On the isomorphism problem, indecomposability and the automorphism groups of Coxeter groups

Koji NUIDA

Abstract

Let (W, S) and (W', S') be Coxeter systems, $\{W_\lambda\}_{\lambda \in \Lambda}$ and $\{W'_{\lambda'}\}_{\lambda' \in \Lambda'}$ the sets of the irreducible components of W relative to S and of W' relative to S' respectively, and let $f : W \to W'$ be an isomorphism of abstract groups. Their Coxeter graphs may not be isomorphic. We show that $f(\prod_{\lambda \in \Lambda, |W_\lambda| < \infty} W_\lambda) = \prod_{\lambda' \in \Lambda', |W'_{\lambda'}| < \infty} W'_{\lambda'}$, and that there is a unique bijection $\varphi : \{\lambda \in \Lambda \mid |W_\lambda| = \infty\} \to \{\lambda' \in \Lambda' \mid |W'_{\lambda'}| = \infty\}$ such that $f(W_\lambda) \equiv W'_{\varphi(\lambda)} \bmod Z(W')$ for every $\lambda \in \Lambda$ with $|W_\lambda| = \infty$, where $Z(W')$ is the center of W'. We also determine which two finite Coxeter groups are isomorphic. Our result reduces the problem of deciding whether two Coxeter groups are isomorphic to the case of infinite irreducible Coxeter groups. As a corollary we determine which irreducible Coxeter group is directly indecomposable as an abstract group. In particular, any infinite irreducible Coxeter group is directly indecomposable.

Soient (W, S) et (W', S') deux systèmes de Coxeter, $\{W_\lambda\}_{\lambda \in \Lambda}$ et $\{W'_{\lambda'}\}_{\lambda' \in \Lambda'}$ les ensembles des composantes irréductibles de W relative à S et de W' relative à S' respectivement, et soit $f : W \to W'$ un isomorphisme de groupes abstraits. Leur graphes de Coxeter peuvent être non isomorphes. Nous montrons que $f(\prod_{\lambda \in \Lambda, |W_\lambda| < \infty} W_\lambda) = \prod_{\lambda' \in \Lambda', |W'_{\lambda'}| < \infty} W'_{\lambda'}$, et qu'il y a une bijection unique $\varphi : \{\lambda \in \Lambda \mid |W_\lambda| = \infty\} \to \{\lambda' \in \Lambda' \mid |W'_{\lambda'}| = \infty\}$ telle que $f(W_\lambda) \equiv W'_{\varphi(\lambda)} \bmod Z(W')$ pour chaque $\lambda \in \Lambda$ avec $|W_\lambda| = \infty$, où $Z(W')$ est le centre du W'. De plus, nous déterminons quel deux groupes de Coxeter finis sont isomorphes. Nôtre résultat ramène le problème de juger si deux groupes de Coxeter sont isomorphes àu cas des groupes de Coxeter infinis irréductibles. Par conséquence, nous déterminous quel groupe de Coxeter irréductible est directement indécomposable comme un groupe abstrait. En particulier, un groupe de Coxeter infini irréductible est directement indécomposable.

Introduction

A pair (W, S) with a group W and its generating set S is called a *Coxeter system* if W has a presentation of the following form:

$$W = \langle S \mid (st)^{m(s,t)} = 1 \text{ (for } s, t \in S \text{ such that } m(s,t) < \infty) \rangle$$

where $m : S \times S \to \{1, 2, 3, \dots\} \sqcup \{\infty\}$ is a symmetric map such that $m(s,t) = 1$ if and only if $s = t$. (Note that we need not assume the finiteness of the set S.) We refer this S as a *Coxeter generating set* of W, and a group W is called a *Coxeter group* if it has a Coxeter generating set. One of the most famous examples of Coxeter groups is a (n-th) symmetric group $\mathcal{S}_n$, where the set of $n-1$ adjacent transpositions forms a Coxeter generating set. Moreover, many of the other important groups, such as elementary abelian 2-groups, dihedral groups, signed-permutation groups, Weyl groups and finite (real) reflection groups, all belong the class of Coxeter groups.

The map m above is usually given in the form of a *Coxeter graph*. This is an unoriented simple graph on the vertex set S, and two vertices s, t are joined by an edge labelled $m(s, t)$ if and only if $3 \leq m(s, t) \leq \infty$ (by convention, the labels '3' are usually omitted). For example, the Coxeter graph corresponding to $\mathcal{S}_n$ (with Coxeter generating set as above) is a simple path with $n-1$ vertices, where all edges are unlabelled. It is nontrivial and crucial that the value $m(s, t)$ for $s, t \in S$ coincides with the order of the element st in W (in particular, every generator $s \in S$ has order 2). This fact implies that the Coxeter graph is indeed determined uniquely by a Coxeter system (W, S); in other words, we have the 1-1 correspondence between Coxeter systems and Coxeter graphs (up to isomorphism).

Now some group-theoretic questions arise naturally. The first one is:

Problem 1. *Given a Coxeter group W, is the corresponding Coxeter graph, with respect to a Coxeter generating set S, determined uniquely by the group W and independent on the choice of S?*

In other words, in which case do two Coxeter graphs define isomorphic Coxeter groups? Or more primitively, when are two Coxeter groups isomorphic? This problem is called the *isomorphism problem* of Coxeter groups.

The second problem relates to the notion of irreducible decompositions of Coxeter groups. Given a Coxeter system (W, S) with Coxeter graph Γ, a subgroup of W of the form $W_I = \langle I \rangle$, where I is the vertex set of a connected component of Γ, is called an *irreducible component* of W (with respect to S). It is well known that W is the (restricted) direct product of all the irreducible components (in the viewpoint, W is called *irreducible* (with respect to S) if W has no proper irreducible component). Now the second problem is:

Problem 2. *Given a decomposition $W = \prod_{\lambda} W_{\lambda}$ of a Coxeter group W into irreducible components, is this a finest decomposition of W as an abstract group? In other words, is each irreducible component of W directly indecomposable as an abstract group?*

It has been well known that these two problems have counterexamples and are never easy or trivial. Here we give two classical examples.

Example 3. *We consider the group $W(B_n)$ of signed-permutations on n letters (or the hyperoctahedral group), the finite irreducible Coxeter group of type B_n. It contains the even signed-permutation group $W(D_n)$ (the finite irreducible Coxeter group of type D_n) and the center $Z(W(B_n))$ (which has order 2 and so is $\simeq \mathcal{S}_2 = W(A_1)$) as normal subgroups. Now it is easily checked that if $n \geq 3$ is odd, then $W(B_n)$ is a direct product of these two subgroups. This means that $W(B_n)$ is directly decomposable, and $W(B_n)$ and $W(D_n) \times W(A_1)$ are isomorphic Coxeter groups defined by non-isomorphic Coxeter graphs. This is a counterexample of Problems 1 and 2.*

Example 4. *We consider the dihedral group $\mathcal{D}_m$ of order $2m$, the finite irreducible Coxeter group $W(I_2(m))$ of type $I_2(m)$. Recall that $\mathcal{D}_m$ is the symmetry group of a regular m-gon Δ. If $m = 2k$ is even, then we can obtain an inscribed regular k-gon Δ' in Δ by joining every other vertex of Δ. In this case, the symmetry group $\mathcal{D}_k$ of Δ' is embedded (as a normal subgroup) into $\mathcal{D}_m$, while $\mathcal{D}_m$ has the center of order 2 generated by the half-rotation (rotation of degree π). Now if k is odd, then $\mathcal{D}_m$ is a direct product of these two subgroups, so that $W(I_2(m)) \simeq W(I_2(k)) \times W(A_1)$. This is the second counterexample.*

Moreover, in a paper [8], Bernhard Mühlherr gives an interesting example of two isomorphic non-finite irreducible Coxeter groups on four generators, which are defined by non-isomorphic Coxeter graphs (this is probably the first counterexample of Problem 1 for non-finite irreducible case).

The aim of this short report is to announce some recent results of the author (and also some other related results) on these topics, which is presented in the poster-session of the FPSAC'05. I would like to express my deep gratitude to the organizers of this conference for giving me the opportunity of the presentation, to the referees for their precise reading and precious comments for my report, and to Prof. Itaru Terada (my supervisor) and Prof. Kazuhiko Koike for their several advice and encouragement (especially for suggestion of application for this conference).

Main results

Recall the well-known classification of finite irreducible Coxeter groups (see [7], Chap. 2, etc.). Given a decomposition $W = \prod_{\lambda} W_{\lambda}$ of a Coxeter group W into irreducible components W_{λ} (with respect to a Coxeter generating set S), we define the *finite part* W_{fin} of W as the product of all finite irreducible components W_{λ} (note that W_{fin} may not be a finite group, in the case that W has infinitely many irreducible components). For Problem 1, the author proved the followings:

Theorem 5 ([10], Theorem 3.4). *Given two Coxeter systems (W, S), (W', S'), we have $W \simeq W'$ (as abstract groups) if and only if the following two conditions are satisfied:*

1. *$W_{\mathrm{fin}} \simeq W'_{\mathrm{fin}}$,*
2. *there is a bijection between the set of non-finite irreducible components of W and the set of those of W', such that the corresponding irreducible components are isomorphic (as abstract groups) to each other.*

Theorem 6 ([10], Theorem 3.4). *Given two Coxeter systems (W, S), (W', S'), let a_n, $b_n, \dots, h_4$, i_m denote the cardinality of the set of all irreducible components of W (with respect to S) of type A_n, $B_n, \dots, H_4$, $I_2(m)$, respectively. Define a'_n, $b'_n, \dots, i'_m$ similarly from (W', S'). Then we have $W_{\mathrm{fin}} \simeq W'_{\mathrm{fin}}$ if and only if all of the following equalities hold:*

$$a_1 + \sum_{n \geq 1} b_{2n+1} + e_7 + h_3 + \sum_{m \geq 1} i_{4m+2} = a'_1 + \sum_{n \geq 1} b'_{2n+1} + e'_7 + h'_3 + \sum_{m \geq 1} i'_{4m+2},$$

$$b_3 + a_3 = b'_3 + a'_3, \ b_{2n+1} + d_{2n+1} = b'_{2n+1} + d'_{2n+1} \text{ for } n \geq 2,$$

$$a_2 + i_6 = a'_2 + i'_6,\ i_{2m+1} + i_{4m+2} = i'_{2m+1} + i'_{4m+2} \textit{ for } m \geq 2,$$
$$a_n = a'_n \textit{ for } n \geq 4,\ b_{2n} = b'_{2n} \textit{ for } n \geq 1,\ d_{2n} = d'_{2n} \textit{ for } n \geq 2,$$
$$e_n = e'_n \textit{ for } n = 6, 7, 8,\ f_4 = f'_4,\ h_3 = h'_3,\ h_4 = h'_4,$$
$$i_{2m+1} = i'_{2m+1} \textit{ for } m \geq 2,\ i_{4m+2} = i'_{4m+2} \textit{ for } m \geq 1.$$

These theorems imply that now the isomorphism problem of Coxeter groups are reduced completely to the case of non-finite irreducible Coxeter groups. Moreover, as a byproduct, Theorem 5 shows that the set W_{fin} (not only its group structure) is uniquely determined by W and independent on the choice of S.

For Problem 2, the author also proved the following:

Theorem 7 ([10], Theorem 3.3). *All nontrivial direct product decompositions of irreducible Coxeter groups are one of the followings:*

1. $W(B_n) \simeq W(D_n) \times W(A_1)$ *for* $n \geq 3$ *odd (where we put* $D_3 = A_3$*),*
2. $W(I_2(2k)) \simeq W(I_2(k)) \times W(A_1)$ *for* $k \geq 3$ *odd (where* $I_2(3) = A_2$*),*
3. $W(E_7) = W(E_7)^+ \times W(A_1)$*, where* W^+ *denotes the subgroup of a Coxeter group* W *of elements of even length,*
4. $W(H_3) = W(H_3)^+ \times W(A_1)$.

In particular, all non-finite irreducible Coxeter groups are directly indecomposable as abstract groups, and the center of a directly decomposable irreducible Coxeter group is always a nontrivial direct factor .

Remark 8. *In view of this theorem, the equalities in Theorem 6 mean that, when we decompose (owing to Theorem 7) each of* W_{fin} *and* W'_{fin} *into directly indecomposable factors, there is a 1-1 correspondence between the factors of* W_{fin} *and those of* W'_{fin} *such that the corresponding factors are isomorphic.*

Note that the factors $W(E_7)^+$ and $W(H_3)^+$ in Theorem 7 are not Coxeter groups (namely the simple groups $S_6(2)$ and A_5, respectively). Thus Examples 3 and 4 are the only nontrivial direct product decompositions of irreducible Coxeter groups into other Coxeter groups.

We give further results related to Problem 1. The proofs of Theorems 5 and 6 given in [10] in fact describe the structure of arbitrary isomorphisms between two isomorphic Coxeter groups. Thus, by taking the Coxeter groups as the same group W, we can obtain a description of the automorphism group $\mathrm{Aut}\, W$ of W. The following results are deduced in this way.

We prepare some notations. For a group G and a group homomorphism $f \in \mathrm{Hom}(G, Z(G))$ from G to its center $Z(G)$, we define an endomorphism $f^\flat \in \mathrm{End}\, G$ of G by
$$f^\flat(w) = w f(w)^{-1} \text{ for } w \in G.$$

Lemma 9 ([10], Lemma 2.2). *The map* $f \mapsto f^\flat$ *is injective. Moreover,* $f^\flat$ *is invertible (i.e.* $f^\flat \in \mathrm{Aut}\, G$*) if and only if the restriction* $f^\flat|_{Z(G)}$ *of* $f^\flat$ *to* $Z(G)$ *is an automorphism of* $Z(G)$.

Given a Coxeter group W (with a Coxeter generating set S), we have a decomposition $W = W_{\mathrm{fin}} \times \prod_{\lambda \in \Lambda} W_\lambda$, where W_λ runs over all non-finite irreducible components of W (with respect to S). Put $W_{\mathrm{inf}} = \prod_{\lambda \in \Lambda} W_\lambda$. Then we give a (unique) partition $\Lambda = \bigsqcup_{\xi \in \Xi} \Lambda_\xi$ of the index set Λ such that $W_\lambda \simeq W_\mu$ (as abstract groups) if and only if λ and μ belong the same part Λ_ξ. Now let H_1 be the set of all $f^\flat$ such that $f \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$. Let $H_2 = \mathrm{Aut}\, W_{\mathrm{fin}}$, H_3 the (complete) direct product of all $\mathrm{Aut}\, W_\lambda$ ($\lambda \in \Lambda$) and H_4 the (complete) direct product of all symmetric groups $\mathrm{Sym}(\Lambda_\xi)$ of the sets Λ_ξ ($\xi \in \Xi$). Note that all of H_2, H_3 and H_4 are naturally embedded into $\mathrm{Aut}\, W$. Now we have:

Theorem 10 ([10], Theorem 3.10). *The group* $\mathrm{Aut}\, W$ *decomposes as*

$$\mathrm{Aut}\, W = H_1 \rtimes (H_2 \times H_3) \rtimes H_4.$$

Moreover, the action of H_4 *fixes* H_2 *pointwise and leaves* H_3 *invariant.*

Note that the group homomorphisms from a Coxeter group W to a group of order 2 (and so the homomorphisms from W to the center $Z(W)$ which is an elementary abelian 2-group) are characterized as follows, by using an odd-Coxeter graph. Here an *odd-Coxeter graph* is the graph obtained from a Coxeter graph by removing all but the edges with label odd. Then the following fact is easy to check (by definition of Coxeter groups).

Lemma 11. *Let* W *be a Coxeter group with odd-Coxeter graph* Γ^{odd} *(with respect to a Coxeter generating set* S*). Then for any map* f *from* W *to a group of order* 2*,* f *is a group homomorphism if and only if* $f(s) = f(t)$ *whenever* $s, t \in S$ *are in the same connected component of* Γ^{odd}*.*

Next, we consider the automorphism group of the finite part W_{fin}, which is the factor H_2 in Theorem 10. Owing to Theorem 7, we obtain a direct product decomposition $W_{\mathrm{fin}} = G_0 \times \prod_{i \in I} G_i$ of W_{fin} such that G_0 is an elementary abelian 2-group (namely the product of all factors $\simeq W(A_1)$) and each G_i ($i \in I$) is either a finite irreducible Coxeter group of type other than A_1, B_n (n odd), $I_2(2k)$ ($k \geq 3$ odd), E_7 and H_3, or a simple group $W(E_7)^+$ or $W(H_3)^+$ (and so each G_i is directly indecomposable as an abstract group). We give a partition $I = \bigsqcup_{\upsilon \in \Upsilon} I_\upsilon$ of the index set I similarly. Let H'_1 be the set of all $f^\flat \in \mathrm{Aut}\, W_{\mathrm{fin}}$ such that $f \in \mathrm{Hom}(W_{\mathrm{fin}}, Z(W_{\mathrm{fin}}))$. Let H'_2 be the (complete) direct product of all $\mathrm{Aut}\, G_i$ ($i \in I$) and H'_3 the (complete) direct product of all $\mathrm{Sym}(I_\upsilon)$ ($\upsilon \in \Upsilon$). Moreover, let H'_4 be the set of all $f^\flat$ such that $f \in \mathrm{Hom}(W_{\mathrm{fin}}, Z(W_{\mathrm{fin}}))$, $f(G_0) = 1$ and $f(G_i) \subset Z(G_i)$ for any $i \in I$. Then we have:

Theorem 12 ([10], Theorem 3.10). *We have*

$$\mathrm{Aut}\, W_{\mathrm{fin}} = (H'_1 H'_2) \rtimes H'_3 \text{ and } H'_1 \cap H'_2 = H'_4.$$

Moreover, H'_1 *is normal in* $\mathrm{Aut}\, W_{\mathrm{fin}}$ *and the action of* H'_3 *leaves* H'_2 *invariant.*

Note that the structure of $\mathrm{Aut}\, W_{\mathrm{fin}}$ is still complicated because of the existence of the intersection H'_4 of the factors H'_1 and H'_2. However, we can compute the order of $\mathrm{Aut}\, W$ for finite Coxeter groups W; see later sections.

Moreover, we consider a decomposition $W = \prod_{\lambda' \in \Lambda'} W_{\lambda'}$ of a Coxeter group W into (possibly finite) irreducible components $W_{\lambda'}$. We give a similar partition $\Lambda' = \sqcup_{\xi' \in \Xi'} \Lambda'_{\xi'}$ of the index set Λ'. Then the (complete) direct product H of all

$\operatorname{Aut} W_{\lambda'}$ ($\lambda' \in \Lambda'$) and all $\mathrm{Sym}(\Lambda'_{\xi'})$ ($\xi' \in \Xi'$) is also embedded naturally into $\operatorname{Aut} W$. (This H can be regarded as the set of automorphisms of W which are easily seen by the decomposition of W.) Then the author proved the following:

Theorem 13 ([10], Theorem 3.10). *The subgroup H of $\operatorname{Aut} W$ has finite index in $\operatorname{Aut} W$ if and only if either $Z(W) = 1$ or the odd-Coxeter graph of W consists of only finitely many connected components.*

Note that Theorems 5, 6, 7 and 13 are also proved independently by Luis Paris in [12], only for finitely generated Coxeter groups (in fact, in his proofs the finiteness of the rank of the Coxeter group is essential).

Outline of the proof

Theorems 5 and 6 are proved by using Theorem 7. To prove Theorem 7, we first show the following property of the centralizers of certain subgroups in Coxeter groups.

Proposition 14 ([10], Theorem 3.1). *Let H be a normal subgroup of a Coxeter group W which is generated by involutions. Then the structure of the centralizer $Z_W(H)$ of H in W is described completely. In particular, if $Z(W) \subsetneq Z_W(H) \subsetneq W$, then there is a proper subgroup of W containing both H and $Z_W(H)$.*

For the proof, it follows from some group theory and properties of Coxeter groups that $Z_W(H)$ is the intersection of core subgroups $\mathrm{Core}_W(N_W(W_I))$ (where $\mathrm{Core}_W(G)$ is defined as the unique largest normal subgroup of W contained in G) of the normalizers $N_W(W_I)$ of certain parabolic subgroups W_I. Then the proof of Proposition 14 is reduced to the computation of the groups $\mathrm{Core}_W(N_W(W_I))$. This is done by a certain graph-theoretical argument about the Coxeter graph owing to some properties of the normalizers $N_W(W_I)$ examined by Brigitte Brink and Robert B. Howlett in a paper [3].

Once Proposition 14 is proved, a part of Theorem 7, namely the direct indecomposability of non-finite irreducible Coxeter groups, is deduced immediately. In fact, if a non-finite irreducible Coxeter group W admits a decomposition $W = G_1 \times G_2$ into subgroups, then G_1 is generated by involutions (since it is a quotient of W) and $W = G_1 Z_W(G_1)$. Since now $Z(W) = 1$, we have (by Proposition 14) either $Z_W(G_1) = 1$ or $Z_W(G_1) = W$ (and so $G_1 = 1$). This implies the desired direct indecomposability of W. The remaining part of the theorem follows from a case-by-case argument based on the classification of finite irreducible Coxeter groups.

By a similar argument based on Proposition 14, if G_1 and G_2 are groups generated by involutions and $f : G_1 \times G_2 \to W$ is a surjective homomorphism from $G_1 \times G_2$ to a Coxeter group W, then either $f(G_1) \subset Z(W)$ or $f(G_2) \subset Z(W)$. Owing to this observation, Theorems 5, 6, 10 and 12 are deduced by similar arguments to the proof of the Remak-Krull-Schmidt Theorem about direct product decompositions of groups (see [15], Section 4.6–4.7). Theorem 13 is deduced from Theorems 10 and 12 together with Lemma 11.

$\operatorname{Aut} W$ of finite Coxeter groups W

As we predicted in a preceding section, we compute the order of the automorphism group $\operatorname{Aut} W$ (or its 'growth' $|\operatorname{Aut} W|/|W|$) of an arbitrary finite Coxeter group W by using Theorem 12. First, since H_1' is normal in $\operatorname{Aut} W$, the product $H_1'H_2'$ consists of elements of the form w_1w_2 ($w_1 \in H_1'$, $w_2 \in H_2'$) and so we have

$$\frac{|\operatorname{Aut} W|}{|W|} = \frac{|H_1'| \cdot |H_2'| \cdot |H_3'|}{|H_4'| \cdot |W|}.$$

Let $a_n, \dots, i_m$ be as in Theorem 6 (now all but finitely many of those are 0), and let $W = G_0 \times \prod_{i \in I} G_i$ be the decomposition of W introduced before the statement of Theorem 12. Note that G_0 is the direct product of the factors $\simeq W(A_1)$, where the number N of factors is (by Theorem 7)

$$N = a_1 + \sum_{n \geq 1} b_{2n+1} + e_7 + h_3 + \sum_{m \geq 1} i_{4m+2}.$$

Similarly, the numbers of factors G_i ($i \in I$) isomorphic to $W(A_2)$, $W(A_3)$, $W(D_{2n+1})$ ($n \geq 2$), $W(I_2(2m+1))$ ($m \geq 2$), $W(E_7)^+$, $W(H_3)^+$ are $a_2 + i_6$, $a_3 + b_3$, $b_{2n+1} + d_{2n+1}$, $i_{2m+1} + i_{4m+2}$, e_7, h_3, respectively.

For the factor H_1' of $\operatorname{Aut} W$, we use the following easy group-theoretic lemma:

Lemma 15. *Let G be a directly indecomposable non-abelian group such that $|Z(G)| \leq 2$. Then we have $f(Z(G)) = 1$ for any homomorphism f from G to a group of order 2.*

Owing to this lemma, we have $f(Z(G_i)) = 1$ for any $f \in \operatorname{Hom}(W, Z(W))$ (since $Z(W)$ is an elementary abelian 2-group and each G_i is either simple or a directly indecomposable non-abelian Coxeter group). Thus we have

$$f^\flat(w) = w \text{ for } w \in Z(G_i),\ f^\flat(w) = wf(w) \text{ for } w \in G_0.$$

By regarding elementary abelian 2-groups as vector spaces over a finite field $\mathbb{F}_2$, this implies (owing to Lemma 9) that the order of H_1' is equal to the number of invertible matrices with coefficients in $\mathbb{F}_2$ of the form

$$\begin{pmatrix} I_N + X & O \\ Y & I_M \end{pmatrix}$$

where N is as above,

$$M = |\{i \in I \mid Z(G_i) \neq 1\}| = \sum_{n \geq 1} b_{2n} + \sum_{n \geq 2} d_{2n} + e_8 + f_4 + h_4 + \sum_{m \geq 2} i_{4m}$$

(see Theorem 7) and X, Y are matrices of appropriate size with coefficients in $\mathbb{F}_2$. Thus we have

$$|H_1'| = |\mathrm{GL}_N(\mathbb{F}_2)| \cdot 2^{NM} = 2^{\binom{N}{2}} \prod_{j=1}^{N} (2^j - 1) \cdot 2^{NM}.$$

For the factor H_2', note that for any finite group G, we have

$$\frac{|\operatorname{Aut} G|}{|G|} = \frac{|\operatorname{Aut} G|}{|\operatorname{Inn} G| \cdot |Z(G)|} = \frac{|\operatorname{Out} G|}{|Z(G)|},$$

where $\operatorname{Inn} G$, $\operatorname{Out} G = \operatorname{Aut} G / \operatorname{Inn} G$ denote the groups of inner, outer automorphisms of G, respectively. Then by Table I in [1] of outer automorphism groups of finite irreducible Coxeter groups, $|\operatorname{Aut} G_i|/|G_i|$ (for $i \in I$) is

$$\begin{cases} 2 & \text{if } G_i \simeq W(A_5), W(B_{2n})\ (n \geq 2), W(D_{2n})\ (n \geq 3), W(H_4) \text{ or } W(H_3)^+ \\ 4 & \text{if } G_i \simeq W(F_4) \\ 6 & \text{if } G_i \simeq W(D_4) \\ \frac{\varphi(m)}{2} & \text{if } G_i \simeq W(I_2(m)) \\ 1 & \text{otherwise} \end{cases}$$

(where φ is the Euler function). Note that $\varphi(4m+2) = \varphi(2m+1)$ since $2m+1$ is odd, and $\varphi(6)/2 = 1$. Thus by the above observation on the numbers of factors G_i of each type, we have

$$\begin{aligned} \frac{|H_2'|}{|W|} &= |G_0|^{-1} \cdot \prod_{i \in I} \frac{|Aut\, G_i|}{|G_i|} \\ &= \frac{1}{2^N} \cdot 2^{a_5 + \sum_{n \geq 2} b_{2n} + \sum_{n \geq 3} d_{2n} + h_3 + h_4} \cdot 4^{f_4} \cdot 6^{d_4} \\ &\quad \cdot \prod_{m \geq 2} \left(\left(\frac{\varphi(2m+1)}{2} \right)^{i_{2m+1} + i_{4m+2}} \left(\frac{\varphi(4m)}{2} \right)^{i_{4m}} \right) \\ &= 2^{-N + a_5 + \sum_{n \geq 2} (b_{2n} + d_{2n}) + 2f_4 + h_3 + h_4 - \sum_{m \geq 5} i_m} \cdot 3^{d_4} \cdot \prod_{m \geq 5} \varphi(m)^{i_m}. \end{aligned}$$

For the order of H_3', it follows immediately from definition that

$$\begin{aligned} |H_3'| = \big(a_2 + i_6\big)!\, (a_3 + b_3)! \prod_{n \geq 4} a_n! \prod_{n \geq 1} b_{2n}! \prod_{n \geq 2} d_{2n}! \prod_{n \geq 2} (b_{2n+1} + d_{2n+1})! \\ \cdot e_6!\, e_7!\, e_8!\, f_4!\, h_3!\, h_4! \prod_{m \geq 2} \big(i_{2m+1} + i_{4m+2}\big)! \prod_{m \geq 2} i_{4m}!. \end{aligned}$$

Moreover, by definition, $|H_4'|$ is equal to the product of the size of all $\operatorname{Hom}(G_i, Z(G_i))$, $i \in I$. Since $|Z(G_i)| \leq 2$, it follows from Lemma 11 that the size of $\operatorname{Hom}(G_i, Z(G_i))$ is

$$\begin{cases} 2 & \text{if } G_i \simeq W(D_{2n})\ (n \geq 2), W(E_8) \text{ or } W(H_4) \\ 4 & \text{if } G_i \simeq W(B_{2n})\ (n \geq 1), W(F_4) \text{ or } W(I_2(4m))\ (m \geq 2) \\ 1 & \text{otherwise.} \end{cases}$$

Thus we have

$$|H_4'| = 2^{2 \sum_{n \geq 1} b_{2n} + \sum_{n \geq 2} d_{2n} + e_8 + 2f_4 + h_4 + 2 \sum_{m \geq 2} i_{4m}}.$$

Now we compress the data $a_n, b_n, \ldots, i_m$ into a symbol $\mathbf{k}$, and write $W = W_{\mathbf{k}}$, $N = N_{\mathbf{k}}$ and $M = M_{\mathbf{k}}$. Then in the following generating function

$$F(\mathbf{X}) = \sum_{\mathbf{k}} \frac{|\operatorname{Aut} W_{\mathbf{k}}|}{|W_{\mathbf{k}}|} {X_{A_1}}^{a_1} \prod_{n \geq 2} \frac{{X_{A_n}}^{a_n}}{a_n!} \cdots \prod_{m \geq 5} \frac{{X_{I_2(m)}}^{i_m}}{i_m!}$$

(where the sum runs over all $\mathbf{k}$ such that all but finitely many contents are 0), the coefficient of $\mathbf{X}^{\mathbf{k}} = \prod_{n\geq 1} {X_{A_n}}^{a_n} \cdots \prod_{m\geq 5} {X_{I_2(m)}}^{i_m}$ is (by the above arguments)

$$2^{\binom{N_{\mathbf{k}}}{2}+N_{\mathbf{k}}M_{\mathbf{k}}-a_1+a_5-b_2-\sum_{n\geq 2} b_n-e_7-e_8-\sum_{m\geq 2}(i_{4m-3}+2i_{4m-2}+i_{4m-1}+3i_{4m})}$$
$$\cdot 3^{d_4} \prod_{j=1}^{N_{\mathbf{k}}} (2^j-1) \prod_{m\geq 5} \varphi(m)^{i_m} \cdot \binom{a_2+i_6}{a_2}\binom{a_3+b_3}{a_3}$$
$$\cdot \prod_{n\geq 2} \binom{b_{2n+1}+d_{2n+1}}{b_{2n+1}} \prod_{m\geq 2} \binom{i_{2m+1}+i_{4m+2}}{i_{2m+1}}.$$

As a simple example, we consider the case that every irreducible component of W is of type A; in other words, W is a Young subgroup of a symmetric group. The corresponding generating function F_Y is obtained from F by substituting $X_{B_n} = \cdots = X_{I_2(m)} = 0$. Put $X_j = X_{A_j}$ $(j \geq 1)$. Then we have

$$\begin{aligned} F_Y(X_1, X_2, \dots) &= \sum_{a_1, a_2, \dots} 2^{\binom{a_1}{2}-a_1+a_5} \prod_{j=1}^{a_1} (2^j-1) {X_1}^{a_1} {X_2}^{a_2} \cdots \\ &= G\Big(\frac{X_1}{2}\Big) \frac{1}{1-2X_5} \sum_{\lambda} m_\lambda(X_2, X_3, X_4, X_6, X_7, \dots) \end{aligned}$$

where $G(x) = \sum_n |\mathrm{GL}_n(\mathbb{F}_2)| x^n$ denotes the generating function of the order of $\mathrm{GL}_n(\mathbb{F}_2)$, m_λ is the monomial symmetric function and the sum runs over all partitions λ.

Further remarks

As we have seen above, we now have a complete solution to Problem 2 (Theorem 7). On the other hand, Problem 1, as well as computation of the automorphism group, is basically reduced to the case of non-finite irreducible Coxeter groups (see Theorems 5, 6, 10 and 12). Nevertheless, it is still true that this problem is difficult to solve generally.

The study of the isomorphism problem of general Coxeter groups has been developing rapidly only in this decade, especially in the last five years. Here we summarize some recent results on this topic.

First we prepare some terminology. For a Coxeter system (W, S), let $R_S(W)$ be the set of *reflections* in W with respect to S; namely,

$$R_S(W) = \{w \in W \mid w \text{ is conjugate to some } s \in S\}.$$

W is said to be *rigid* if all Coxeter generators S' of W yield isomorphic Coxeter graphs. W is said to be *strongly rigid* if all Coxeter generators S' of W are conjugate in W with each other (note that this name is valid, since strong rigidity is actually stronger than rigidity). W (precisely, (W, S)) is said to be (*strongly*) *reflection rigid* if the conclusion in definition of (strong) rigidity holds for all Coxeter generators S' contained in $R_S(W)$. Moreover, W is said to be *reflection independent* if the set $R_S(W)$ of reflections is independent on the choice of the Coxeter generating set S.

The followings are examples of the known results on these properties:

Theorem 16. *Let (W, S) be a Coxeter system.*

1. *([2], Theorem 3.10) If W is finite, then (W, S) is reflection rigid.*
2. *([2], Theorem 3.9) If (W, S) is 'even' (that is, $m(s, t)$ is not odd for any distinct $s, t \in S$), then (W, S) is reflection rigid.*
3. *([13]) If $|S| < \infty$ and (W, S) is 'right-angled' (that is, $m(s, t) \in \{2, \infty\}$ for any distinct $s, t \in S$), then W is rigid.*
4. *([4]) If $|S| < \infty$ and W is capable of acting effectively, properly and co-compactly on some contractible manifold, then W is strongly rigid. In particular, affine Coxeter groups are strongly rigid.*
5. *([9]) If $|S| < \infty$ and W is non-finite, irreducible and '2-spherical' (that is, $m(s, t) < \infty$ for any $s, t \in S$), then W is strongly rigid.*

On the other hand, for the counterexamples for these properties, we summarize the properties of finite irreducible Coxeter groups in Table 1.

Table 1: List of properties of finite irreducible Coxeter groups

type		strongly rigid	rigid	strongly reflection rigid	reflection independent
A_n	$(n \neq 5)$	○	○	○	○
A_5		×	○	○	×
B_2		○	○	○	○
B_n	$(n \geq 3$ odd$)$	×	×	○	×
B_n	$(n \geq 4$ even$)$	×	○	○	×
D_n	$(n \geq 5$ odd$)$	○	○	○	○
D_n	$(n \geq 4$ even$)$	×	○	○	×
E_6, E_7		○	○	○	○
E_8		×	○	○	×
F_4		×	○	○	×
H_3		×	○	×	○
H_4		×	○	×	×
$I_2(m)$	$(m \not\equiv 2 \pmod 4)$	×	○	×	○
$I_2(6)$		×	×	○	×
$I_2(4k+2)$	$(k \geq 2)$	×	×	×	×

In this list, the reflection rigidity is omitted since it always holds.
The description of $\operatorname{Aut} W$ for finite irreducible W given in [1] is used.

We introduce an important operation, *diagram twisting*, on Coxeter graphs and a related conjecture for the isomorphism problem. Let (W, S) be a Coxeter system with Coxeter graph Γ. A diagram twisting on Γ, with respect to two subsets $I, J \subset S$ satisfying certain conditions, is an operation which changes, for each edge e of Γ between I and J, the terminal vertex $s \in J$ of e to the new terminal vertex $w_0(J)sw_0(J)$ (where $w_0(J)$ denotes the longest element of W_J; its existence is due to the definition of diagram twistings). For the precise definition, see the paper [2] in which diagram twistings are first introduced. It is shown in [2] that, if we obtain another Coxeter graph Γ' from Γ by a diagram twisting, then there is canonically another Coxeter generating set S' of W which corresponds to Γ'. In this case, we have $R_S(W) = R_{S'}(W)$ but Γ and Γ' may be non-isomorphic. This means that, by using diagram twistings, we can obtain many examples of non-rigid Coxeter groups (note that Mühlherr's example given in [8] which we mentioned above is one of such examples).

For the converse direction, the following conjecture is presented in [2]:

Conjecture 17 ([2], Conjecture 8.1). *Let W be a Coxeter group and S, S' two finite Coxeter generating sets of W. Let Γ, Γ' be the Coxeter graphs of W with respect to S, S', respectively. If $R_S(W) = R_{S'}(W)$, then Γ' is obtained from Γ by a finite number of consecutive diagram twistings. In other words, finitely-generated Coxeter groups are reflection rigid up to diagram twistings.*

Moreover, it is hoped by many researchers (including the author) that there are (not too many) classes of special isomorphisms, described explicitly, between Coxeter groups (such as diagram twistings) such that, an arbitrary isomorphism between two Coxeter groups is made up from those special ones. If this is fortunately true, then we can reduce the study of relations between some combinatorial properties of two isomorphic Coxeter groups to the study of those special isomorphisms.

Finally, we state some more recent results of the author. Note that, most of the results on the isomorphism problem of Coxeter groups which have been known now are only limited to the case of finitely generated Coxeter groups. One of the reason is that now a main strategy for this problem is to analize the maximal finite subgroups of given Coxeter groups, but in the non-finitely generated cases, it happens very often that the Coxeter group has no maximal finite subgroups. In contrast with those cases, the results of the author are applicable to non-finitely generated cases as well as finitely generated cases.

We prepare some notations. For a generator $x \in S$ of a Coxeter system (W, S), let $W^{\perp x}$ be the subgroup of W generated by all reflections $t \in R_S(W)$, $t \neq x$, which commutes with x. By a general result of Vinay V. Deodhar [5] or of Matthew Dyer [6], $W^{\perp x}$ forms a Coxeter group with a canonical Coxeter generating set. Let ${W^{\perp x}}_{\mathrm{fin}}$ denote the finite part of this Coxeter group $W^{\perp x}$.

Theorem 18 ([11]). *Let (W, S), (W', S') be two Coxeter system (where S or S' may be infinite), $f : W \xrightarrow{\sim} W'$ an isomorphism and $x \in S$.*

1. *The structure of ${W^{\perp x}}_{\mathrm{fin}}$ is completely determined.*
2. *(See [14]) There are an inner automorphism g of W' and a finite subset $I' \subset S'$ of (-1)-type (i.e. every irreducible component of $W'_{I'}$ has nontrivial center) such that $g \circ f(x)$ is the longest element $w_0(I')$ of $W'_{I'}$.*
3. *Suppose that there is a finite subset $I' \subset S'$ of (-1)-type such that $f(x) = w_0(I')$. Then $f^{-1}(W'_{I'}) \subset \langle x \rangle \times {W^{\perp x}}_{\mathrm{fin}}$.*
4. *Suppose that ${W^{\perp x}}_{\mathrm{fin}}$ is either trivial or generated by a single reflection $t \in R_S(W)$ conjugate to x. Then $f(x) \in R_{S'}(W')$.*

As an application of this theorem, the author obtained recently the following results on reflection independence of some (possibly non-finitely generated) Coxeter groups.

Theorem 19 ([11]). *Let (W, S) be a Coxeter system (where S may be infinite).*

1. *If W is non-finite, irreducible and 2-spherical, then W is reflection independent.*
2. *If W is non-finite and 'odd-connected' (that is, the odd-Coxeter graph Γ^{odd} is connected), then W is reflection independent.*

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Koji Nuida
Graduate School of Mathematical Sciences, University of Tokyo, Japan

Supported by JSPS Research Fellowship (No. 16-10825)

E-mail: *nuida@ms.u-tokyo.ac.jp*