diagram from [16] and give the index set in the K-theoretic stable component formula (6) from [4, 21].

We define a *K-theoretic factor sequence* for the rank conditions r to be any sequence $(T_1, \dots, T_n)$ of decreasing tableaux, such that $(w(T_1), \dots, w(T_n))$ is a KMS-factorization for r. As noted in the introduction, this definition means that Theorem 3 is a consequence of Theorem 1 combined with the stable component formula (6). To obtain the inductive definition of factor sequences we need the following result proved in [4, Thm. 7], which shows that KMS-factorizations can themselves be defined as 'factor sequences'.

Theorem 19. *(a) If $(w_1, \dots, w_n)$ is a KMS-factorization for r, then each permutation w_i has a reduced factorization $w_i = v_{i-1} \cdot W_{i-1,i} \cdot u_i$ with $v_{i-1} \in S_{e_{i-1}}$ and $u_i \in S_{e_i}$, such that $v_0 = u_n = 1$.*

(b) Let $u_1, v_1, \dots, u_{n-1}, v_{n-1}$ be permutations with $u_i, v_i \in S_{e_i}$. Then the sequence $(W_{01} \cdot u_1, v_1 \cdot W_{12} \cdot u_2, \dots, v_{n-1} \cdot W_{n-1,n})$ is a KMS-factorization for r if and only if $(u_1 \cdot v_1, u_2 \cdot v_2, \dots, u_{n-1} \cdot v_{n-1})$ is a KMS-factorization for $\overline{r}$.

We also need the following statement.

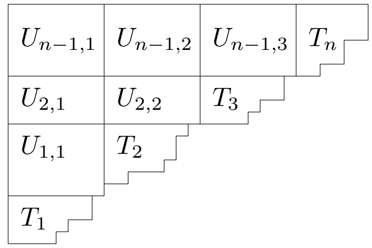
Lemma 20. *Let T be any decreasing tableau such that $w(T) \in S_m$, and for some integers $a, b < m$ we have $w(T)(p) \le b$ for all $a < p \le m$. Then T contains the rectangle $R = (m-a) \times (m-b)$ in its upper left corner. The upper-left box of R equals $m-1$, and the boxes of R decrease by one for each step down or to the right.*

Let $(U, T) \mapsto U \cdot T$ be the product of decreasing tableaux defined in section 3.5.

Corollary 21. *A sequence of decreasing tableaux $(T_1, \dots, T_n)$ is a K-theoretic factor sequence for the rank conditions r if and only if there exist decreasing tableaux A_i, B_i for $1 \le i \le n-1$, such that $T_i = B_{i-1} \cdot U_{i-1,i} \cdot A_i$ for each i (with $B_0 = A_n = \emptyset$) and $(A_1 \cdot B_1, \dots, A_{n-1} \cdot B_{n-1})$ is a K-theoretic factor sequence for $\overline{r}$.*

Given a sequence $(T_1, \dots, T_n)$ of decreasing tableaux, such that each tableau T_i can be contained in the rectangle $e_i \times e_{i-1}$ and all entries of T_i are smaller than $e_{i-1} + e_i$, we let $\Phi(T_1, \dots, T_n)$ denote the decreasing tableau constructed from this sequence as well as the tableaux U_{ij} for $i \geq j$ as follows.

$$\Phi(T_1, \dots, T_n) \quad =$$



Notice that the upper-left box of $U_{n-1,1}$ is equal to $N-1$, and the boxes in the union of tableaux U_{ij} decrease by one for each step down or to the right. Theorem 4 follows from the following proposition combined with Theorems 1 and 3.

Proposition 22. *The map $(T_1, \dots, T_n) \mapsto \Phi(T_1, \dots, T_n)$ gives a bijection of the set of all K-theoretic factor sequences for r with the set of all decreasing tableaux representing $z(r)$.*

Proof. Since the permutation of a decreasing tableau can be defined as the south-west to north-east Hecke product of the simple reflections given by the boxes of the tableau, it follows from the definition of KMS-factorizations that $(T_1, \dots, T_n)$ is a factor sequence if and only if $\Phi(T_1, \dots, T_n)$ represents the Zelevinsky permutation $z(r)$. It remains to show that any decreasing tableau T representing $z(r)$ contains the arrangement of rectangular tableaux U_{ij} in its upper-left corner, and has no boxes strictly south-east of the tableaux U_{ii} for $1 \leq i \leq n-1$. The inclusion of the tableaux U_{ij} in T follows from Lemma 20 because $z(r) \in S_N$ and for each $0 < i \leq n$ and $p > r_{ni}$ we have $z(r)(p) \leq r_{i0}$, see [16, Prop 1.6] or [4, Lemma 3.1].

To see that T contains no boxes strictly south-east of U_{ii}, we use that the Grothendieck polynomial $\mathfrak{G}_{\widehat{z}(r)}(x_1, \dots, x_N)$ is separately symmetric in each group of variables $\{x_p \mid r_{n,i} < p \leq r_{n,i-1}\}$, where $\widehat{z}(r) = w_0^{(N)} z(r)^{-1} w_0^{(N)}$ and $w_0^{(N)}$ is the longest permutation in S_N. This is true because the descent positions of $\widehat{z}(r)$ are contained in the set $\{r_{nj} \mid 0 < j \leq n\}$. It follows that the exponent of $x_{r_{ni}+1}$ in any monomial of $\mathfrak{G}_{\widehat{z}(r)}(x_1, \dots, x_N)$ is less than or equal to $N - r_{n,i-1} = r_{i-2,0}$. Now T can be used to construct a unique compatible pair (a, k) for $\widehat{z}(r)$, such that T contains the integer p in some box of row q if and only if $(a_l, k_l) = (N-p, q)$ for some l. Since this pair contributes the monomial x^k to $\mathfrak{G}_{\widehat{z}(r)}(x_1, \dots, x_N)$, it follows that row $r_{ni}+1$ of T has at most $r_{i-2,0}$ boxes. This means exactly that T contains no boxes south-east of $U_{i-1,i-1}$, as required. $\square$

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