NEW EXPLICIT EXPRESSION FOR $A_n^{(1)}$ SUPERNOMIALS

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Abstract. A new fermionic formula for type $A_{n-1}^{(1)}$ supernomials is presented. This formula is different from the one given by Hatayama et al. [6]. A new set of unrestricted rigged configurations is introduced which is in bijection with the unrestricted crystal paths.

Résumé. On présente une nouvelle formule fermionique pour les supernomiales de type $A_{n-1}^{(1)}$. Cette formule est différente de celle donnée par Hatayama et al. [6]. On présente un nouvel ensemble de configurations 'gréées' sans restriction qui est en bijection avec l'ensemble des chemins cristallins sans restriction.

1. INTRODUCTION

Supernomial coefficients were first introduced in [15] as q-analogs of the coefficient of x^a in the expansion of $\prod_{j=1}^{N}(1+x+x^2+\cdots+x^j)^{L_j}$. They are generalizations of q-multinomial coefficients and were used to prove Bose-Fermi or Rogers–Ramanujan-type identities. The supernomials of [15] can be naturally associated with the algebra $A_1^{(1)}$. Motivated by crystal base theory, supernomials can be defined for any affine Kac–Moody Lie algebra as generating functions of unrestricted paths with energy statistics [16, 6, 7, 5]. An explicit formula for the $A_n^{(1)}$ supernomials for completely symmetric and completely antisymmetric crystals was proved in [6]. This formula is called fermionic as it is a manifestly positive expression. The purpose of this note is to give a new explicit expression for supernomials of type $A_n^{(1)}$.

Our motivation to seek an explicit expression for supernomials is their appearance in generalizations of the Bailey lemma [2]. Bailey's lemma is a very powerful method to prove Rogers–Ramanujan-type identities. In [16] a type A_n generalization of Bailey's lemma was conjectured which was subsequently proven in [18]. A type A_2 Bailey chain, which yields an infinite family of identities, was given in [1].

Recently, fermionic expressions for generating functions of unrestricted paths for type $A_1^{(1)}$ have also surfaced in connection with box-ball systems. Takagi [17] establishes a bijection between box-ball systems and a new set of rigged configurations to prove a fermionic formula for the q-binomial coefficient. This bijection extends a bijection of Kirillov and Reshetikhin [9] between semi-standard Young tableaux and rigged configurations to unrestricted paths.

In this note we define a new set of unrestricted rigged configurations for type $A_n^{(1)}$. A bijection between this new set and the set of unrestricted crystal paths is given which preserves the statistics. In particular this yields a new fermionic expression for the supernomial coefficients of type $A_n^{(1)}$. Subsequently, a crystal structure on the new set of rigged configurations has been defined [12] which can be used to establish the bijection and the correct properties of the statistics. These results generalize to other affine simply-laced root systems. In this note we give an algorithmic definition of the bijection by extending the definition in [10]. Details will be available in [4].

This paper is structured as follows. In section 2 we review crystals of type $A_n^{(1)}$, the definition of unrestricted paths and the definition of supernomials as generating functions of unrestricted paths with energy statistics. In section 3 we give our new definition of unrestricted rigged configurations and derive from this a fermionic expression for the generating function of unrestricted rigged configurations graded by cocharge. Our main results are stated in section 4. The fermionic formula

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of section 3 yield an explicit expression for the supernomials. This result is based on a bijection between unrestricted paths and unrestricted rigged configurations.

2. UNRESTRICTED PATHS AND SUPERNOMIALS

$A^{(1)}_{n-1}$-supernomials were first introduced in [16] as generating functions of unrestricted paths graded by an energy function. An unrestricted path is an element in the tensor product of crystals $B = B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$. As a set the crystal $B^{r,s}$ of type $A^{(1)}_{n-1}$ is the set of all column-strict Young tableaux of shape (s^r) over the alphabet $\{1, 2, \ldots, n\}$. Kashiwara [8] introduced the notion of crystals and crystal graphs as a combinatorial means to study representations of quantum algebras. In particular, there are Kashiwara operators e_i, f_i defined on the elements in $B^{r,s}$ for $0 \le i \le n$. However, for the purpose of this note, we do not require the explicit action of e_i and f_i.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-tuple of nonnegative integers. The set of **unrestricted paths** is defined as

$$\mathcal{P}(B, \lambda) = \{b \in B \mid \mathrm{wt}(b) = \lambda\}.$$

Here $\mathrm{wt}(b) = (w_1, \ldots, w_n)$ is the weight of b where w_i counts the number of letters i in b.

Example 2.1. For $B = B^{1,1} \otimes B^{2,2} \otimes B^{3,1}$ of type A_3 and $\lambda = (2, 3, 1, 2)$ the path

$$b \;=\; \begin{array}{|c|}\hline 2 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

is in $\mathcal{P}(B, \lambda)$.

There exists a crystal isomorphism $R : B^{r,s} \otimes B^{r',s'} \to B^{r',s'} \otimes B^{r,s}$, called the **combinatorial** R**-matrix**. Combinatorially it is given as follows. Let $b \in B^{r,s}$ and $b' \in B^{r',s'}$. The product $b \cdot b'$ of two tableaux is defined as the Schensted insertion of b' into b. Then $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$ is the unique pair of tableaux such that $b \cdot b' = \tilde{b}' \cdot \tilde{b}$.

The **local energy function** $H : B^{r,s} \otimes B^{r',s'} \to \mathbb{Z}$ is defined as follows. For $b \otimes b' \in B^{r,s} \otimes B^{r',s'}$, $H(b \otimes b')$ is the number of boxes of the shape of $b \cdot b'$ outside the shape obtained by concatenating (s^r) and $({s'}^{r'})$.

Example 2.2. For

$$b \otimes b' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

we have

$$b \cdot b' = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} = \tilde{b}' \cdot \tilde{b}.$$

so that

$$R(b \otimes b') = \tilde{b}' \otimes \tilde{b} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}.$$

Since the concatentation of $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$ is $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$, the local energy function $H(b \otimes b') = 0$.

Now let $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ be a k-fold tensor product of crystals. The **tail energy function** $\overleftarrow{D} : B \to \mathbb{Z}$ is given by

$$\overleftarrow{D} = \sum_{1 \le i < j \le k} H_{j-1} R_{j-2} \cdots R_{i+1} R_i,$$

where H_i (resp. R_i) is the local energy function (resp. combinatorial R-matrix) acting on the i-th and $(i+1)$-th tensor factors.

Definition 2.3. The q-supernomial coefficient is the generating function of unrestricted paths graded by the tail energy function

$$S_{B,\lambda}(q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{\overleftarrow{D}(b)}.$$

3. UNRESTRICTED RIGGED CONFIGURATIONS AND FERMIONIC FORMULA

Rigged configurations are combinatorial objects invented to label the solutions of the Bethe equations, which give the eigenvalues of the Hamiltonian of the underlying physical model [3]. Motivated by the fact that representation theoretically the eigenvectors and eigenvalues can also be labelled by Young tableaux, Kirillov and Reshetikhin [9] gave a bijection between tableaux and rigged configurations. This result and generalizations thereof were proven in [10].

In terms of crystal base theory, the bijection is between highest weight paths and rigged configurations. The new result of this note is an extension of this bijection to a bijection between unrestricted paths and a new set of rigged configurations, which we define in this section. In [12] we also define a crystal structure on this new set of rigged configurations.

Let $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ and denote by $L = (L_i^{(a)} \mid (a,i) \in \mathcal{H})$ the multiplicity array of B, where $L_i^{(a)}$ is the multiplicity of $B^{a,i}$ in B. Here $\mathcal{H} = \overline{I} \times \mathbb{Z}_{>0}$ and $\overline{I} = \{1,2,\ldots,n-1\}$ is the index set of the Dynkin diagram A_{n-1}. The sequence of partitions $\nu = \{\nu^{(a)} \mid a \in \overline{I}\}$ is a (L,λ)-**configuration** if

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \Lambda_a - \lambda, \tag{3.1}$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$. Note that we do not require λ to be a dominant weight here. The **(quasi-)vacancy number** of a configuration is defined as

$$p_i^{(a)} = \sum_{j\geq 1} \min(i,j) L_j^{(a)} - \sum_{(b,j)\in\mathcal{H}} (\alpha_a|\alpha_b) \min(i,j) m_j^{(b)}.$$

Here $(\cdot|\cdot)$ is the normalized invariant form on the weight lattice P such that $(\alpha_i|\alpha_j)$ is the Cartan matrix. Let $\mathrm{C}(L,\lambda)$ be the set of all (L,λ)-configurations. We call $p_i^{(a)}$ quasi-vacancy number to indicate that they can actually be negative in our setting. For the rest of the paper we will simply call them vacancy numbers.

In the usual setting a rigged configuration (ν, J) consists of a configuration $\nu \in \mathrm{C}(L,\lambda)$ together with a double sequence of partitions $J = \{J^{(a,i)} \mid (a,i) \in \mathcal{H}\}$ such that the partition $J^{(a,i)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle. In particular this requires that $p_i^{(a)} \geq 0$. For unrestricted paths we need a bigger set, where the lower bound on the parts in $J^{(a,i)}$ can be less than zero.

To define the lower bounds we need the following notation. Let $\lambda' = (c_1, c_2, \ldots, c_{n-1})^t$ where $c_k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$. We also set $c_0 = c_1$. Let $\mathcal{A}(\lambda')$ be the set of tableaux of shape λ' such that the entries in column k are from the set $\{1,2,\ldots,c_{k-1}\}$ and are strictly decreasing along each column.

Example 3.1. For $n=4$ and $\lambda = (0,1,1,1)$, the set $\mathcal{A}(\lambda')$ consists of the following tableaux

$$\begin{array}{|c|c|c|}\hline 3&3&2\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&3&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&3&1\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&3&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}.$$

Given $t \in \mathcal{A}(\lambda')$, we define the **lower bound** as

$$M_i^{(a)}(t) = -\sum_{j=1}^{c_a} \chi(i \geq t_{j,a}) + \sum_{j=1}^{c_{a+1}} \chi(i \geq t_{j,a+1}),$$

where $t_{j,a}$ denotes the entry in row j and column a of t, and $\chi(S) = 1$ if the the statement S is true and $\chi(S) = 0$ otherwise.

Let $M, p, m \in \mathbb{Z}$ such that $m \geq 0$. A (M, p, m)-quasipartition μ is a tuple of integers $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ such that $M \leq \mu_m \leq \mu_{m-1} \leq \cdots \leq \mu_1 \leq p$. Each μ_i is called a part of μ. Note that for $M = 0$ this would be a partition with at most m parts each not exceeding p.

Definition 3.2. An **unrestricted rigged configuration** (ν, J) is a configuration $\nu \in \mathrm{C}(L, \lambda)$ together with a sequence $J = \{J^{(a,i)} \mid (a,i) \in \mathcal{H}\}$ where $J^{(a,i)}$ is a $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-quasipartition for some $t \in \mathcal{A}(\lambda')$. Denote the set of all unrestricted rigged configurations corresponding to (L, λ) by $\mathrm{RC}(L, \lambda)$.

Remark 3.3.

(1) Note that this definition is similar to the definition of level-restricted rigged configurations [13, Definition 5.5]. Whereas for level-restricted rigged configurations the vacancy number had to be modified according to tableaux in a certain set, here the lower bounds are modified.

(2) For type A_1 we have $\lambda = (\lambda_1, \lambda_2)$ so that $\mathcal{A} = \{t\}$ contains just the single tableau

$$t = \begin{array}{|c|} \hline \lambda_2 \\ \hline \lambda_2 - 1 \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}.$$

In this case $M_i(t) = -\sum_{j=1}^{\lambda_2} \chi(i \geq t_{j,1}) = -i$. This agrees with the findings of [17].

The quasipartition $J^{(a,i)}$ is called **singular** if it has a part of size $p_i^{(a)}$. It is often useful to view an (unrestricted) rigged configuration (ν, J) as a sequence of partitions ν where the parts of size i in $\nu^{(a)}$ are labeled by the parts of $J^{(a,i)}$. The pair (i, x) where i is a part of $\nu^{(a)}$ and x is a part of $J^{(a,i)}$ is called a **string** of the a-th rigged partition $(\nu, J)^{(a)}$. The label x is called a **rigging**.

Example 3.4. Let $n = 4$, $\lambda = (2, 2, 1, 1)$, $L_1^{(1)} = 6$ and all other $L_i^{(a)} = 0$. Then

$$(\nu, J) = \begin{array}{|c|c|c|l} \hline & & & -2 \\ \hline & \multicolumn{3}{|l}{0} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|l} \hline & & 0 \\ \hline \end{array} \quad \begin{array}{|c|l} \hline & -1 \\ \hline \end{array}$$

is an unrestricted rigged configuration in $\mathrm{RC}(L, \lambda)$, where we have written the parts of $J^{(a,i)}$ next to the parts of length i in partition $\nu^{(a)}$. To see that the riggings form quasipartitions, let us write the vacancy numbers $p_i^{(a)}$ next to the parts of length i in partition $\nu^{(a)}$:

$$\begin{array}{|c|c|c|l} \hline & & & 0 \\ \hline & \multicolumn{3}{|l}{3} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|l} \hline & & 0 \\ \hline \end{array} \quad \begin{array}{|c|l} \hline & -1 \\ \hline \end{array}.$$

This shows that the labels are indeed all weakly below the vacancy numbers. For

$$\begin{array}{|c|c|c|} \hline 4 & 4 & 1 \\ \hline 3 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \in \mathcal{A}(\lambda')$$

we get the lower bounds

$$\begin{array}{|c|c|c|l} \hline & & & -2 \\ \hline & \multicolumn{3}{|l}{-1} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|l} \hline & & 0 \\ \hline \end{array} \quad \begin{array}{|c|l} \hline & -1 \\ \hline \end{array},$$

which are less or equal to the riggings in (ν, J).

The following statistics can be defined on the set of unrestricted rigged configurations. For $(\nu, J) \in \mathrm{RC}(L, \lambda)$ let

$$cc(\nu, J) = cc(\nu) + \sum_{(a,i) \in \mathcal{H}} |J^{(a,i)}|,$$

where $|J^{(a,i)}|$ is the sum of all parts of the quasipartition $J^{(a,i)}$ and

$$cc(\nu) = \frac{1}{2} \sum_{a,b \in \overline{I}} \sum_{j,k \geq 1} (\alpha_a \mid \alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}.$$

Definition 3.5. The RC polynomial is defined as

$$\mathrm{RC}_{L,\lambda}(q) = \sum_{(\nu,J)\in\mathrm{RC}(L,\lambda)} q^{cc(\nu,J)}.$$

The RC polynomial is in fact S_n-symmetric in the weight λ. This is not obvious from its definition as both (3.1) and the lower bounds are not symmetric with respect to λ.

Let $\mathcal{SA}(\lambda')$ be the set of all nonempty subsets of $\mathcal{A}(\lambda')$ and set

$$M_i^{(a)}(S) = \max\{M_i^{(a)}(t) \mid t \in S\} \qquad \text{for } S \in \mathcal{SA}(\lambda').$$

By inclusion-exclusion the set of all allowed riggings for a given $\nu \in \mathrm{C}(L,\lambda)$ is

$$\bigcup_{S\in\mathcal{SA}(\lambda')} (-1)^{|S|+1}\{J \mid J^{(a,i)} \text{ is a } (M_i^{(a)}(S), p_i^{(a)}, m_i^{(a)})\text{-quasipartition}\}.$$

The q-binomial coefficient $\begin{bmatrix} m+p \\ m \end{bmatrix}$, defined as

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p}$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$, is the generating function of partitions with at most m parts each not exceeding p. Hence the polynomial $\mathrm{RC}_{L,\lambda}(q)$ may be rewritten as

$$\mathrm{RC}_{L,\lambda}(q) = \sum_{S\in\mathcal{SA}(\lambda')} (-1)^{|S|+1} \sum_{\nu\in\mathrm{C}(L,\lambda)} q^{cc(\nu)+\sum_{(a,i)\in\mathcal{H}} m_i^{(a)} M_i^{(a)}(S)} \times \prod_{(a,i)\in\mathcal{H}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} - M_i^{(a)}(S) \\ m_i^{(a)} \end{bmatrix}$$

called **fermionic formula**.

4. Main results

In this section we relate the fermionic formula for the RC polynomial of section 3 and the q-supernomial coefficients of section 2.

Theorem 4.1. *If L is the multiplicity array for the crystal B, then $S_{B,\lambda}(q) = \mathrm{RC}_{L,\lambda}(q)$.*

This theorem follows immediately from the following result.

Theorem 4.2. *There exists a bijection $\Phi : \mathcal{P}(B,\lambda) \to \mathrm{RC}(L,\lambda)$ which preserves the statistics, that is, $\overleftarrow{D}(b) = cc(\Phi(b))$ for all $b \in \mathcal{P}(B,\lambda)$.*

A proof of Theorem 4.2 is given in [4, 12].

In [12] a crystal structure is defined on the set of unrestricted rigged configurations which is the same as the crystal structure on paths. The highest weight elements are given by the usual rigged configurations and highest weight paths, respectively, for which Theorem 4.2 is known to hold by [10]. Since the statistics is constant on all classical crystal components, the proof of Theorem 4.2 follows in general. It should be noted that the results in [12] hold for all for all simply-laced types, not just type $A_{n-1}^{(1)}$. Hence Theorem 4.2 holds whenever there is a corresponding bijection for the highest weight elements (for example for type $D_n^{(1)}$ for symmetric powers [14] and antisymmetric powers [11]). It is expected that using virtual crystals and the method of folding Dynkin diagrams, these results can be extended to other affine root systems.

In this note, which is a summary of [4], we take a different approach and define the map Φ algorithmically which generalizes the bijection of [10]. To define Φ we first need to define certain maps on paths and rigged configurations. These maps correspond to the following operations on crystals:

(1) If $B = B^{1,1} \otimes B'$, let $\mathrm{lh}(B) = B'$. This operation is called **left-hat**.

(2) If $B = B^{r,s} \otimes B'$ with $s \geq 2$, let $\mathrm{ls}(B) = B^{r,1} \otimes B^{r,s-1} \otimes B'$. This operation is called **left-split**.
(3) If $B = B^{r,1} \otimes B'$ with $r \geq 2$, let $\mathrm{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$. This operation is called **box-split**.

In analogy we define $\mathrm{lh}(L)$ (resp. $\mathrm{ls}(L)$, $\mathrm{lb}(L)$) to be the multiplicity array of $\mathrm{lh}(B)$ (resp. $\mathrm{ls}(B)$, $\mathrm{lb}(B)$), if L is the multiplicity array of B. The corresponding maps on crystal elements are given by:

(1) Let $b = c \otimes b' \in B^{1,1} \otimes B'$. Then $\mathrm{lh}(b) = b'$.
(2) Let $b = c \otimes b' \in B^{r,s} \otimes B'$, where $c = c_1 c_2 \cdots c_s$ and c_i denotes the i-th column of c. Then $\mathrm{ls}(b) = c_1 \otimes c_2 \cdots c_s \otimes b'$.
(3) Let $b = \begin{array}{|c|}\hline b_1 \\ \hline b_2 \\ \hline \vdots \\ \hline b_r \\ \hline \end{array} \otimes b' \in B^{r,1} \otimes B'$, where $b_1 < \cdots < b_r$. Then $\mathrm{lb}(b) = \begin{array}{|c|}\hline b_r \\ \hline \end{array} \otimes \begin{array}{|c|}\hline b_1 \\ \hline \vdots \\ \hline b_{r-1} \\ \hline \end{array} \otimes b'$.

In the next subsection we define the corresponding maps on rigged configurations, and give the bijection in subsection 4.2.

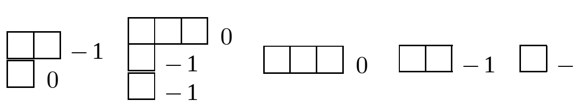
4.1. **Operations on rigged configurations.** Suppose $L_1^{(1)} > 0$. The main algorithm on rigged configurations as defined in [9, 10] for admissible rigged configurations can be extended to our setting. For a tuple of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$, let λ^- be the set of all nonnegative tuples $\mu = (\mu_1, \ldots, \mu_n)$ such that $\lambda - \mu = e_r$ for some $1 \leq r \leq n$ where e_r is the canonical r-th unit vector in $\mathbb{Z}^n$. Define $\delta : \mathrm{RC}(L, \lambda) \to \bigcup_{\mu \in \lambda^-} \mathrm{RC}(\mathrm{lh}(L), \mu)$ by the following algorithm. Let $(\nu, J) \in \mathrm{RC}(L, \lambda)$. Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n-1$ or until stopped. Find the smallest index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a+1$. Set all undefined $\ell^{(a)}$ to ∞.

The new rigged configuration $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly

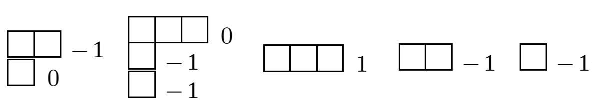
$$m_i^{(a)}(\tilde{\nu}) = m_i^{(a)}(\nu) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu)$ for $i = \ell^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu})$ for $i = \ell^{(a)} - 1$, and leaving it unchanged otherwise. Then $\delta(\nu, J) \in \mathrm{RC}(\mathrm{lh}(L), \mu)$ where $\mu = \lambda - e_{\mathrm{rk}(\nu,J)}$.

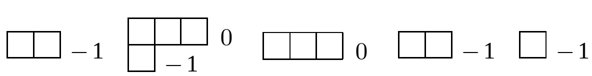
Example 4.3. Let L be the multiplicity array of $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,3}$ and $\lambda = (2, 2, 2, 1, 1, 1)$. Then

$(\nu, J) =$  $\in \mathrm{RC}(L, \lambda)$.

Writing the vacancy numbers next to each part instead of the riggings we get



Hence $\ell^{(1)} = \ell^{(2)} = 1$ and all other $\ell^{(a)} = \infty$, so that

$\delta(\nu, J) =$ 

Also $cc(\nu, J) = 2$.

Let $s \geq 2$. Suppose $B = B^{r,s} \otimes B'$ and L the corresponding multiplicity array. Note that $\mathrm{C}(L,\lambda) \subset \mathrm{C}(\mathrm{ls}(L),\lambda)$. Under this inclusion map, the vacancy number $p_i^{(a)}$ for ν increases by $\delta_{a,r}\chi(i<s)$. Hence there is a well-defined injective map $i : \mathrm{RC}(L,\lambda) \to \mathrm{RC}(\mathrm{ls}(L),\lambda)$ given by $i(\nu,J) = (\nu,J)$.

Suppose $r \geq 2$ and $B = B^{r,1} \otimes B'$ with multiplicity array L. Then there is an injection $j : \mathrm{RC}(L,\lambda) \to \mathrm{RC}(\mathrm{lb}(L),\lambda)$ defined by adding singular strings of length 1 to $(\nu,J)^{(a)}$ for $1 \leq a < r$. Moreover the vacancy numbers stay the same.

4.2. **Bijection.** The map $\Phi : \mathcal{P}(B,\lambda) \to \mathrm{RC}(L,\lambda)$ is defined by various commutative diagrams. Note that it is possible to go from $B = B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$ to the empty crystal via successive application of lh, ls and lb.

Definition 4.4. Define that map $\Phi : \mathcal{P}(B,\lambda) \to \mathrm{RC}(L,\lambda)$ such that the empty path maps to the empty rigged configuration, and:

(1) Suppose $B = B^{1,1} \otimes B'$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L,\lambda) \\ {\scriptstyle \mathrm{lh}}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ \displaystyle\bigcup_{\mu \in \lambda^-} \mathcal{P}(\mathrm{lh}(B),\mu) & \xrightarrow[\Phi]{} & \displaystyle\bigcup_{\mu \in \lambda^-} \mathrm{RC}(\mathrm{lh}(L),\mu) \end{array}$$

commutes.

(2) Suppose $B = B^{r,s} \otimes B'$ with $s \geq 2$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L,\lambda) \\ {\scriptstyle \mathrm{ls}}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathcal{P}(\mathrm{ls}(B),\lambda) & \xrightarrow[\Phi]{} & \mathrm{RC}(\mathrm{ls}(L),\lambda) \end{array}$$

(3) Suppose $B = B^{r,1} \otimes B'$ with $r \geq 2$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L,\lambda) \\ {\scriptstyle \mathrm{lb}}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \mathcal{P}(\mathrm{lb}(B),\lambda) & \xrightarrow[\Phi]{} & \mathrm{RC}(\mathrm{lb}(L),\lambda) \end{array}$$

It is shown in [4] that the map Φ of Definition 4.4 is indeed a well-defined bijection.

Example 4.5. Let $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,3}$ and $\lambda = (2,2,2,1,1,1)$. Then

$$b = \boxed{3} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \in \mathcal{P}(B,\lambda)$$

and $\Phi(b)$ is the rigged configuration (ν,J) of Example 4.3. We have $\overleftarrow{D}(b) = cc(\nu,J) = 2$.

Example 4.6. Let $n = 4$, $B = B^{2,2} \otimes B^{2,1}$ and $\lambda = (2,2,1,1)$. Then the multiplicity array is $L_1^{(2)} = 1, L_2^{(2)} = 1$ and $L_i^{(a)} = 0$ for all other (a,i). There are 7 possible unrestricted paths in $\mathcal{P}(B,\lambda)$. For each path $b \in \mathcal{P}(B,\lambda)$ the corresponding rigged configuration $(\nu,J) = \Phi(b)$ together with the tail energy and cocharge is summarized below.

$$b = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \qquad (\nu, J) = \quad \square\ 0 \quad \begin{array}{l}\square\ -1\\ \square\ -1\end{array} \quad \square\ 0 \qquad \overleftarrow{D}(b) = 0 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \qquad (\nu, J) = \quad \square\ -1 \quad \begin{array}{l}\square\ 0\\ \square\ 0\end{array} \quad \square\ 0 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \qquad (\nu, J) = \quad \square\ 0 \quad \begin{array}{l}\square\ 0\\ \square\ 0\end{array} \quad \square\ -1 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \qquad (\nu, J) = \quad \square\ 0 \quad \begin{array}{l}\square\ 0\\ \square\ -1\end{array} \quad \square\ 0 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \qquad (\nu, J) = \quad \square\ 0 \quad \begin{array}{l}\square\ 0\\ \square\ 0\end{array} \quad \square\ 0 \qquad \overleftarrow{D}(b) = 2 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \qquad (\nu, J) = \quad \square\ -1 \quad \square\square\ 0 \quad \square\ -1 \qquad \overleftarrow{D}(b) = 0 = cc(\nu, J)$$

$$b = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \qquad (\nu, J) = \quad \square\ -1 \quad \square\square\ 1 \quad \square\ -1 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J)$$

The supernomial in this case is $\mathrm{RC}_{L,\lambda}(q) = 2 + 4q + q^2 = S_{B,\lambda}(q)$.

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