THE NUMBER OF MONOTONE TRIANGLES WITH PRESCRIBED BOTTOM ROW

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ABSTRACT. We show that the number of monotone triangles with prescribed bottom row $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, $k_1 < k_2 < \ldots < k_n$, is given by a simple product formula which remarkably involves (shift) operators. Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices.

1. INTRODUCTION

An *alternating sign matrix* is a square matrix of 0s, 1s and -1s for which the sum of entries in each row and in each column is 1 and the non-zero entries of each row and of each column alternate in sign. For instance,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is an alternating sign matrix. In the early 1980s, Robbins and Rumsey [8] introduced alternating sign matrices in the course of generalizing a determinant evaluation algorithm. Out of curiosity they posed the question for the number of alternating sign matrices of fixed size and, together with Mills, they came up with the appealing conjecture [7] that the number of $n \times n$ alternating sign matrices is

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1.1}$$

This turned out to be one of the hardest problems in enumerative combinatorics within the last decades. In 1996, Zeilberger [10] finally succeeded in proving their conjecture. Then, some months later, Kuperberg [5] realized that alternating sign matrices are equivalent to a model in statistical physics for two-dimensional square ice. Using a determinental expression for the partition function of this model discovered earlier by physicists, he was able to provide a shorter proof of the formula. For a nice exposition on this topic see [1].

Alternating sign matrices can be translated into certain triangular arrays of positive integers, called *monotone triangles*. Monotone triangles are probably the right guise of alternating sign matrices for a recursive treatment [1, Section 2.3]. In order to obtain the monotone triangle corresponding to a given alternating sign matrix, replace every entry in the matrix by the sum of entries in the same column above, the entry itself included. In our running example we obtain

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Row by row we record the columns that contain a 1 and obtain the following triangular array.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

This is the monotone triangle corresponding to the alternating sign matrix above. Observe that it is weakly increasing in northeast direction and in southeast direction. Moreover, it is strictly increasing along rows. In general, a monotone triangle with n rows is a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ such that $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$ and $a_{i,j} < a_{i,j+1}$ for all i, j. It is not too hard to see that monotone triangles with n rows and bottom row $(1, 2, \ldots, n)$, i.e. $a_{n,j} = j$, are in bijection with $n \times n$ alternating sign matrices. Our main theorem provides a formula for the number of monotone triangles with prescribed last row $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

Theorem 1. *The number of monotone triangles with n rows and bottom row $k_1, k_2, \ldots, k_n$ is given by*

$$\left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where E_x denotes the shift operator, defined by $E_x \, p(x) = p(x+1)$, and $\Delta_x := E_x - \mathrm{id}$ denotes the difference operator.

In order to understand this formula, there are a few things to remark. The product of operators is understood as the composition. Moreover note that the shift operators commute, and consequently, it does not matter in which order the operators in the product $\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right)$ are applied. In order to use this formula to compute the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$, one first has to apply the operator $\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{x_p} \Delta_{x_q} \right)$ to the polynomial $\prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i}$ and then set $x_i = k_i$. Thus, it is not so clear how to derive (1.1) from this formula.

What is the significance of the formula? In the last decades, the enumeration of plane partitions, alternating sign matrices and related objects subject to a variety of different constraints has attracted a lot of interest. This attraction stems from the fact that now and then these enumerations lead to appealing product formula or hypergeometric series, which are, in spite of their simplicity, pretty hard to prove. At the moment the search for these simple product formulas seems to be a bit exhausted. Therefore, a new challenge is the search for possibilities to give enumeration formulas for the vast majority of enumeration problems for which there exists no closed formula in a traditional sense. The formula in Theorem 1 contributes to this issue.

Also note that the second product in the formula in Theorem 1, i.e. $\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$, is the number of semistandard tableaux of shape $(k_n - n, k_{n-1} - n, \ldots, k_1 - 1)$ and, equivalently, the number of columnstrict plane partitions of this shape, see [9, p. 375, in (7.105) $q \to 1$]. In fact, these objects are in bijection with monotone triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n)$ that are strictly increasing in southeast direction, see [2, Section 5]. Thus, our formula once more gives an indication of the relation between plane partitions and alternating sign matrices manifested by a number of enumeration formulas which show up in both fields, a phenomenon which is not yet well (i.e. bijectively) understood.

In this extended abstract we sketch the proof of Theorem 1. (See [4] for the full version of this paper.) The method can roughly be described as follows. In the first step, we introduce a recursion, which relates monotone triangles with n rows to monotone triangles with $n-1$ rows. This recursion immediately implies that the enumeration formula is a polynomial in $k_1, k_2, \ldots, k_n$. In the next step we compute the degree of the polynomial. Finally, we deduce enough properties of the polynomial in order to compute it. The polynomial's degree determines how much information is in fact needed. This method is related to the method for proving polynomial enumeration formulas we have introduced in [2] and extended in [3]. In the final section we mention some problems around Theorem 1 we plan to consider next.

2. The recursion

In the following let $\alpha(n; k_1, \ldots, k_n)$, $n \ge 1$, denote the number of monotone triangles with $(k_1, \ldots, k_n)$ as bottom row. If we delete the last row of such a monotone triangle we obtain a monotone triangle with $n-1$ rows and bottom row, say, $(l_1, l_2, \ldots, l_{n-1})$. By the definition of a monotone triangle $k_1 \le l_1 \le k_2 \le l_2 \le \ldots \le k_{n-1} \le l_{n-1} \le k_n$ and $l_i \ne l_{i+1}$. Thus

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \le l_1 \le k_2 \le \ldots \le k_{n-1} \le l_{n-1} \le k_n, l_i \ne l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1}). \tag{2.1}$$

We introduce the following abbreviation

$$\sum_{\substack{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \le l_1 \le k_2 \le \ldots \le k_{n-1} \le l_{n-1} \le k_n, l_i \ne l_{i+1}}} =: \sum_{(l_1, \ldots, l_{n-1})}^{(k_1, \ldots, k_n)}$$

for $n \ge 2$. This summation operator is well-defined for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \ldots < k_n$. We extend the definition to arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by induction with respect to n. If $n = 2$ then

$$\sum_{(l_1)}^{(k_1, k_2)} A(l_1) := \sum_{l_1 = k_1}^{k_2} A(l_1),$$

where here and in the following we use the extended definition of the summation over an interval, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \cdots + f(b) & \text{if } a \leq b \\ 0 & \text{if } b = a-1 \\ -f(b+1) - f(b+2) - \cdots - f(a-1) & \text{if } b+1 \leq a-1 \end{cases} . \tag{2.2}$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I containing $\mathbb{Q}$ there exists a unique polynomial $q(X)$ over I such that $\sum\limits_{x=0}^{y} p(x) = q(y)$ for all integers y. We usually write $\sum\limits_{x=0}^{y} p(x)$ for $q(y)$. (We also use the analog extended definition for the product symbol.) If $n > 2$ then

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) :=$$
$$\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1}).$$

We renew the definition of $\alpha(n;k_1,\ldots,k_n)$ after this extension by setting $\alpha(1;k_1) = 1$ and

$$\alpha(n;k_1,\ldots,k_n) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1;l_1,\ldots,l_{n-1}).$$

This gives us an extension of our original function $\alpha(n;k_1,\ldots,k_n)$ to arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$. The recursion implies that $\alpha(n;k_1,\ldots,k_n)$ is a polynomial in $k_1,\ldots,k_n$. We have used this recursion (and a computer) to compute $\alpha(n;k_1,\ldots,k_n)$ for $n = 1,2,3,4$ and obtain the following

$$
\begin{aligned}
&1, 1-k_1+k_2, \frac{1}{2}(-3k_1+k_1^2+2k_1k_2-k_1^2k_2-2k_2^2+k_1k_2^2+3k_3-4k_1k_3+k_1^2k_3+\\
&2k_2k_3-k_2^2k_3+k_3^2-k_1k_3^2+k_2k_3^2), \frac{1}{12}(20k_2+11k_1k_2-16k_1^2k_2+3k_1^3k_2+4k_1k_2^2+3k_1^2k_2^2-k_1^3k_2^2+\\
&4k_2^3-5k_1k_2^3+k_1^2k_2^3-20k_3+16k_1k_3-4k_1^2k_3-27k_2k_3+9k_1^2k_2k_3-2k_1^3k_2k_3-3k_1^2k_2^2k_3+k_1^3k_2^2k_3-\\
&3k_2^3k_3+4k_1k_2^3k_3-k_1^2k_2^3k_3+16k_1k_3^2-12k_1^2k_3^2+2k_1^3k_3^2-9k_1k_2k_3^2+6k_1^2k_2k_3^2-k_1^3k_2k_3^2+9k_2^2k_3^2-\\
&3k_1k_2^2k_3^2-3k_2^3k_3^2+k_1k_2^3k_3^2-4k_3^3+8k_1k_3^3-2k_1^2k_3^3-3k_2k_3^3-2k_1k_2k_3^3+k_1^2k_2k_3^3+3k_2^2k_3^3-k_1k_2^2k_3^3-\\
&27k_1k_4+20k_1^2k_4-3k_1^3k_4+16k_2k_4+24k_1k_2k_4-24k_1^2k_2k_4+4k_1^3k_2k_4-16k_2^2k_4+9k_1k_2^2k_4+\\
&3k_1^2k_2^2k_4-k_1^3k_2^2k_4+8k_2^3k_4-6k_1k_2^3k_4+k_1^2k_2^3k_4+11k_3k_4-24k_1k_3k_4+15k_1^2k_3k_4-2k_1^3k_3k_4-\\
&9k_2^2k_3k_4+2k_2^3k_3k_4-4k_3^2k_4+9k_1k_3^2k_4-6k_1^2k_3^2k_4+k_1^3k_3^2k_4+3k_2^2k_3^2k_4-k_2^3k_3^2k_4-5k_3^3k_4+\\
&6k_1k_3^3k_4-k_1^2k_3^3k_4-4k_2k_3^3k_4+k_2^2k_3^3k_4-20k_1k_4^2+9k_1^2k_4^2-k_1^3k_4^2+4k_2k_4^2+15k_1k_2k_4^2-9k_1^2k_2k_4^2+\\
&k_1^3k_2k_4^2-12k_2^2k_4^2+6k_1k_2^2k_4^2+2k_2^3k_4^2-k_1k_2^3k_4^2+16k_3k_4^2-24k_1k_3k_4^2+9k_1^2k_3k_4^2-k_1^3k_3k_4^2+\\
&9k_2k_3k_4^2-6k_2^2k_3k_4^2+k_2^3k_3k_4^2+3k_3^2k_4^2-3k_1k_3^2k_4^2+3k_2k_3^2k_4^2-k_3^3k_4^2+k_1k_3^3k_4^2-k_2k_3^3k_4^2-3k_1k_4^3+\\
&k_1^2k_4^3+2k_1k_2k_4^3-k_1^2k_2k_4^3-2k_2^2k_4^3+k_1k_2^2k_4^3+3k_3k_4^3-4k_1k_3k_4^3+k_1^2k_3k_4^3+2k_2k_3k_4^3-k_2^2k_3k_4^3+\\
&\qquad k_3^2k_4^3-k_1k_3^2k_4^3+k_2k_3^2k_4^3).
\end{aligned}
$$

From this data it is obviously hard to guess a general formula for $\alpha(n; k_1, \ldots, k_n)$. However, it seems plausible that the degree of $\alpha(n; k_1, \ldots, k_n)$ in k_i is $n-1$. In the following two sections we prove that this is indeed true. Note that at first glance the linear growth of the degree is quite surprising: suppose $A(l_1, \ldots, l_{n-1})$ is a polynomial of degree no greater than R in each of l_{i-1} and l_i. Then

$$
\deg_{k_i}\left(\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1})\right) =
$$

$$
\deg_{k_i}\left(\sum_{l_{i-1}=k_{i-1}}^{k_i}\sum_{l_i=k_i}^{k_{i+1}} A(l_1,\ldots,l_{n-1}) - A(l_1,\ldots,l_{i-2},k_i,k_i,l_{i+1},\ldots,l_{n-1})\right) \le 2R+2
$$

and there exist polynomials $A(l_1, \ldots, l_{n-1})$ such that the upper bound $2R+2$ is attained. Consequently, $\alpha(n; k_1, \ldots, k_n)$ must be of a very specific shape.

3. Sketch of the proof of Theorem 1

In this section we sketch the proof of the main theorem by presenting the relevant lemmas without proofs.

Recall that the *shift operator*, denoted by E_x, is defined as $E_x p(x) = p(x+1)$. Clearly E_x is invertible in the algebra of operators of $\mathbb{C}[X]$ and we denote its inverse by E_x^{-1}. Observe that the shift operators with respect to different variables commute, i.e. $E_x E_y = E_y E_x$. The difference operator Δ_x is defined as $\Delta_x = E_x - \mathrm{id}$. However, the difference operator Δ_x is not invertible since it decreases the degree of a polynomial.

If we apply the shift operator or the delta operator to the i-th variable of a function, we sometimes write E_i or Δ_i, respectively, i.e. $\Delta_{k_i} f(k_1,\ldots,k_n) = \Delta_i f(k_1,\ldots,k_n)$. Moreover, $\Delta_2 f(k_3,k_3,k_3)$, for instance, is shorthand for

$$(\Delta_{l_2} f(l_1,l_2,l_3))|_{l_1=k_3,l_2=k_3,l_3=k_3} .$$

The *swapping operator* $S_{x,y}$ is applicable to functions in (at least) two variables and defined as $S_{x,y} f(x,y) = f(y,x)$. If we apply it to the i-th and j-th variable of a function we sometimes write $S_{i,j}$.

In the following we consider rational functions in shift operators. In order to guarantee that the inverse of the denominator always exists, we need the following lemma.

Lemma 1. *Let $p(x_1,\ldots,x_n)$ be a polynomial in $x_1,x_1^{-1},x_2,x_2^{-1},\ldots,x_n,x_n^{-1}$ over $\mathbb{C}$, and fix an integer i, $1 \le i \le n$. Consider the operator*

$$\operatorname{id} + \Delta_{k_i} p(E_{k_1},E_{k_2},\ldots,E_{k_n}) =: \operatorname{Op}$$

on $\mathbb{C}[k_1,\ldots,k_n]$. Then Op is invertible and the inverse is

$$\operatorname{Op}^{-1} = \sum_{l=0}^{\infty} (-1)^l \Delta_{k_i}^l p(E_{k_1},E_{k_2},\ldots,E_{k_n})^l,$$

where $\Delta_{k_i}^0 p(E_{k_1},E_{k_2},\ldots,E_{k_n})^0 = \operatorname{id}$. Moreover

$$\deg_{k_i} G(k_1,\ldots,k_n) = \deg_{k_i} \operatorname{Op} G(k_1,\ldots,k_n) = \deg_{k_i} \operatorname{Op}^{-1} G(k_1,\ldots,k_n).$$

We define two operators applicable to polynomials $G(k_1,\ldots,k_n) \in \mathbb{C}[k_1,\ldots,k_n]$. We set

$$V_{k_i,k_j} = \operatorname{id} + E_{k_i}^{-1} \Delta_{k_i} \Delta_{k_j} = E_{k_i}^{-1}(\operatorname{id} + E_{k_j} \Delta_{k_i})$$

and

$$T_{k_i,k_{i+1}} = (\operatorname{id} + E_{k_{i+1}} E_{k_i}^{-1} S_{k_i,k_{i+1}}) \frac{V_{k_i,k_{i+1}}}{V_{k_i,k_{i+1}} + V_{k_{i+1},k_i}}.$$

By Lemma 1, the inverse $(V_{k_i,k_{i+1}} + V_{k_{i+1},k_i})^{-1}$ is well-defined. The following lemma explains the significance of $T_{k_i,k_{i+1}}$ for the recursion (2.1).

Lemma 2. *Let $A(l_1,l_2)$ be a polynomial in l_1 and l_2 which is of degree at most R in each of l_1 and l_2. Moreover assume that $T_{l_1,l_2} A(l_1,l_2)$ is of degree at most R as a polynomial in l_1 and l_2, i.e. a linear combination of monomials $l_1^m l_2^n$ with $m+n \le R$. Then*

$$\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1,l_2) = \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1,l_2) - A(k_2,k_2)$$

is of degree at most $R+2$ in k_2. Moreover, if $T_{l_1,l_2} A(l_1,l_2) = 0$ then $\sum\limits_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1,l_2)$ is of degree at most $R+1$ in k_2.

In order to use Lemma 2 to compute the degree of $\sum\limits_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1,l_2)$ in k_2 one has to compute the degree of $T_{l_1,l_2} A(l_1,l_2)$ in l_1 and l_2. However, the operator T_{l_1,l_2} is complicated and thus it is convenient to consider a simplified version of T_{l_1,l_2} for this purpose, which we obtain by multiplying an operator that preserves the degree.

$$T'_{k_i,k_{i+1}} = E_{k_i}(V_{k_i,k_{i+1}} + V_{k_{i+1},k_i})T_{k_i,k_{i+1}} = (\mathrm{id} + S_{k_i,k_{i+1}})E_{k_i}V_{k_i,k_{i+1}} = (\mathrm{id} + S_{k_i,k_{i+1}})(\mathrm{id} + E_{k_{i+1}}\Delta_{k_i})$$

Observe that $\deg_{k_i,k_{i+1}} T_{k_i,k_{i+1}} G(k_1,\ldots,k_n) = \deg_{k_i,k_{i+1}} T'_{k_i,k_{i+1}} G(k_1,\ldots,k_n)$, since

$$V_{k_i,k_{i+1}} + V_{k_{i+1},k_i} = 2\,\mathrm{id} + (E_{k_i}^{-1} + E_{k_{i+1}}^{-1})\Delta_{k_i}\Delta_{k_{i+1}}$$

and $\Delta_{k_i}\Delta_{k_{i+1}}$ decreases the degree of a polynomial in k_i and k_{i+1}. In particular, $T_{k_i,k_{i+1}}G(k_1,\ldots,k_n) = 0$ if and only if $T'_{k_i,k_{i+1}}G(k_1,\ldots,k_n) = 0$.

Suppose $A(l_1,\ldots,l_n)$ is a function on $\mathbb{Z}^n$. Next we aim to express

$$T'_{k_i,k_{i+1}} \left(\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} A(l_1,\ldots,l_n) \right)(k_1,k_2,\ldots,k_{n+1})$$

in terms of $T'_{l_{i-1},l_i}A(l_1,\ldots,l_n)$ and $T'_{l_i,l_{i+1}}A(l_1,\ldots,l_n)$. In particular, this shows that if $T'_{l_i,l_{i+1}}A(l_1,\ldots,l_n) = 0$ for all $i = 1,2,\ldots,n-1$ then

$$T'_{k_i,k_{i+1}} \left(\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} A(l_1,\ldots,l_n) \right)(k_1,\ldots,k_{n+1}) = 0$$

for all $i = 1,2,\ldots,n$.

Lemma 3. *Let $f(k_1,k_2,k_3)$ be a function from $\mathbb{Z}^3$ to $\mathbb{C}$ and define*

$$g(k_1,k_2,k_3,k_4) := \sum_{(l_1,l_2,l_3)}^{(k_1,k_2,k_3,k_4)} f(l_1,l_2,l_3).$$

Then

$$\begin{aligned} T'_{2,3}\, g(k_1,k_2,k_3,k_4) = \\ -\frac{1}{2}\left(\sum_{l_1=k_2+1}^{k_3} \sum_{l_2=k_2+1}^{k_3} \sum_{l_3=k_2}^{k_4} T'_{1,2}\, f(l_1,l_2,l_3) + \sum_{l_1=k_1}^{k_2+1} \sum_{l_2=k_2}^{k_3-1} \sum_{l_3=k_2}^{k_3-1} T'_{2,3}\, f(l_1,l_2,l_3) \right) \\ + \frac{1}{2}\left(\sum_{l_1=k_2}^{k_3-1} \sum_{l_2=k_2}^{k_3-1} \Delta_2(\mathrm{id} + E_1)T'_{1,2}\, f(l_1,l_2,k_2) - \sum_{l_2=k_2}^{k_3-1} \sum_{l_3=k_2}^{k_3-1} \Delta_2(\mathrm{id} + E_3)T'_{2,3}\, f(k_2+1,l_2,l_3) \right) \\ + \frac{1}{2}\Big(T'_{1,2}\, f(k_2,k_2,k_2+1) - T'_{1,2}\, f(k_2,k_2,k_3+1) + T'_{2,3}\, f(k_2,k_2,k_2) - T'_{2,3}\, f(k_3,k_2,k_2) \Big) \\ - T'_{1,2}\, f(k_2,k_3,k_2+1) - T'_{2,3}\, f(k_2,k_2,k_3). \end{aligned}$$

Moreover, for a function $h(l_1,l_2)$ on $\mathbb{Z}^2$,

$$T'_{1,2} \left(\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} h(l_1,l_2) \right)(k_1,k_2,k_3) = -\frac{1}{2} \sum_{l_1=k_1}^{k_2-1} \sum_{l_2=k_1}^{k_2-1} T'_{1,2}\, h(l_1,l_2).$$

This proves the statement preceding the lemma for $n = 2, 3$. It can easily be extended to general n by deriving a merging rule for the recursion (2.1). For this purpose we need another operator. Let $f(x, z)$ be a function on $\mathbb{Z}^2$. Then the operator $I^y_{x,z}$ transforms $f(x, z)$ into a function on $\mathbb{Z}$ by

$$I^y_{x,z} f(x, z) := f(y-1, y) + f(y, y+1) - f(y-1, y+1) = V_{x,z} f(x, z)\big|_{x=y=z}.$$

With this definition we have

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) = I^{k_i}_{k'_i,k''_i} \sum_{(l_1,\ldots,l_{i-1})}^{(k_1,\ldots,k_{i-1},k'_i)} \sum_{(l_i,\ldots,l_{n-1})}^{(k''_i,k_{i+1},\ldots,k_n)} A(l_1, \ldots, l_n).$$

If we combine Lemma 3 with this merging rule we obtain formulas for general n. These formulas imply the following corollary.

Corollary 1. *Suppose $A(l_1, \ldots, l_n)$ is a function on $\mathbb{Z}^n$ with $T'_{l_i,l_{i+1}} A(l_1, \ldots, l_n) = 0$ for all i, $1 \le i < n$. Then*

$$T'_{k_i,k_{i+1}} \left(\sum_{(l_1,\ldots,l_n)}^{(k_1,\ldots,k_{n+1})} A(l_1, \ldots, l_n) \right) (k_1, \ldots, k_{n+1}) = 0$$

for all i, $1 \le i \le n$.

We come back to $\alpha(n; k_1, \ldots, k_n)$. By induction with respect to n we conclude that $T'_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0$ for all i, $1 \le i < n$, if $n \ge 2$. (Note that $\alpha(2; k_1, k_2) = k_2 - k_1 + 1$.) Thus $T_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0$ for all i. Therefore, by Lemma 2 and by induction with respect to n, the polynomial $\alpha(n; k_1, \ldots, k_n)$ is of degree no greater than $n - 1$ in every k_i.

In the following we demonstrate that the property that $T'_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0$ for all i is not only fundamental for the computation of the polynomial's degree but already determines $\alpha(n; k_1, \ldots, k_n)$ up to a multiplicative constant. Observe that $T'_{k_i,k_{i+1}} A(k_1, \ldots, k_n) = 0$ is equivalent with the fact that $(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) A(k_1, \ldots, k_n)$ is antisymmetric in k_i and k_{i+1}. In the following lemma we characterize polynomials $A(k_1, \ldots, k_n)$ with the property that $(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) A(k_1, \ldots, k_n)$ is antisymmetric in k_i and k_{i+1} for all i.

Lemma 4. *Let $A(k_1, \ldots, k_n)$ be a polynomial in $(k_1, \ldots, k_n)$. Then*

$$(\mathrm{id} + E_{k_{i+1}} \Delta_{k_i}) A(k_1, \ldots, k_n)$$

is antisymmetric in k_i and k_{i+1} for all i, $1 \le i \le n-1$, if and only if

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_q} \Delta_{k_p}) \right) A(k_1, \ldots, k_n)$$

is antisymmetric in $k_1, \ldots, k_n$.

Using this lemma we see that

$$\left(\prod_{1 \le p < q \le n} (\mathrm{id} + E_{k_q} \Delta_{k_p}) \right) \alpha(n; k_1, \ldots, k_n) \tag{3.1}$$

is an antisymmetric polynomial in $k_1, \dots, k_n$. The product of shift operators does not increase the polynomial's degree and thus the degree of (3.1) in every k_i is no greater than $n-1$. Every antisymmetric function in $k_1, \dots, k_n$ is a multiple of $\prod_{1 \le i < j \le n} (k_j - k_i)$ and since this product is of degree $n-1$ in every k_i, the expression in (3.1) is equal to $C \cdot \prod_{1 \le i < j \le n} (k_j - k_i)$, where C is a rational constant. Therefore,

$$\alpha(n; k_1, \dots, k_n) = \left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) C \prod_{1 \le i < j \le n} (k_j - k_i).$$

It is not too hard to show that the coefficient of $k_1^0 k_2^1 \dots k_n^{n-1}$ in $\alpha(n; k_1, \dots, k_n)$ is $C = \prod_{1 \le i < j \le n} \frac{1}{j-i}$. Consequently,

$$\alpha(n; k_1, \dots, k_n) = \left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}. \tag{3.2}$$

We need a final lemma in order to derive Theorem 1 from that.

Lemma 5. *Let $P(X_1, \dots, X_n)$ be a polynomial in $(X_1, \dots, X_n)$ over $\mathbb{C}$ which is symmetric in $(X_1, \dots, X_n)$. Then*

$$P(E_{k_1}, \dots, E_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = P(1, \dots, 1) \cdot \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Observe that $\prod_{1 \le p, q \le n} (1 + X_q (X_p - 1))$ is symmetric in $(X_1, \dots, X_n)$. Thus, by Lemma 5,

$$\prod_{1 \le p, q \le n} \left(\mathrm{id} + E_{k_q} \Delta_{k_p} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Therefore, by (3.2),

$$\begin{aligned} \alpha(n; k_1, \dots, k_n) &= \left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \\ &= \left(\prod_{1 \le p < q \le n} \frac{1}{\mathrm{id} + E_{k_q} \Delta_{k_p}} \right) \left(\prod_{1 \le p, q \le n} \left(\mathrm{id} + E_{k_q} \Delta_{k_p} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \\ &= \left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} \end{aligned}$$

and this completes the proof of Theorem 1.

4. Some further projects

In this section we list some further projects around the formula given in Theorem 1 we plan to pursue.

(1) A natural question to ask is whether it is possible to derive the formula for the number of $n \times n$ alternating sign matrices (1.1) from Theorem 1, i.e. to show that

$$\left[\left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_q}\Delta_{k_p})\right) \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i}\right]\Bigg|_{(k_1,k_2,\ldots,k_n)=(1,2,\ldots,n)} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

More general, one could try to reprove the refined alternating sign matrix theorem [11], which states that the number of $n \times n$ alternating sign matrices in which the unique 1 in the top row is in the k-th column is given by

$$\frac{(k)_{n-1}(1+n-k)_{n-1}}{(n-1)!} \prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!}. \tag{4.1}$$

An analysis of the correspondence between alternating sign matrices and monotone triangles shows that $\alpha(n-1;1,2,\ldots,k-1,k+1,\ldots,n)$ is the number of $n \times n$ alternating sign matrices in which the unique 1 in the bottom row is in the k-th column and this is by symmetry equal to (4.1). This could be a consequence of an even more general theorem: computer experiments suggest that there are other $(k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$ "near" $(1,2,\ldots,n)$ for which $\alpha(n;k_1,\ldots,k_n)$ has small prime factors. Small prime factors are an indication for a simple product formula. A similar phenomenon can be observed for some $(k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$ "near" $(1,3,\ldots,2n-1)$. It is not too hard to see that $\alpha(n;1,3,\ldots,2n-1)$ is the number of $(2n+1)\times(2n+1)$ alternating sign matrices, which are symmetric with respect to reflection along the vertical axis. Kuperberg [6] showed that the number of these objects is given by

$$\frac{n!}{(2n)!2^n} \prod_{j=1}^{n} \frac{(6j-2)!}{(2n+2j-1)!}.$$

(2) Let $\beta(n;k_1,\ldots,k_n)$ denote the number of monotone triangles with prescribed bottom row $(k_1,\ldots,k_n)$ that are strictly increasing in southeast direction. With this notation, Theorem 1 states that

$$\alpha(n;k_1,\ldots,k_n) = \left(\prod_{1\le p<q\le n} (\mathrm{id} + E_{k_p}\Delta_{k_q})\right) \beta(n;k_1,\ldots,k_n). \tag{4.2}$$

It would be interesting to find a bijective proof of this formula in the following sense: if we expand the product of operators on the left hand side we obtain a sum of expressions of the form

$$E_{k_1}^{a_1} E_{k_2}^{a_2} \ldots E_{k_n}^{a_n} \Delta_{k_1}^{b_1} \Delta_{k_2}^{b_2} \ldots \Delta_{k_n}^{b_n} \beta(n;k_1,\ldots,k_n)$$

with $a_i, b_i \in \{0,1,2,\ldots\}$. We can interpret these expressions as sums and differences of cardinalities of certain subsets of the set of monotone triangles with n rows. For instance,

$$\Delta_{k_q} \beta(n;k_1,\ldots,k_n)$$

is the number of monotone triangles that are strictly increasing in southeast direction and with bottom row $(k_1, \ldots, k_q + 1, \ldots, k_n)$ such that the $(q-1)$-st part of the $(n-1)$-st row is equal to k_q minus the number of monotone triangles that are strictly increasing in southeast direction and with bottom row $(k_1, \ldots, k_n)$ such that the q-th part of the $(n-1)$-st row is equal to k_q. In order to prove (4.2), one has to show that these cardinalities add up to the number of monotone triangles.

(3) This is more a remark than another project: to prove Theorem 1 I have more or less carried out an analysis of the recursion (2.1). I originally started this analysis when considering a somehow reversed situation: let an (r,n) monotone trapezoid be a monotone triangle with the top $n-r$ rows cut off and bottom row $(1,2,\ldots,n)$. Let $\gamma(r,n;k_1,\ldots,k_{n-r+1})$ denote the number of (r,n) monotone trapzoids with prescribed top row $(k_1,\ldots,k_{n-r+1})$. In particular, $\gamma(n,n;k)$ is the number of monotone triangles with n rows, bottom row $(1,2,\ldots,n)$ and k as entry in the top row. In the bijection between alternating sign matrices and monotone triangles, the entry in the top row of the monotone triangle corresponds to the column of the unique 1 in the first row of the alternating sign matrix. Thus, $\gamma(n,n;k)$ must be equal to (4.1). On the other hand, we can also use (2.1) to compute $\gamma(r,n;k_1,\ldots,k_{n-r+1})$: $\gamma(1,n;k_1,\ldots,k_n)=1$ and

$$\gamma(r,n;k_1,\ldots,k_{n-r+1}) = \sum_{(l_1,\ldots,l_{n-r+2})}^{(1,k_1,\ldots,k_{n-r+1},n)} \gamma(r-1,n;l_1,\ldots,l_{n-r+2}).$$

With this extended definition, $\gamma(n,n;k)$ is a polynomial in k. In the following we list it for $n=1,2,\ldots,6$.

$$\begin{aligned}
\gamma(1,1;k) &= 1\\
\gamma(2,2;k) &= -1+3\,k-k^2\\
\gamma(3,3;k) &= \frac{1}{12}(48-92\,k+103\,k^2-40\,k^3+5\,k^4)\\
\gamma(4,4;k) &= \frac{1}{72}(-2160+5910\,k-5407\,k^2+2940\,k^3\\
&\quad -919\,k^4+150\,k^5-10\,k^6)\\
\gamma(5,5;k) &= \frac{1}{1440}(584640-1644072\,k+1970008\,k^2\\
&\quad -1211172\,k^3+456863\,k^4-111708\,k^5\\
&\quad +17462\,k^6-1608\,k^7+67\,k^8)\\
\gamma(6,6;k) &= \frac{1}{7560}(-73316880+225502200\,k\\
&\quad -284097336\,k^2+204504097\,k^3\\
&\quad -91897169\,k^4+27466950\,k^5\\
&\quad -5651016\,k^6+805518\,k^7\\
&\quad -77646\,k^8+4655\,k^9-133\,k^{10})
\end{aligned}$$

Unfortunately, these polynomials are not equal to (4.1). (For instance, they do not factor over $\mathbb{Z}$.) They only coincide on the combinatorial range $\{1, 2, \ldots, n\}$ of k. However, it might still be possible to compute $\gamma(n, n; k)$ for general n.

Strikingly the degree of $\gamma(n, n; k)$ in k is $2n - 2$ as the degree of (4.1). This linear growth is again unexpected because the application of (2.1) can more than double a polynomial's degree, see Section 2. However, one can use Lemma 2 and an extension of Lemma 3 to show that, more generally, the degree of $\gamma(r, n; k_1, \ldots, k_{n-r+1})$ is $2r - 2$ in every k_i.

(4) Finally we have started to investigate a q-version of the formula in Theorem 1, i.e. a weighted enumeration of monotone triangles with prescribed bottom row $(k_1, \ldots, k_n)$ which reduces to our formula as q tends to 1.

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