

# A $q$ -analogue of the Kostant partition function and twisted representations

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November 21, 2004

## Abstract

We discuss a family of representations of Lie groups related to quantization with respect to the Dirac signature operator. The combinatorics of these twisted representations is similar to that of the usual irreducible representations, but involve a specialization of a  $q$ -analogue of the Kostant partition function. In particular, we prove signature analogues of the Kostant formula for weight multiplicities and the Steinberg formula for tensor product multiplicities. Using symmetric functions, we also find, for type  $A$ , analogues of the Weyl branching rule and the Gelfand-Tsetlin theorem.

## Résumé

Nous étudions une famille de représentations de groupes de Lie liée à la quantisation par rapport à l'opérateur de signature de Dirac. Ces représentations obéissent à des règles combinatoires semblables à celles qui régissent le cas classique des représentations irréductibles, mais font appel à une spécialisation d'un  $q$ -analogue de la fonction de partition de Kostant. Nous donnons des analogues des formules de Kostant, pour les multiplicités de poids, et de Steinberg, pour les multiplicités de facteurs dans les produits tensoriels. À l'aide de fonctions symétriques, nous trouvons aussi en type  $A$  des analogues de la règle de bifurcation de Weyl et de la théorie de Gelfand-Tsetlin.

## Introduction

The results described in this note are closely related to an article of Guillemin, Sternberg and Weitsman [1] on signature quantization.

A symplectic manifold  $(M, \omega)$  is *pre-quantizable* if the cohomology class of  $\omega$  is an integral class, i.e. is in the image of the map  $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$ . This assumption implies the existence of a *pre-quantum structure* on  $M$ : a line bundle,  $\mathbb{L}$ , and a connection,  $\nabla$ , such that  $\text{curv}(\nabla) = \omega$ . If  $g$  is a Riemannian metric compatible with  $\omega$ , then, from  $g$  and  $\omega$ , one gets an elliptic operation  $\not{D}_{\mathbb{C}} : S^+ \rightarrow S^-$ , the *spin- $\mathbb{C}$  Dirac operator*, and, by twisting this operator with  $\mathbb{L}$ , an operator  $\not{D}_{\mathbb{C}}^{\mathbb{L}} : S^+ \otimes \mathbb{L} \rightarrow S^- \otimes \mathbb{L}$ . If  $M$  is compact one can “quantize” it by associating with it the virtual vector space

$$Q(M) = \text{Index } \not{D}_{\mathbb{C}}^{\mathbb{L}}. \quad (1)$$

Moreover if  $G$  is a compact Lie group and  $\tau$  a Hamiltonian action of  $G$  on  $M$  one gets from  $\tau$  a representation of  $G$  on  $Q(M)$  which is well-defined up to isomorphism (independent of the choice of  $g$ ).

The results described in this note are closely related to two theorems in the article [1]. In this article the authors study the signature analogue of spin- $\mathbb{C}$  quantization: i.e. they define the virtual vector space (1) by replacing  $\not{D}_{\mathbb{C}}$  by the signature operator  $\not{D}_{\text{sig}}$ , and prove signature versions of a number of standard theorems about quantized symplectic manifolds. The two theorems we'll be concerned with in this paper are the following.

1. Let  $G = (S^1)^n$  and let  $M$  be a  $2n$ -dimensional toric variety with moment polytope  $\Delta \subseteq \mathbb{R}^n$ . Then, for spin- $\mathbb{C}$  quantization, the weights of the representation of  $G$  on  $Q(M)$  are the lattice points,  $\beta \in \Delta \cap \mathbb{Z}^n$ , and each weight occurs with multiplicity 1. For signature quantization the weights are the same; however, the weight  $\beta$  occurs with multiplicity  $2^n$  if  $\beta$  lies in  $\text{Int}(\Delta)$ , with multiplicity  $2^{n-1}$  if it lies on a facet, and, in general, with multiplicity  $2^{n-i}$  if it lies on  $i$  facets. Further details can be found in the work of Agapito [2].

2. Let  $G$  be a compact simply connected Lie group,  $\lambda$  a dominant weight and  $O_\lambda = M$  the coadjoint orbit of  $G$  through  $\lambda$ . In the spin- $\mathbb{C}$  theory, the representation of  $G$  on  $Q(M)$  is the unique irreducible representation  $V_\lambda$  of  $G$  with highest weight  $\lambda$ ; however, in the signature theory, it is the representation

$$\tilde{V}_\lambda = V_{\lambda-\rho} \otimes V_\rho, \quad (2)$$

where  $\rho$  is half the sum of the positive roots. (This is modulo the proviso that  $\lambda - \rho$  be dominant.)

The article [1] also contains a signature version of the Kostant multiplicity formula. We recall that the Kostant multiplicity formula computes the multiplicity with which a weight,  $\mu$ , of  $T$  occurs in  $V_\lambda$  by the formula

$$\sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K(\sigma(\lambda + \rho) - (\mu + \rho)) \quad (3)$$

where  $\mathcal{W}$  is the Weyl group,  $|\sigma|$  is the length of  $\sigma$  in  $\mathcal{W}$ , and  $K$ , the *Kostant partition function* (described below in Definition 1). The signature version of the Kostant multiplicity formula computes the multiplicity  $\tilde{m}_\lambda(\mu)$  with which the weight  $\mu$  appears in  $\tilde{V}_\lambda$  by a similar formula:

$$\tilde{m}_\lambda(\mu) = \sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K_2(\sigma(\lambda) - \mu) \quad (4)$$

where  $K_2$  is the  $q = 2$  specialization of a new  $q$ -analogue of the Kostant partition function, described below.

Our initial goal in writing this paper was to give a purely algebraic derivation of this result; however we noticed that there are  $\tilde{V}_\lambda$  analogues of a number of other basic formulas in the representation theory of compact semisimple Lie groups, in particular, an analogue of the Steinberg formula and, for  $\mathrm{GL}_k(\mathbb{C})$ , analogues of the Weyl branching rule and the Gelfand-Tsetlin theorem. Some of the proofs are sketched but details can be found in [3].

## The Kostant partition function and its $q$ -analogues

We start by introducing the Kostant partition function.

**Definition 1** *The Kostant partition function for a root system  $\Phi$ , given a choice of positive roots  $\Phi_+$ , is the function*

$$K(\mu) = \left| \left\{ (k_\alpha)_{\alpha \in \Phi_+} \in \mathbb{N}^{|\Phi_+|} : \sum_{\alpha \in \Phi_+} k_\alpha \alpha = \mu \right\} \right|, \quad (5)$$

*i.e.  $K(\mu)$  is the number of ways that  $\mu$  can be written as a sum of positive roots (see [4]).*

Note that  $K(\mu)$  can also be computed as the number of integer points inside the polytope

$$Q_\mu = \left\{ (k_\alpha)_{\alpha \in \Phi_+} \in \mathbb{R}_{\geq 0}^{|\Phi_+|} : \sum_{\alpha \in \Phi_+} k_\alpha \alpha = \mu \right\}. \quad (6)$$

We can write down a generating function for the  $K(\mu)$  that is very similar to Euler's generating function for the number of partitions (see [4, Section 25.2]):

$$\sum_{\mu} K(\mu) e^\mu = \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^\alpha}. \quad (7)$$

The classical  $q$ -analogue  $\hat{K}_q(\mu)$  of  $K(\mu)$ , due to Lusztig [5], keeps track of how many times the roots appear:

$$\hat{K}_q(\mu) = \sum_{(k_\alpha)_\alpha \in Q_\mu} q^{\sum k_\alpha}, \quad (8)$$

corresponding to the generating function

$$\sum_{\mu} \hat{K}_q(\mu) e^\mu = \prod_{\alpha \in \Phi_+} \left( \sum_{m \geq 0} q^m e^{m\alpha} \right) = \prod_{\alpha \in \Phi_+} \frac{1}{1 - qe^\alpha}. \quad (9)$$

The  $q$ -analogue  $K_q(\mu)$  that interests us here is the one that counts the integer points of  $Q_\mu$  according to how many of the  $k_\alpha$ 's are nonzero:

$$K_q(\mu) = \sum_{(k_\alpha)_\alpha \in Q_\mu} q^{|\{k_\alpha > 0\}|}. \quad (10)$$

In terms of generating functions, this translates to

$$\sum_{\mu} K_q(\mu) e^{\mu} = \prod_{\alpha \in \Phi_+} \left( 1 + q \sum_{m \geq 1} e^{\alpha} \right) = \prod_{\alpha \in \Phi_+} \frac{1 + (q-1)e^{\alpha}}{1 - e^{\alpha}}. \quad (11)$$

## The representations $\tilde{V}_\lambda = V_{\lambda-\rho} \otimes V_\rho$

We are working in the context of a complex semisimple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ , choice of positive roots  $\Phi_+$ , and Weyl group  $\mathcal{W}$ ;  $\rho$  is half the sum of the positive roots (or the sum of the fundamental weights). For a dominant weight  $\lambda$ , we denote by  $V_\lambda$  the irreducible representation of  $\mathfrak{g}$  with highest weight  $\lambda$ . We will call a weight  $\lambda$  *strictly dominant* if  $\lambda - \rho$  is dominant. We will use the notation  $\Lambda^+$  for the set of dominant weights, and  $\Lambda_S^+$  for the set of strictly dominant weights. For a strictly dominant weight, we define the representation

$$\tilde{V}_\lambda = V_{\lambda-\rho} \otimes V_\rho \quad (12)$$

and its character

$$\tilde{\chi}_\lambda = \chi_{V_{\lambda-\rho} \otimes V_\rho} = \chi_{\lambda-\rho} \cdot \chi_\rho. \quad (13)$$

The following theorem of Guillemin, Sternberg, and Weitsman [1] provides a formula for the multiplicities of the weights in the weight space decomposition of  $\tilde{V}_\lambda$ . This formula is very similar to the Kostant multiplicity formula (3), but uses the  $q = 2$  specialization of the  $q$ -analogue of the Kostant partition function  $K_q(\mu)$  introduced above, instead of the usual Kostant partition function. The formula for the  $\tilde{V}_\lambda$  multiplicities further distinguishes itself from the Kostant formula by being free of the  $\rho$  factors.

## An analogue of the Kostant multiplicity formula for the $\tilde{V}_\lambda$

**Theorem 2 (Guillemin-Sternberg-Weitsman [1])** *Let  $\lambda$  be a strictly dominant weight. Then the multiplicity of the weight  $\nu$  in the tensor product  $\tilde{V}_\lambda = V_{\lambda-\rho} \otimes V_\rho$  is given by*

$$\tilde{m}_\lambda(\nu) = \dim (\tilde{V}_\lambda)_\nu = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} K_2(\omega(\lambda) - \nu), \quad (14)$$

where  $|\omega|$  is the length of  $\omega$  in the Weyl group.

**Proof.** We give a simple proof here using the Weyl character formula. This formula expresses the character  $\chi_\lambda$  of  $V_\lambda$  as the quotient

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}, \quad (15)$$

where  $A_\mu = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\mu)}$ . For  $\rho$ , we get the nice expression [4, Lemma 24.3]

$$A_\rho = \prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}), \quad (16)$$

which means, in particular, that we get

$$\chi_\rho = \frac{A_{2\rho}}{A_\rho} = \frac{e^{2\rho} \prod_{\alpha \in \Phi_+} (1 - e^{-2\alpha})}{e^\rho \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})} = e^\rho \prod_{\alpha \in \Phi_+} (1 + e^{-\alpha}). \quad (17)$$

Thus, for  $\lambda$  strictly dominant,

$$\tilde{\chi}_\lambda = \chi_{\lambda-\rho} \cdot \chi_\rho = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \quad (18)$$

$$\begin{aligned} &= \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \sum_{\mu} K_2(\mu) e^{-\mu} \\ &= \sum_{\mu} \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} K_2(\mu) e^{\omega(\lambda)-\mu}. \end{aligned} \quad (19)$$

Extracting the coefficient of  $e^\nu$  on both sides gives (14). ■

The next step will be to use a formula due to Atiyah and Bott for the characters of the  $V_\lambda$  and  $\tilde{V}_\lambda$  to break down  $\tilde{V}_\lambda$  into its irreducible components and find their multiplicities. The Atiyah-Bott formula [6, 7] gives the character of  $V_\mu$  as

$$\chi_\mu = \sum_{\omega \in \mathcal{W}} e^{\omega(\mu)} \prod_{\alpha \in \Phi_+} \frac{1}{1-e^{-\omega(\alpha)}}. \quad (20)$$

**Remark 3** We can deduce this formula from the Weyl character formula (equation (15)) by first observing that

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\omega(\alpha)}) = (-1)^{|\omega|} e^{\sum\{\alpha \in \Phi_+ : \omega(\alpha) \in \Phi_-\}} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}) \quad (21)$$

Also,

$$\rho - \omega(\rho) = \sum\{\alpha \in \Phi_+ : \omega(\alpha) \in \Phi_-\}. \quad (22)$$

Combining (21) with (22) gives

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\omega(\alpha)}) = (-1)^{|\omega|} e^{\rho - \omega(\rho)} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}), \quad (23)$$

and we can translate Weyl's character formula into the Atiyah-Bott formula using this equation.

For any  $\omega \in \mathcal{W}$ ,

$$\begin{aligned} \chi_\rho &= e^\rho \prod_{\alpha \in \Phi_+} (1 + e^{-\alpha}) \\ &= e^{\omega(\rho)} \prod_{\alpha \in \Phi_+} (1 + e^{-\omega(\alpha)}), \end{aligned} \quad (24)$$

since characters are invariant under the Weyl group action. Using this and the Atiyah-Bott formula, we can write<sup>1</sup>

$$\begin{aligned} \tilde{\chi}_\lambda &= \chi_{\lambda-\rho} \cdot \chi_\rho = \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1+e^{-\omega(\alpha)}}{1-e^{-\omega(\alpha)}} \\ &= \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1}{1-e^{-\omega(\alpha)}} \sum_{I \subseteq \Phi_+} e^{-\omega(\alpha_I)} \end{aligned} \quad (25)$$

<sup>1</sup>Alternatively, we can obtain equation (25) from equation (18) by observing that for  $\omega \in \mathcal{W}$ ,

$$\omega \cdot \left( \prod_{\alpha \in \Phi_+} \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \right) = \prod_{\alpha \in \Phi_+} \frac{1+e^{-\omega(\alpha)}}{1-e^{-\omega(\alpha)}} = (-1)^{|\omega|} \prod_{\alpha \in \Phi_+} \frac{1+e^{-\alpha}}{1-e^{-\alpha}}.$$

where as before,  $\alpha_I = \sum_{\alpha \in I} \alpha$ . This gives

$$\tilde{\chi}_\lambda = \sum_{I \subseteq \Phi_+} \left( \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda - \alpha_I)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}} \right). \quad (26)$$

Letting,  $\lambda_I = \lambda - \alpha_I$ , we observe that if  $\lambda_I$  is dominant, the Atiyah-Bott formula tells us that

$$\sum_{\omega \in \mathcal{W}} e^{\omega(\lambda - \alpha_I)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}} \quad (27)$$

is the character  $\chi_{\lambda_I}$  of the irreducible representation  $V_{\lambda_I}$ , so that

$$\tilde{\chi}_\lambda = \sum_{I \subseteq \Phi_+} \chi_{\lambda_I} \quad \text{and} \quad \tilde{V}_\lambda = V_{\lambda - \rho} \otimes V_\rho = \bigoplus_{I \subseteq \Phi_+} V_{\lambda_I} \quad (28)$$

if all the  $\lambda_I$  are dominant.

Finally, since  $\alpha_I$  and  $\alpha_{I'}$  can be equal for different subsets  $I$  and  $I'$ , certain highest weights appear multiple times in the above sums. For the weight  $\mu = \lambda_I = \lambda - \alpha_I$ , we will get  $V_\mu$  as many times as we can write  $\alpha_I = \lambda - \mu$  as a sum of positive roots, where each positive root appears at most once. Hence

$$\tilde{V}_\lambda = \sum_{\mu} P(\lambda - \mu) V_\mu, \quad (29)$$

where the sum is over all  $\mu$  such that  $\mu = \lambda_I$  for some  $I$ , and  $P(\nu)$  is given by

$$\sum_{\nu} P(\nu) e^\nu = \prod_{\alpha \in \Phi_+} (1 + e^\alpha). \quad (30)$$

**Remark 4** *David Vogan pointed out to us that this decomposition is well-known and can be deduced from the Steinberg formula. For type  $A_n$ , the number of distinct  $\mu$ 's in the above sum is the number of forests of labelled unrooted tree on  $n + 1$  vertices [8, 9].*

### A tensor product formula for the $\tilde{V}_\lambda$

We will derive here an analogue of the Steinberg formula for the  $\tilde{V}_\lambda$ . Given two representations  $\tilde{V}_\lambda$  and  $\tilde{V}_\mu$ , the problem is to determine whether their tensor product  $\tilde{V}_\lambda \otimes \tilde{V}_\mu$  can be decomposed in terms of  $\tilde{V}_\nu$ 's. This is readily seen to be the case, as

$$\tilde{V}_\lambda \otimes \tilde{V}_\mu = (V_{\lambda - \rho} \otimes V_\rho) \otimes (V_{\mu - \rho} \otimes V_\rho) = (V_{\lambda - \rho} \otimes V_\rho \otimes V_{\mu - \rho}) \otimes V_\rho. \quad (31)$$

Breaking up  $V_{\lambda - \rho} \otimes V_\rho \otimes V_{\mu - \rho}$  into irreducibles  $V_\gamma$  and tensoring each factor with  $V_\rho$  yields factors  $V_\gamma \otimes V_\rho = \tilde{V}_{\gamma + \rho}$ . Thus for strictly dominant weights  $\lambda$  and  $\mu$ , we can write

$$\tilde{V}_\lambda \otimes \tilde{V}_\mu = \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \tilde{V}_\nu \quad (32)$$

for some nonnegative integers  $\tilde{N}_{\lambda\mu}^\nu$ .

**Theorem 5** *For  $\lambda, \mu$  and  $\nu$  strictly dominant weights, the tensor product multiplicity  $\tilde{N}_{\lambda\mu}^\nu$  of  $\tilde{V}_\nu$  in  $\tilde{V}_\lambda \otimes \tilde{V}_\mu$  is given by*

$$\tilde{N}_{\lambda\mu}^\nu = \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\omega(\lambda) + \sigma(\mu) - \nu). \quad (33)$$

**Proof.** Starting from the equation  $\tilde{V}_\lambda \otimes \tilde{V}_\mu = \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \tilde{V}_\nu$ , we can use equation (18) to write

$$\sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \cdot \tilde{\chi}_\mu = \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} e^{\tau(\nu)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}}.$$

Cancelling terms and using Theorem 2 to write down the character  $\tilde{\chi}_\mu$  yields

$$\begin{aligned} \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \cdot \sum_{\beta} \sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K_2(\sigma(\mu) - \beta) e^\beta &= \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} e^{\tau(\nu)} \\ \sum_{\beta} \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega| + |\sigma|} K_2(\sigma(\mu) - \beta) e^{\omega(\lambda) + \beta} &= \sum_{\nu \in \Lambda_S^+} \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^\nu e^{\tau(\nu)} \end{aligned}$$

Substituting  $\gamma = \omega(\lambda) + \beta$  on the left hand side, and  $\gamma = \tau(\nu)$  on the right hand side gives

$$\sum_{\gamma} \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\sigma(\mu) + \omega(\lambda) - \gamma) e^\gamma = \sum_{\substack{\gamma \text{ conjugate} \\ \text{to a strictly} \\ \text{dominant weight}}} \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)} e^\gamma,$$

and extracting the coefficient of  $e^\gamma$  on both sides yields

$$\sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\sigma(\mu) + \omega(\lambda) - \gamma) = \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)}. \quad (34)$$

Now, since  $\tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)}$  vanishes unless  $\tau^{-1}(\gamma)$  is strictly dominant, all the terms in the sum on the right hand side vanish except for the one where  $\tau$  is the identity (i.e. the term where  $\gamma = \nu$ ), and we get the result. ■

If we denote by  $N_{\lambda\mu}^\nu$  the multiplicities of the irreducible representations  $V_\nu$  in the tensor product  $V_\lambda \otimes V_\mu$ , defined by

$$V_\lambda \otimes V_\mu = \sum_{\nu \in \Lambda^+} N_{\lambda\mu}^\nu V_\nu, \quad (35)$$

then we can write down the tensor product multiplicities  $\tilde{N}_{\lambda\mu}^\nu$  for the decomposition of  $\tilde{V}_\lambda \otimes \tilde{V}_\mu$  into  $\tilde{V}_\nu$ 's in terms of the  $N_{\lambda\mu}^\nu$  as follows:

$$\begin{aligned} \tilde{V}_\lambda \otimes \tilde{V}_\mu &= V_{\lambda-\rho} \otimes V_\rho \otimes V_{\mu-\rho} \otimes V_\rho \\ &= \left( \left( \sum_{\beta \in \Lambda^+} N_{\lambda-\rho, \rho}^\beta V_\beta \right) \otimes V_{\mu-\rho} \right) \otimes V_\rho \\ &= \left( \sum_{\beta \in \Lambda^+} \sum_{\gamma \in \Lambda^+} N_{\lambda-\rho, \rho}^\beta N_{\beta, \mu-\rho}^\gamma V_\gamma \right) \otimes V_\rho \\ &= \sum_{\beta \in \Lambda^+} \sum_{\gamma \in \Lambda^+} N_{\lambda-\rho, \rho}^\beta N_{\beta, \mu-\rho}^\gamma \tilde{V}_{\gamma+\rho} \\ &= \sum_{\nu \in \Lambda_S^+} \sum_{\beta \in \Lambda^+} N_{\lambda-\rho, \rho}^\beta N_{\beta, \mu-\rho}^{\nu-\rho} \tilde{V}_\nu, \end{aligned}$$

so that for strictly dominant  $\nu$ ,

$$\tilde{N}_{\lambda\mu}^\nu = \sum_{\beta \in \Lambda^+} N_{\lambda-\rho, \rho}^\beta N_{\beta, \mu-\rho}^{\nu-\rho}. \quad (36)$$

**Remark 6** In type  $A$ , there is a combinatorial interpretation for the coefficients  $N_{\lambda\mu}^\nu$  in terms of shifted Young tableaux: they are given by a shifted analogue of the Littlewood-Richardson rule (see [10])

## Links with symmetric functions in type $A$

As for the weight multiplicities and Clebsch-Gordan coefficients, there is a link between the character products  $\tilde{\chi}_\lambda = \chi_{\lambda-\rho} \cdot \chi_\rho$  and symmetric functions in type  $A$ , again in terms of Schur functions.

The character of the irreducible polynomial representation  $V_\lambda$  of  $\mathrm{GL}_k\mathbb{C}$ , where we now think of  $\lambda$  as a partition with  $k$  parts (allowing the empty part) is the Schur function  $s_\lambda(x_1, \dots, x_k)$ . We will call a partition *strict* if all its parts are distinct (corresponding to a strictly dominant weight). Thus we have that, for  $\mathrm{GL}_k\mathbb{C}$ ,

$$\tilde{\chi}_\lambda = \chi_{\lambda-\rho} \cdot \chi_\rho = s_{\lambda-\rho}(x_1, \dots, x_k) s_\rho(x_1, \dots, x_k), \quad (37)$$

for any strict partition  $\lambda$ . It is readily checked that the weight  $\rho$  corresponds to the partition  $(k-1, k-2, \dots, 1, 0)$ .

**Remark 7** *We can also write the characters of  $\tilde{V}_\lambda$  in terms of Hall-Littlewood polynomials (see [11, III, 1. and 2.]). The results of the following sections can be deduced from this link with Hall-Littlewood polynomials, but we will rather use the Schur function expression (37) for the characters. This makes the proofs a bit more technical but avoids the heavier machinery of Hall-Littlewood polynomials.*

## A branching rule for the $\tilde{V}_\lambda$ in type $A$

We have seen that the representations  $\tilde{V}_\lambda$  behave somewhat like irreducible representations, in that tensor products of them can be broken down into direct sums of  $\tilde{V}_\nu$ 's again, and that the multiplicities in those decompositions as well as in the weight space decomposition are given by formulas very similar to those of Kostant and Steinberg in the irreducible case. The Weyl branching rule (see [4] for example) describes how to restrict a representation  $V_\lambda$  from  $\mathrm{GL}_k\mathbb{C}$  to  $\mathrm{GL}_{k-1}\mathbb{C}$ . This rule can be applied iteratively and provides a way to index one-dimensional subspaces of  $V_\lambda$  by diagrams (Gelfand-Tsetlin diagrams [12]) that is compatible with the weight space decomposition. It is natural to ask whether the representations  $\tilde{V}_\lambda$  of  $\mathrm{GL}_k\mathbb{C}$  are also well-behaved under restriction, or in other words, if there is an analogue of the Weyl branching rule for the  $\tilde{V}_\lambda$  in type  $A$ .

For two partitions  $\mu = (\mu_1, \dots, \mu_m)$  and  $\gamma = (\gamma_1, \dots, \gamma_{m-1})$ , we say that  $\gamma$  interlaces  $\mu$ , and write  $\gamma \triangleleft \mu$ , if

$$\mu_1 \geq \gamma_1 \geq \mu_2 \geq \gamma_2 \geq \mu_3 \geq \dots \geq \mu_{m-1} \geq \gamma_{m-1} \geq \mu_m.$$

For two such partitions  $\mu$  and  $\gamma$  such that  $\gamma \triangleleft \mu$ , we define

$$\nabla(\mu, \gamma) = \left| \left\{ i \in \{1, 2, \dots, m-1\} : \mu_i > \gamma_i > \mu_{i+1} \right\} \right|. \quad (38)$$

In other words,  $\nabla(\mu, \gamma)$  is the number of  $\gamma_i$  that are wedged strictly between  $\mu_i$  and  $\mu_{i+1}$ .

**Theorem 8** *The decomposition of the restriction of the representation  $\tilde{V}_\lambda$  of  $\mathrm{GL}_k\mathbb{C}$  to  $\mathrm{GL}_{k-1}\mathbb{C}$  into irreducible representations of  $\mathrm{GL}_{k-1}\mathbb{C}$  is given by*

$$\mathrm{Res}_{\mathrm{GL}_{k-1}\mathbb{C}}^{\mathrm{GL}_k\mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\nu \in \Lambda_S^+ : \nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \tilde{V}_\nu. \quad (39)$$

**Proof.** We give here a sketch of the proof. We argue using characters and the fact that those can be written in terms of Schur functions. We saw above (equation (37)) that the character of the representation  $\tilde{V}_\lambda$  of  $\mathrm{GL}_k\mathbb{C}$  is the product of Schur functions  $s_{\lambda-\rho}(x_1, \dots, x_k) s_\rho(x_1, \dots, x_k)$ . We obtain the character of the restriction of  $\tilde{V}_\lambda$  to  $\mathrm{GL}_{k-1}\mathbb{C}$  by setting the last variable  $x_k$  equal to 1. Using well-known identities on Schur functions (see [13, Section 7.15] for example), we have that

$$s_\lambda(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda} s_\mu(x_1, \dots, x_{k-1}). \quad (40)$$

and

$$s_\rho(x_1, \dots, x_k) = \prod_{1 \leq i < j \leq k} (x_i + x_j). \quad (41)$$

Thus,

$$s_{\lambda-\rho}(x_1, \dots, x_{k-1}, 1) s_\rho(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda-\rho} s_\mu(x_1, \dots, x_{k-1}) \prod_{1 \leq i < j \leq k-1} (x_i + x_j) \prod_{i=1}^{k-1} (x_i + 1). \quad (42)$$

We recognize the product  $\prod_{1 \leq i < j \leq k-1} (x_i + x_j)$  as the Schur function  $s_\rho(x_1, \dots, x_{k-1})$  (where  $\rho$  now corresponds to the partition  $(k-2, k-3, \dots, 1, 0)$  with  $k-1$  parts), and the product  $\prod_{i=1}^{k-1} (x_i + 1)$  as the sum  $(e_0 + e_1 + \dots + e_{k-1})$  of elementary symmetric functions in the variables  $x_1, \dots, x_{k-1}$ . A dual version of the Pieri rule [13, Section 7.15] describes how to break down the product of a Schur function with an elementary symmetric function into Schur functions:

$$s_\mu e_m = \sum_{\nu} s_\nu, \quad (43)$$

where the sum is over all  $\nu$  obtained from  $\mu$  by adding a vertical strip of size  $m$ , i.e. over the  $\nu$  such that  $\mu \subseteq \nu$  and the skew-shape  $\nu/\mu$  consists of  $m$  boxes, no two of which are in the same row. As we are working in  $k-1$  variables, the  $s_\nu$  with more than  $k-1$  parts vanish, so we can add the further constraint that the vertical strip be confined to the first  $k-1$  rows (we will say such a vertical strip has height at most  $k-1$ ). This gives

$$\begin{aligned} s_{\lambda-\rho}(x_1, \dots, x_{k-1}, 1) s_\rho(x_1, \dots, x_{k-1}, 1) &= \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} s_\nu(x_1, \dots, x_{k-1}) s_\rho(x_1, \dots, x_{k-1}) \\ \tilde{\chi}_\lambda(x_1, \dots, x_{k-1}, 1) &= \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} \tilde{\chi}_{\nu+\rho}(x_1, \dots, x_{k-1}) \end{aligned} \quad (44)$$

where the sum is over all the  $\nu$  that can be obtained from  $\mu$  by adding a vertical strip of size and height at most  $k-1$ . We can rewrite this as

$$\tilde{\chi}_\lambda(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} \tilde{\chi}_\nu(x_1, \dots, x_{k-1}) \quad (45)$$

where the sum is over all strict partitions  $\nu$  such that  $\nu - \rho$  can be obtained from  $\mu$  by adding a vertical strip of size and height at most  $k-1$ . Since the  $s_\nu s_\rho$  are linearly independent, we can lift this to the level of representations to get

$$\text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\mu \triangleleft \lambda-\rho} \bigoplus_{\nu} \tilde{V}_\nu, \quad (46)$$

with the sum over the same set of  $\nu$  as before.

In order to compute the multiplicity of a given  $\tilde{V}_\nu$  in  $\text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda$ , we define, for strict partitions  $\lambda$  and  $\nu$ ,  $n(\lambda, \nu)$  to be the number of ways that  $\nu - \rho$  can be obtained by adding a vertical strip of size and height at most  $k-1$  to some partition  $\mu$  such that  $\mu \triangleleft \lambda - \rho$ , so that

$$\tilde{V}_\lambda = \bigoplus_{\nu \in \Lambda_S^+} n(\lambda, \nu) \tilde{V}_\nu. \quad (47)$$

It can be checked that

$$n(\lambda, \nu) = \begin{cases} 2^{\nabla(\lambda, \nu)} & \text{if } \nu \triangleleft \lambda \text{ and } \nu \in \Lambda_S^+, \\ 0 & \text{otherwise.} \end{cases} \quad (48)$$

■

## Gelfand-Tsetlin theory for the $\tilde{V}_\lambda$

After restricting to  $\text{GL}_{k-1}\mathbb{C}$ , we can further restrict to  $\text{GL}_{k-2}\mathbb{C}$ . From now on, we will assume that all partitions are strict. We can write

$$\begin{aligned} \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda &= \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \left( \text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda \right) \\ &= \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \left( \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \tilde{V}_\nu \right) \\ &= \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \tilde{V}_\nu \end{aligned} \quad (49)$$

$$= \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \left( \bigoplus_{\mu \triangleleft \nu} 2^{\nabla(\nu, \mu)} \tilde{V}_\mu \right) \quad (50)$$

$$= \bigoplus_{\mu \triangleleft \nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu) + \nabla(\nu, \mu)} \tilde{V}_\mu. \quad (51)$$



Denoting by  $\lambda^{(m)} = \lambda_1^{(m)} \geq \dots \geq \lambda_m^{(m)} \geq 0$  the strict partitions indexing the representations  $\tilde{V}$  of  $\mathrm{GL}_m \mathbb{C}$ , we can iterate the branching rule until we get to  $\mathrm{GL}_1 \mathbb{C}$ :

$$\mathrm{Res}_{\mathrm{GL}_1 \mathbb{C}}^{\mathrm{GL}_k \mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\lambda^{(1)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda} 2^{\nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \nabla(\lambda^{(k-1)}, \lambda^{(k-2)}) + \dots + \nabla(\lambda^{(2)}, \lambda^{(1)})} V_{\lambda^{(1)}}. \quad (52)$$

We will call a sequence of strict partitions of the form  $\lambda^{(1)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda$  a *twisted Gelfand-Tsetlin diagram* for  $\lambda$ , which can be viewed schematically as

$$\begin{array}{ccccccc} \lambda_1^{(k)} & & \lambda_2^{(k)} & & \dots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\ & \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \dots & & \lambda_{k-1}^{(k-1)} & \\ & & \ddots & & \vdots & & \ddots & & \\ & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array} \quad (53)$$

with  $\lambda_j^{(k)} = \lambda_j$  and each  $\lambda_j^{(i)}$  is a nonnegative integer satisfying

$$\lambda_j^{(i)} > \lambda_{j+1}^{(i)} \quad (54)$$

and

$$\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)} \quad (55)$$

for all  $1 \leq j \leq i, 1 \leq i \leq k-1$ .

Let  $\tilde{V}_{\mathcal{D}}$  be the subspace of  $\tilde{V}_\lambda$  corresponding to a twisted Gelfand-Tsetlin diagram  $\mathcal{D}$ . This subspace has dimension  $2^{\nabla(\mathcal{D})}$ , where

$$\nabla(\mathcal{D}) = \nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \nabla(\lambda^{(k-1)}, \lambda^{(k-2)}) + \dots + \nabla(\lambda^{(2)}, \lambda^{(1)}). \quad (56)$$

We can also think of  $\nabla(\mathcal{D})$  as the number of triangles

$$\begin{array}{ccc} \lambda_j^{(i)} & & \lambda_{j+1}^{(i)} \\ & \lambda_j^{(i+1)} & \end{array}$$

with strict inequalities  $\lambda_j^{(i+1)} > \lambda_j^{(i)} > \lambda_{j+1}^{(i+1)}$  in the diagram  $\mathcal{D}$ .

We show here that  $\tilde{V}_{\mathcal{D}}$  lies completely within the same weight space of the weight space decomposition of  $\tilde{V}_\lambda$ .

We will think of the groups  $\mathrm{GL}_k \mathbb{C}$  as included into one another by identifying  $\mathrm{GL}_m \mathbb{C}$  with

$$\left( \begin{array}{c|c} \mathrm{GL}_m \mathbb{C} & \mathbf{0} \\ \hline \mathbf{0} & id_{k-m} \end{array} \right)$$

Consider the element  $I \in \mathfrak{gl}_m \mathbb{C}$  and a representation  $\tilde{V}_\mu$  of  $\mathrm{GL}_m \mathbb{C}$ . We have the representation  $\mathrm{GL}_k \mathbb{C} \rightarrow \mathfrak{gl}(V_\mu \otimes V_\rho)$ . For  $v \in V_{\mu-\rho}$  and  $w \in V_\rho$ , we have

$$\begin{aligned} I \cdot (v \otimes w) &= (I \cdot v) \otimes w + v \otimes (I \cdot w) \\ &= \left( \left( \sum_{j=1}^m (\mu - \rho)_j \right) v \right) \otimes w + v \otimes \left( \left( \sum_{j=1}^m \rho_j \right) w \right) \\ &= \left( \sum_{j=1}^m ((\mu - \rho)_j + \rho_j) \right) v \otimes w \\ &= \left( \sum_{j=1}^m \mu_j \right) v \otimes w, \end{aligned}$$

since  $V_{\mu-\rho}$  has highest weight  $\mu - \rho$  and  $V_\rho$  has highest weight  $\rho$ . So  $I \in \mathfrak{gl}_m \mathbb{C}$  gets represented as  $(\sum_{j=1}^m \mu_j) I$  in  $\tilde{V}_\mu$ . In general, for

$$\text{Res}_{\text{GL}_m \mathbb{C}}^{\text{GL}_k \mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\lambda^{(m)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda} 2^{\nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \nabla(\lambda^{(k-1)}, \lambda^{(k-2)}) + \dots + \nabla(\lambda^{(m+1)}, \lambda^{(m)})} \tilde{V}_{\lambda^{(m)}},$$

we will find that  $I \in \mathfrak{gl}_m \mathbb{C}$  gets represented as  $(\sum_{i=1}^m \lambda_i^{(m)}) I$  in  $\tilde{V}_{\lambda^{(m)}}$ .

Therefore, in the basis  $I_1, \dots, I_k$ , the subspace  $\tilde{V}_{\mathcal{D}}$  corresponding to a twisted Gelfand-Tsetlin diagram  $\mathcal{D}$  has weight

$$\left( \sum_{i=1}^1 \lambda_i^{(1)}, \sum_{i=1}^2 \lambda_i^{(2)}, \dots, \sum_{i=1}^k \lambda_i^{(k)} \right)$$

or

$$\left( \sum_{i=1}^1 \lambda_i^{(1)}, \sum_{i=1}^2 \lambda_i^{(2)} - \sum_{i=1}^1 \lambda_i^{(1)}, \dots, \sum_{i=1}^k \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)} \right)$$

in the usual basis  $J_1, \dots, J_k$ .

In other words,  $\tilde{V}_{\mathcal{D}} \subseteq (\tilde{V}_\lambda)_\beta$  if

$$\beta_m = \sum_{i=1}^m \lambda_i^{(m)} - \sum_{i=1}^{m-1} \lambda_i^{(m-1)}, \quad (57)$$

or, equivalently,

$$\beta_1 + \dots + \beta_m = \sum_{i=1}^m \lambda_i^{(m)}. \quad (58)$$

Hence twisted Gelfand-Tsetlin diagrams for  $\lambda$  correspond to the same weight if all their row sums are the same. So we have proved the following analogue of the Gelfand-Tsetlin theorem [12].

**Theorem 9** *Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a strictly dominant weight. The dimension of the representation  $\tilde{V}_\lambda$  of  $\text{GL}_k \mathbb{C}$  is given by*

$$\dim \tilde{V}_\lambda = \sum_{\mathcal{D}} 2^{\nabla(\mathcal{D})} \quad (59)$$

where the sum is over all twisted Gelfand-Tsetlin diagrams with top row  $\lambda$ .

Furthermore, the multiplicity  $\tilde{m}_\lambda(\beta)$  of the weight  $\beta$  in  $\tilde{V}_\lambda$  is given by

$$\tilde{m}_\lambda(\beta) = \dim (\tilde{V}_\lambda)_\beta = \sum_{\mathcal{D}} 2^{\nabla(\mathcal{D})} \quad (60)$$

where the sum is over all twisted Gelfand-Tsetlin diagrams with top row  $\lambda$  and row sums satisfying equation (57) (or (58)).

**Remark 10** *We can also prove that  $\tilde{V}_{\mathcal{D}}$  lies completely within a weight space of  $\tilde{V}_\lambda$  using characters and Schur function identities.*

## Acknowledgements

We would like to thank Richard Stanley for suggesting that the Schur function approach might work, rather than our more complicated approach in terms of Hall-Littlewood polynomials, and also for the observation that the tensor product of two twisted representations can be written as a positive sum (rather than as a virtual sum) of twisted representations. We would also like to thank Shlomo Sternberg and David Vogan for useful discussions and comments.

## References

- [1] V. Guillemin, S. Sternberg, and J. Weitsman. Signature quantization. *Proc. Natl. Acad. Sci. USA*, 100(22):12559–12560, 2003.
- [2] Jose Agapito. A weighted version of quantization commutes with reduction principle for a toric manifold. arXiv:math.SG/0307318, 2003.
- [3] Etienne Rassart. *Geometric approaches to computing Kostka numbers and Littlewood-Richardson coefficients*. PhD thesis, Massachusetts Institute of Technology, 2004.
- [4] William Fulton and Joe Harris. *Representation Theory*, volume 129 of *GTM*. Springer, 1991.
- [5] George Lusztig. Singularities, character formulas, and a  $q$ -analogue of weight multiplicities. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque 101-102, pages 208–229. Soc. Math. France, Paris, 1983.
- [6] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.
- [7] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. *Ann. of Math. (2)*, 88:451–491, 1968.
- [8] Daniel J. Kleitman and Kenneth J. Winston. Forests and score vectors. *Combinatorica*, 1:49–54, 1981.
- [9] Richard P. Stanley. Decompositions of rational convex polytopes. *Annals of Discrete Mathematics*, 6:333–342, 1980.
- [10] John R. Stembridge. Shifted tableaux and the projective representations of the symmetric groups. *Advances in Mathematics*, 74:87–134, 1989.
- [11] I. G. Macdonald. *Symmetric functions and Hall polynomials, second edition*. Oxford Mathematical Monographs. Clarendon Press, Oxford Science Publications, 1995.
- [12] I. M. Gel’fand and M. L. Cetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk. SSSR*, 71:825–828, 1950. (In Russian; an English translation appeared in Gelfand’s *Collected papers*, Springer-Verlag, 1987-1989).
- [13] Richard P. Stanley. *Enumerative Combinatorics, volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

