THE COMBINATORICS OF TRANSLATION FUNCTORS FOR THE VIRASORO ALGEBRA

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ABSTRACT. In this paper, I consider the representation theory of the Virasoro algebra Vir from a combinatorial perspective. For the Virasoro algebra, the irreducible modules in Category $\mathcal{O}$ are indexed by the weights $\lambda \in \mathfrak{h}^\star = \text{Hom}(\mathfrak{h}, \mathbb{C})$ of Vir, where $\mathfrak{h}$ is a certain abelian subalgebra of Vir. The weights of Vir can be naturally partitioned in blocks. Using the results of Feigin and Fuchs [1], I give a combinatorial characterization of the blocks for the Virasoro algebra. The module $M(\lambda) \otimes L(\mu)$, where $M(\lambda)$ is the Verma module indexed by the weight $\lambda \in \mathfrak{h}^\star$ and $L(\mu)$ is the irreducible module indexed by the weight $\mu \in \mathfrak{h}^\star$, decomposes into a direct sum of submodules according to blocks. I give a description of the submodules corresponding to each block. The main tools I use to describe blocks and the decomposition of $M(\lambda) \otimes L(\mu)$ are the Shapovalov form and the Shapovalov determinant.

Ce travail est une étude de la théorie combinatoire des représentations de l'algèbre de Lie Virasoro Vir. Les modules simples de la catégorie $\mathcal{O}$ pour Vir sont parametrisés par des poids $\mathfrak{h}^\star = \text{Hom}(\mathfrak{h}, \mathbb{C})$ de Vir, où $\mathfrak{h}$ est une sous-algèbre commutative spécifique. Les poids sont partitionnés en blocs. En utilisant le travail de Feigin et Fuchs [1], je donne une caractérisation combinatoire pour les blocs de l'algèbre de Virasoro. Si $M(\lambda)$ est un module Verma et $L(\mu)$ le module simple de poids maximal μ, le produit $M(\lambda) \otimes L(\mu)$ a une décomposition en somme directe de sous-modules indexés par les blocs. Je donne une description des sous-modules correspondant aux blocs. Les techniques principales utilisées sont la forme de Shapovalov et le déterminant de Shapovalov.

1. INTRODUCTION

The motivation for this paper comes from similar results for semisimple Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Shapovalov [7] showed that the Shapovalov determinant associated to a Verma module $M(\lambda)$ of $\mathfrak{g}$ reflects the block structure of Category $\mathcal{O}$ for the Lie algebra $\mathfrak{g}$. Jantzen [3] used the Shapovalov determinant to analyze translation functors for $\mathfrak{g}$, that is, the decomposition of $M(\lambda) \otimes L(\mu)$ as a direct sum of submodules each corresponding to a block. Specifically, he produced determinant formulas for each of these submodules. Using these formulas, Gabber and Joseph [2] were able to contruct a Hecke algebra action on the blocks of $\mathfrak{g}$ using certain translation functors. They suggest this approach may lead to a proof of the Kazhdan-Lusztig Conjecture.

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My results are analogous to the results of Jantzen and may allow for contructions for the Virasoro algebra similar to those of Gabber and Joseph. In particular, it may be possible to formulate a Hecke algebra and Kazhdan-Lusztig polynomials for the Virasoro algebra.

2. The Virasoro algebra and a definition of blocks

Lie algebras with triangular decomposition generalize complex finite-dimensional semisimple Lie algebras. The Virasoro algebra is an important example of this class of Lie algebras. The *Virasoro algebra*, Vir, is the complex vector space with basis

$$z \quad \text{and} \quad d_k, k \in \mathbb{Z}$$

and with relations

$$[d_j, d_k] = (j-k)d_{j+k} + \delta_{j,-k}\frac{j^3-j}{12}z, \qquad [z, d_k] = 0.$$

The triangular decomposition of Vir is given by

$$Vir = Vir_- \oplus \mathfrak{h} \oplus Vir_+,$$

where Vir_- is the subalgebra of Vir generated by $d_k, k \in \mathbb{Z}_{<0}$; $\mathfrak{h}$ is the subalgebra generated by z and d_0; and Vir_+ is the subalgebra generated by $d_k, k \in \mathbb{Z}_{>0}$.

Note that $\mathfrak{h}$ acts diagonally on $Vir_\pm$:

$$[d_0, d_k] = -kd_k, \quad [z, d_k] = 0.$$

The *enveloping algebra* $U(Vir)$ *of* Vir is the associative algebra with generators

$$z \quad \text{and} \quad d_k, k \in \mathbb{Z}$$

and relations

$$\begin{aligned} d_j d_k - d_k d_j &= (j-k)d_{j+k} + \delta_{j,-k}\frac{j^3-j}{12}z \\ z d_k - d_k z &= 0 \end{aligned}$$

The algebra $U(Vir_+)$ is the subalgebra of $U(Vir)$ generated by d_k, $k > 0$, and $U(Vir_-)$ is the subalgebra of $U(Vir)$ generated by d_k, $k < 0$. The algebra $U(Vir_-)$ has a basis

$$d_{-\lambda} = d_{-\lambda_1} d_{-\lambda_2} \cdots d_{-\lambda_k}, \quad \lambda = (\lambda_1 \geq \cdots \geq \lambda_k) \text{ a partition.}$$

For example, if $\lambda = (5,5,3,2)$, then $d_{-\lambda} = d_{-5}d_{-5}d_{-3}d_{-2}$.

Let $\mathfrak{h}^\star = \mathrm{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$, linear maps from $\mathfrak{h}$ to $\mathbb{C}$. There is a bijection

$$\begin{aligned} \mathfrak{h}^\star &\rightarrow \mathbb{C}^2 \\ \lambda &\mapsto (\lambda(d_0), \lambda(z)) . \end{aligned}$$

For $\lambda, \mu \in \mathfrak{h}^\star$, define $\lambda > \mu$ if $\lambda(d_0) - \mu(d_0) \in \mathbb{Z}_{<0}$ and $\lambda(z) = \mu(z)$. This partial order on $\mathfrak{h}^\star$ is the analogue for Vir of the classical dominance order on partitions.

Category $\mathcal{O}$ consists of those Vir-modules M such that

- M is $\mathfrak{h}$-diagonalizable:

$$M = \bigoplus_{\mu \in \mathfrak{h}^\star} M^\mu, \quad \text{where} \quad M^\mu = \{m \in M | hm = \mu(x)m \text{ for all } x \in \mathfrak{h}\};$$

- with respect to dominance, the $\mu \in \mathfrak{h}^\star$ such that $M^\mu \neq 0$ lie in a lower order ideal with a finite number of maximal elements;
- $\dim M^\mu < \infty$ for all $\mu \in \mathfrak{h}^\star$.

Category $\mathcal{O}$ reflects much of the structure of Vir. Within Category $\mathcal{O}$, Verma modules are of special interest. For $(h, c) \in \mathbb{C}^2$, define the Verma module $M(h, c)$ to be the $U(Vir)$-module generated by v^+ with relations

$$d_0 v^+ = hv^+, \quad zv^+ = cv^+, \quad d_k v^+ = 0 \quad \text{for } k > 0.$$

Then

$$M(h, c) = \bigoplus_{(h',c') \leq (h,c)} M(h, c)^{(h',c')} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(h, c)^{(h+n,c)}.$$

Proposition 1. [6] *The set* $\{ d_{-\lambda} v^+ | \, \lambda \vdash n\}$ *is a basis for* $M(h, c)^{(h+n,c)}$.

For each $\lambda \in \mathfrak{h}^\star$, $M(\lambda)$ has a unique proper maximal submodule $J(\lambda)$. This implies that $M(\lambda)$ has a unique irreducible quotient

$$L(\lambda) = M(\lambda)/J(\lambda).$$

Theorem 1. [6] *Up to isomorphism, the modules $L(\lambda)$ are all of the irreducible modules in Category $\mathcal{O}$.*

In general, modules M in Category $\mathcal{O}$ do not have finite composition series. However, one can make appropriate definitions to say that "$L(\mu)$ appears in M" if $L(\mu)$ is a factor in a "local" composition series for M. Let $\sim$ be the equivalence relation generated by $\lambda \sim \mu$ if $L(\mu)$ appears in $M(\lambda)$. The *blocks* are the equivalence classes $[\lambda]$ of $\mathfrak{h}^\star$ with respect to this equivalence relation.

3. A Description of Blocks

In this section we explicitly describe the blocks for the Virasoro algebra. The main tools we use are the Shapovalov form and the Shapovalov determinant.

The following result provides an alternative way to define blocks.

Proposition 2. [6] *For $\mu, \lambda \in \mathfrak{h}^\star$, $L(\mu)$ appears in $M(\lambda)$ if and only if $M(\mu) \subseteq M(\lambda)$.*

Verma module embeddings prove to be a useful way to characterize blocks. In particular, the Shapovalov form and the Shapovalov determinant can be used to provide a complete description of Verma module embeddings.

For $(h,c) \in \mathbb{R}^2$ the *Shapovalov form*, $\langle , \rangle : M(h,c) \times M(h,c) \to \mathbb{R}$, is the form defined by the conditions

$$\begin{aligned} \langle v^+, v^+ \rangle &= 1 \\ \langle d_{-\lambda_1} \cdots d_{-\lambda_k} v^+, d_{-\mu_1} \cdots d_{-\mu_j} v^+ \rangle &= \langle v^+, d_{\lambda_k} \cdots d_{\lambda_1} d_{-\mu_1} \cdots d_{-\mu_j} v^+ \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \alpha v + u, w \rangle &= \overline{\alpha} \langle v, w \rangle + \langle u, w \rangle \\ \langle w, \alpha v + u \rangle &= \alpha \langle w, v \rangle + \langle w, u \rangle \end{aligned}$$

for $\alpha \in \mathbb{C}$, $u, v, w \in M(h,c)$. For example,

$$\langle d_{-1} v^+, d_{-1} v^+ \rangle = \langle v^+, d_1 d_{-1} v^+ \rangle = \langle v^+, (d_{-1} d_1 + 2d_0) v^+ \rangle = \langle v^+, 2h v^+ \rangle = 2h.$$

Define the *radical* of $\langle , \rangle$ by

$$Rad\langle , \rangle = \{ v \in M(h,c) | \langle v, w \rangle = 0 \ \text{ for all } w \in M(h,c) \}.$$

Lemma 1. [6]

- *$Rad\langle , \rangle = J(h,c)$.*
- *$M(h,c)^{(h+n,c)} \perp M(h,c)^{(h+m,c)}$ with respect to $\langle , \rangle$ if $m \neq n$.*

This lemma has the following implications: the form provides information about submodules of $M(h,c)$, and we can consider how the form acts weight space by weight space. Also note that this lemma implies the Shapovalov form is well-defined on the quotient $L(h,c)$.

The *Shapovalov determinant* for $M(h,c)^{(h+n,c)}$ is

$$detM(h,c)^{(h+n,c)} = det \left(\langle d_{-\lambda} v^+, d_{-\mu} v^+ \rangle \right)_{\lambda, \mu \vdash n}.$$

The determinant $detM(h,c)^{(h+n,c)}$ gives information about the submodule structure of $M(h,c)$:

$$(J(h,c))^{(h+n,c)} \neq 0 \quad \text{if and only if} \quad detM(h,c)^{(h+n,c)} = 0.$$

Kac [5] and Feigin and Fuchs [1] have found explicit formulas for $detM(h,c)^{(h+n,c)}$.

Theorem 2. [4], [1] *Define*

$$\begin{aligned} \mathcal{C}_{r,s}(h,c) &= \left(48h - \left((13-c)(r^2+s^2) - 24rs - 2 + 2c\right)\right)^2 - (c-1)(c-25)(r^2-s^2)^2 \text{ if } r \neq s, \\ \mathcal{C}_{r,r}(h,c) &= 48h - \left(2r^2(13-c) - 24r^2 - 2 + 2c\right) = 48h - 2(r^2-1)(1-c), \end{aligned}$$

and

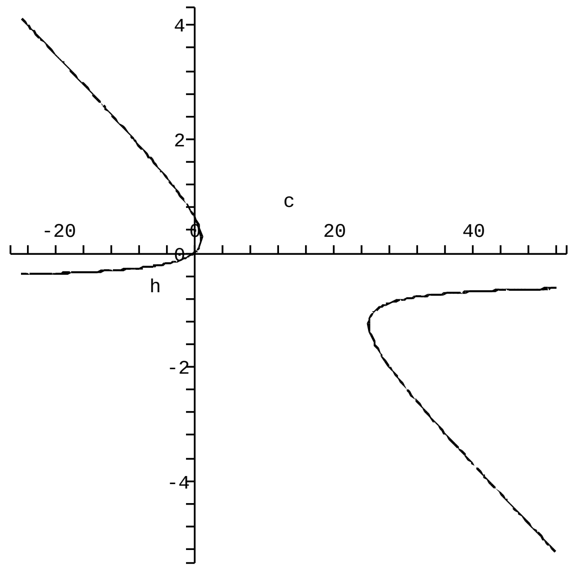
$$K_n = \prod_{\substack{r,s \in \mathbb{Z}_{>0}, \\ 1 \le r \le s \le n}} \left((2r)^s s!\right)^{p(n-rs) - p(n-r(s+1))},$$

where $p(k)$ is the number of partitions of k. Then for $(h,c) \in \mathbb{R}^2$,

$$detM(h,c)^{(h+n,c)} = K_n \prod_{\substack{r,s \in \mathbb{Z}_{>0}, \\ 1 \le r \le s \le n}} (\mathcal{C}_{r,s}(h,c))^{p(n-rs)}.$$

Feigin and Fuchs [1] have shown that $detM(h,c)^{(h+n,c)}$ actually contains *all* the information about the submodule structure of $M(h,c)$. That is, the formulas for $detM(h,c)^{(h+n,c)}$ can be used to completely describe the submodule structure of $M(h,c)$. The statement and proof of this result (theorem 3) depend on the following analysis of the curves $\mathcal{C}_{r,s}(h,c) = 0$.

For fixed r, s, $r \neq s$, the curves $\mathcal{C}_{r,s}(h,c) = 0$ are hyperbolas. Below is the curve $\mathcal{C}_{1,2}(h,c) = 0$.



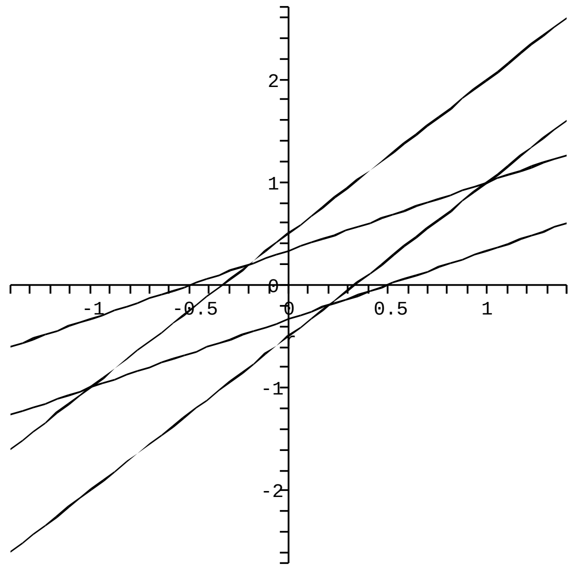
For fixed (h,c), the equation $\mathcal{C}_{r,s}(h,c) = 0$ can be factored into linear terms

$$0 = \mathcal{C}_{r,s}(h,c) = K(pr+qs+m)(pr+qs-m)(ps+qr+m)(ps+qr-m)$$

where $K, p, q, m \in \mathbb{C}$ such that

$$\frac{p}{q} + \frac{q}{p} = \frac{c-13}{6} \quad \text{and} \quad 4pqh + (p+q)^2 = m^2.$$

Thus, for fixed (h,c),the solutions to the equation $\mathcal{C}_{r,s}(h,c)$ form two sets of parallel lines. The figure below illustrates the example $\mathcal{C}_{r,s}(0,0) = 0$.



To find all integer solutions to $\mathcal{C}_{r,s}(h,c) = 0$, we only need to consider one line, say $pr+qs+m = 0$. (If (r,s) is a point on any of the other lines, $(-r,-s)$, (s,r) or $(-s,-r)$ will lie on the line $pr+qs+m = 0$.) We fix one of the lines and call it $\mathcal{L}_{(h,c)}$. The integer points (r,s) on this line encode the embeddings $M(h',c') \subseteq M(h,c) \subseteq M(h'',c'')$.

The line $\mathcal{L}_{(h,c)}$ passes through 0, 1, or infinitely many integer points. (If the line passes through two integer points, it has rational slope and therefore passes through infinitely many integer points.) Also, the line $\mathcal{L}_{(h,c)}$ has nonzero slope. Therefore, if it passes through infinitely many integer points (r,s) with $rs > 0$ it must pass through finitely many points (r,s) with $rs < 0$, and vice versa.

Theorem 3. [1] *Fix a pair $(h,c) \in \mathbb{R}^2$, and let $\mathcal{L}_{(h,c)}$ be one of the lines defined by this pair. Then the Verma module embeddings involving $M(h,c)$, and thus the block containing (h,c), are described by one of the following four cases.*

(1) *Suppose $\mathcal{L}_{(h,c)}$ passes through no integer points. Then the block $[(h,c)]$ is given by*

$$[(h,c)] = \{(h,c)\}.$$

The Verma module embeddings for $M(h,c)$ look like

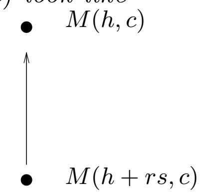
$\bullet$ $M(h,c)$

i.e, the Verma module $M(h,c)$ is irreducible and does not embed in any other Verma modules.

(2) *Suppose $\mathcal{L}_{(h,c)}$ passes through exactly one integer point (r,s). Then the block $[(h,c)]$ is given by*

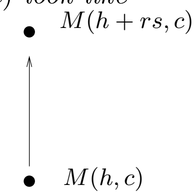
$$[(h,c)] = \{(h,c), (h+rs,c)\}.$$

- *If $rs > 0$, the embeddings for $M(h,c)$ look like*

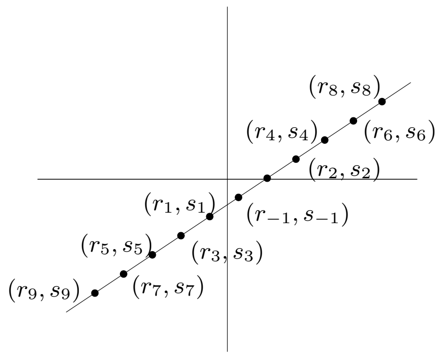


where the arrow indicates inclusion.

- *If $rs < 0$, the embeddings for $M(h,c)$ look like*



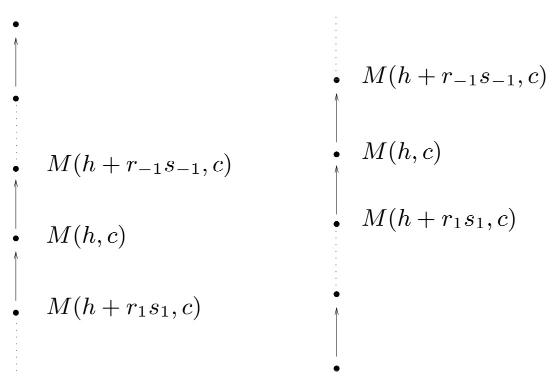
(3) *Suppose $\mathcal{L}_{(h,c)}$ passes through infinitely many integer points and crosses an axis at an integer point. Label these points (r_i, s_i) so that $\ldots < r_{-2}s_{-2} < r_{-1}s_{-1} < 0 < r_1 s_1 < r_2 s_2 \ldots$. (We exclude points (r,s) where $r = 0$ or $s = 0$.)*



Then, the block $[(h,c)]$ is given by

$$[(h,c)] = \{(h,c), (h + r_i s_i, c)\}.$$

The embeddings between the corresponding Verma modules take one of the following forms:

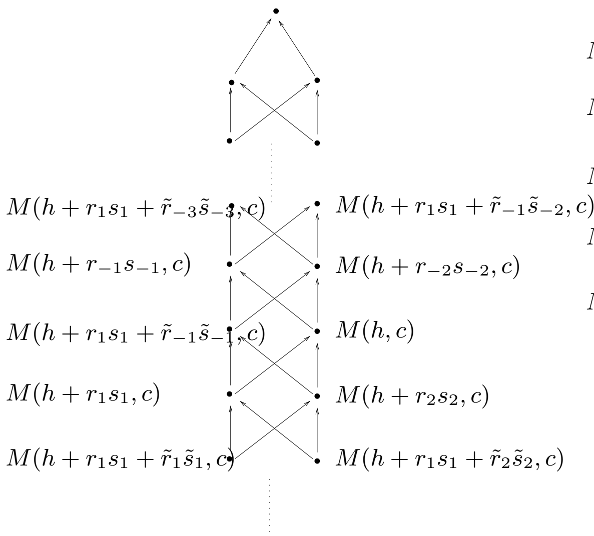


slope $(\mathcal{L}_{(h,c)}) > 0$ *slope* $(\mathcal{L}_{(h,c)}) < 0$

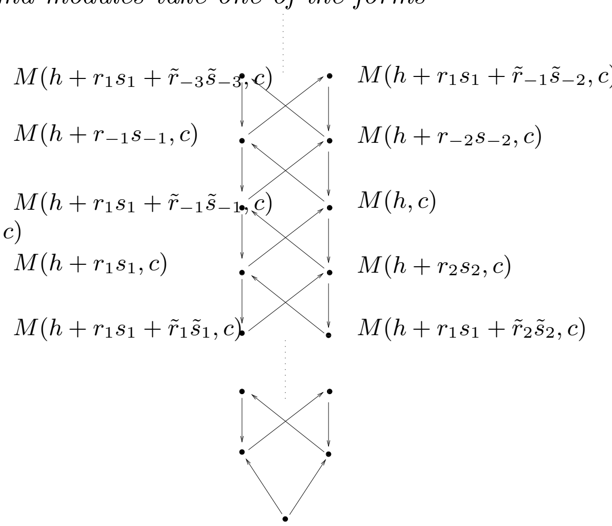
(4) *Suppose $\mathcal{L}_{(h,c)}$ passes through infinitely many integer points and does not cross either axis at an integer point. Again label the integer points (r_i, s_i) on $\mathcal{L}_{(h,c)}$ so that $\ldots < r_{-2}s_{-2} < r_{-1}s_{-1} < 0 < r_1 s_1 < r_2 s_2 \ldots$. Also consider the auxiliary line $\tilde{\mathcal{L}}_{(h,c)}$ with the same slope as $\mathcal{L}_{h,c}$ passing through the point $(-r_1, s_1)$. Label the integer points on this line $(\tilde{r}_j, \tilde{s}_j)$ as above. Then,*

$$[(h,c)] = \{(h + r_i s_i, c), (h + r_1 s_1 + \tilde{r}_j \tilde{s}_j, c)\}.$$

The embeddings between the corresponding Verma modules take one of the forms



slope $(\mathcal{L}_{(h,c)}) > 0$



slope $(\mathcal{L}_{(h,c)}) < 0$

We can use the line $\mathcal{L}_{(h,c)}$ to generate lines corresponding to the entire block $[(h,c)]$. If (r,s) is an integer point on the line $\mathcal{L}_{(h,c)}$, let $\tilde{\mathcal{L}}_{(h,c)}$ be the line with the same slope as $\mathcal{L}_{(h,c)}$ and passing through the point $(-r, s)$. Then $\tilde{\mathcal{L}}_{(h,c)}$ corresponds to the weight $(h + rs, c) \in [(h,c)]$. This allows us to identify maximal or minimal elements of a block. The weight (h,c) is a maximal (respectively minimal) element of its block if and only if there are no integer points (r,s) on $\mathcal{L}_{(h,c)}$ such that $rs < 0$ (respectively $rs >$).

In the following proposition, we identify a line $\mathcal{L}$ with the triple (μ, a, b), where μ is the slope of the line and (a,b) is a point on the line. Then $\mathcal{L}$ determines a weight (h,c) by

$$h = \frac{(a\mu - b)^2 - (\mu - 1)^2}{4\mu}, \quad c = 13 - 6\left(\mu + \frac{1}{\mu}\right).$$

Theorem 4.

- *Blocks of size two are indexed by triples*

$$\{(\mu, a, b) | \mu \in \mathbb{R} - \mathbb{Q} \textit{ with } |\mu| \leq 1 \textit{ and } a, b \in \mathbb{Z}_{>0}\}.$$

The weights in a block of size two are determined by triples $\{(\mu, \pm a, b)\}$.

- *Infinite blocks with a maximal element are indexed by triples*

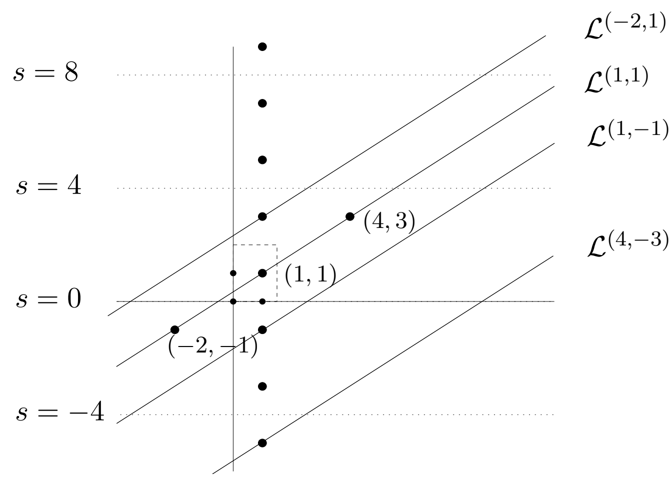
$$\left\{ \left(\frac{p}{q}, a, b \right) \middle| \begin{array}{l} p, q \in \mathbb{Z}_{>0}, \textit{ with } gcd(p,q) = 1, p < q \\ \textit{If } 2 \not| q, \textit{ then } 0 \le a < \frac{q}{2}, \ 0 \le b < p \\ \textit{If } 2|q, \textit{ then } 0 \le a < q, \ 0 \le b < \frac{p}{2} \end{array} \right\}.$$

Infinite blocks with a minimal element are indexed by triples

$$\left\{ \left(-\frac{p}{q}, -a, b \right) \middle| \begin{array}{l} p, q \in \mathbb{Z}_{>0}, \textit{ with } gcd(p,q) = 1, p < q \\ \textit{If } 2 \not| q, \textit{ then } 0 \le a < \frac{q}{2}, \ 0 \le b < p \\ \textit{If } 2|q, \textit{ then } 0 \le a < q, \ 0 \le b < \frac{p}{2} \end{array} \right\}.$$

The weights in an infinite block are determined by triples $\left\{ \left(\frac{p}{q}, a, \pm b + 2kp \right) |k \in \mathbb{Z} \right\}$ *(for blocks with a maximal element) and* $\left\{ \left(-\frac{p}{q}, -a \pm b + 2kp \right) |k \in \mathbb{Z} \right\}$ *(for blocks with a minimal element).*

Consider the example with $\mu = \frac{2}{3}$. We denote the line with slope $\frac{2}{3}$ and passing through the point (a, b) by $\mathcal{L}^{(a,b)}$.



The set of integer points $\{(a, b) \in \mathbb{Z}^2 | 0 \le a < \frac{3}{2}, 0 \le b < 2\}$ indexes the infinite blocks with $c = 13 - 6(\frac{2}{3} + \frac{3}{2}) = 0$. The line $\mathcal{L}^{(1,1)}$ determines the weight $(0, 0)$. From the integer points $(1, 1)$, $(-2, -1)$, and $(4, 3)$ on the line $\mathcal{L}^{(1,1)}$, we get the lines $\mathcal{L}^{(1,-1)}$, $\mathcal{L}^{(-2,1)}$, and $\mathcal{L}^{(4,-3)}$; these lines determine the weights $(1, 0)$, $(2, 0)$, and $(12, 0)$ respectively. In general, the set of points $\{(1, 4k \pm 1) | k \in \mathbb{Z}\}$ correspond to the block

$$\left\{ \left(\frac{(12k + 2 \pm 3)^2 - 1}{24}, 0 \right) |k \in \mathbb{Z} \right\} = \left\{ \left(\frac{j(3j \pm 1)}{2}, 0 \right) |j \in \mathbb{Z}_{\ge 0} \right\}.$$

4. Translation Functors

We now consider $M(h,c)\otimes L(h',c')$ and its decomposition by blocks. The module $M(h,c)\otimes L(h',c')$ can be written as a direct sum of submodules

$$M(h,c)\otimes L(h',c') = \bigoplus_{[\mu]\in[\mathfrak{h}^\star]} \left(M(h,c)\otimes L(h',c')\right)^{[\mu]}$$

where $(M(h,c)\otimes L(h',c'))^{[\mu]}$ is such that $L(\nu)$ appears in $(M(h,c)\otimes L(h',c'))^{[\mu]}$ only if ν is in the block $[\mu]$. For a given choice of $(h',c')\in\mathbb{C}^2$ and $[\mu]\in[\mathfrak{h}^\star]$, a *translation functor* is the map sending $M(h,c)$ to $(M(h,c)\otimes L(h',c'))^{[\mu]}$. Our goal in this section is to describe the submodules $(M(h,c)\otimes L(h',c'))^{[\mu]}$ corresponding to each block $[\mu]$. The Shapovalov determinant will again prove to be a useful tool.

We begin by observing that for all $(h,c),(h',c')\in\mathbb{C}^2$, the weights of $M(h,c)\otimes L(h',c')$ are of the form $(h+h'+n, c+c')$, $n\in\mathbb{Z}_{\geq 0}$. Therefore, $(M(h,c)\otimes L(h',c'))^{[\mu]}\neq 0$ only if $[\mu]=[(h+h'+n,c+c')]$ for some $n\in\mathbb{Z}_{\geq 0}$.

Proposition 3. [6] *For* $(h,c),(h',c')\in\mathbb{C}^2$ *and* $[\mu]\in[\mathfrak{h}^\star]$, $(M(h,c)\otimes L(h',c'))^{[\mu]}$ *has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

where

- $(M(h,c)\otimes L(h',c'))^{[\mu]} = \bigcup_i M_i$;
- $M_{i+1}/M_i \cong M(h+h'+\tilde{n}, c+c')$ *for some* $(h+h'+\tilde{n},c+c')\in[\mu]$;
- *the multiplicity of the factor* $M(h+h'+\tilde{n},c+c')$ *is* $\dim L(h',c')^{(h'+\tilde{n},c')}$.

We will use the Shapovalov form and the Shapovalov determinant to say more about $(M(h,c)\otimes L(h',c'))^{[\mu]}$.

Let $(h,c),(h',c')\in\mathbb{R}^2$. Recall that the Shapovalov form is defined on both $M(h,c)$ and $L(h',c')$. Define a form on $M(h,c)\otimes L(h',c')$ by

$$\langle v\otimes w, \tilde{v}\otimes\tilde{w}\rangle = \langle v,\tilde{v}\rangle\langle w,\tilde{w}\rangle.$$

Observe that we still have

$$\langle d_k(v\otimes w), \tilde{v}\otimes\tilde{w}\rangle = \langle v\otimes w, d_{-k}(\tilde{v}\otimes\tilde{w})\rangle$$

for all $x\in Vir$. For each weight space of $L(h',c')^{(h'+j,c')}$, we fix a basis $\{w_{j,i} | 1\leq i\leq \dim L(h',c')^{(h'+j,c')}\}$ and define $det L(h',c')^{(h'+j,c')} = det\left(\langle w_{j,i}, w_{j,k}\rangle\right)$. Then

$$\bigcup_{j=0}^{n}\{d_{-\lambda}v^+\otimes w_{j,i} | 1\leq i\leq \dim L(h',c')^{(h'+j,c')}, \lambda\vdash n-j\}$$

is a basis for $(M(h,c)\otimes L(h',c'))^{(h+h'+n,c+c')}$. Define

$$det\left(M(h,c)\otimes L(h',c')\right)^{(h+h'+n,c+c')} = det\left(\langle d_{-\lambda}v^+\otimes w_{j,i}, d_{-\mu}\otimes w_{j',k}\rangle\right)$$

where $0\leq j,j'\leq n$, $1\leq i\leq \dim L(h',c')^{(h'+j,c')}$, $1\leq k\leq \dim L(h',c')^{(h'+j',c')}$, $\lambda\vdash n-j$, and $\mu\vdash n-j'$. The following lemma is a straightforward calculation, given this definition.

Lemma 2. *Let $(h,c),(h',c')\in\mathbb{R}^2$ and $n\in\mathbb{Z}_{\geq 0}$. Then*

$$det\left(M(h,c)\otimes L(h',c')\right)^{(h+h'+n,c+c')}$$

is given by

$$\prod_{0\leq j\leq n}\left(a_j^{(h',c')}(h,c)\ detL(h',c')^{(h'+j,c')}\right)^{p(n-j)} \times\left(detM(h+h'+j,c+c')^{(h+h'+n,c+c')}\right)^{\dim L(h',c')^{(h'+j,c')}},$$

where

$$a_j^{(h',c')}(h,c)=\prod_{\substack{1\leq r\leq s\\ rs\leq j}}\left(\frac{\mathcal{C}_{r,s}(h,c)}{\mathcal{C}_{r,s}(h+h'+j-rs,c+c')}\right)^{\dim L(h',c')^{(h'+j-rs,c')}}.$$

Proposition 4. *Let $(h',c')\in\mathbb{R}^2$ and let $(h,c)\in\mathbb{R}^2$ be generic. For a block $[\mu]$ and $n\in\mathbb{Z}_{\geq 0}$,*

$$det\left(\left(M(h,c)\otimes L(h',c')\right)^{[\mu]}\right)^{(h+h'+n,c+c')}$$

is given by

$$\prod_j\left(a_j^{(h',c')}(h,c)detL(h',c')^{(h'+j,c')}\right)^{p(n-j)}\left(detM(h+h'+j,c+c')^{(h+h'+n,c+c')}\right)^{\dim L(h',c')^{(h'+j,c')}}.$$

where the product is over j such that $0\leq j\leq n$ and $(h+h'+j,c+c')\in[\mu]$.

Proof. With respect to the Shapovalov form

$$(1)\qquad \left(M(h,c)\otimes L(h',c')\right)^{[\lambda]}\perp\left(M(h,c)\otimes L(h',c')\right)^{[\mu]}\quad\text{for }[\lambda]\neq[\mu].$$

Now fix $n\in\mathbb{Z}_{\geq 0}$ and $(h',c')\in\mathbb{R}^2$.

For generic (h,c), $det\,(M(h,c)\otimes L(h',c'))^{(h+h'+n,c+c')}$ is nondegenerate. Given (1), the Shapovalov form can be used to construct explicit projection maps

$$Pr_{[\mu]}:M(h,c)\otimes L(h',c')\to\left(M(h,c)\otimes L(h',c')\right)^{[\mu]}.$$

By applying these maps to the basis for $(M(h,c)\otimes L(h',c'))^{(h+h'+n,c+c')}$ given above, we obtain bases for the weight spaces of each block . We use these bases to define $det\left((M(h,c)\otimes L(h',c'))^{[\mu]}\right)^{(h+h'+n,c+c')}$. From (1), we have

$$det\left(M(h,c)\otimes L(h',c')\right)^{(h+h'+n,c+c')}=\prod_{[\mu]\in[\mathfrak{h}^\star]}det\left(\left(M(h,c)\otimes L(h',c')\right)^{[\mu]}\right)^{(h+h'+n,c+c')}.$$

Suppose $(h,c)\in\mathbb{R}^2$ is such that the block $[\mu]=[(h+h'+j,c+c')]$ $(j\leq n)$ has size one. Proposition 3 then implies that

$$\left(M(h,c)\otimes L(h',c')\right)^{[\mu]}\cong M(h+h'+j,c+c')^{\oplus\dim L(h',c')^{(h'+j,c')}}.$$

By inducting on n, we show that in this case $det\left(\left((M(h,c)\otimes L(h',c'))^{[\mu]}\right)^{(h+h'+n,c+c')}\right)$ is given by

$$(2)\qquad \left(a_j^{(h',c')}(h,c) detL(h',c')^{(h'+j,c')}\right)^{p(n-j)} \times \left(detM(h+h'+j,c+c')^{(h+h'+n,c+c')}\right)^{\dim L(h',c')^{(h'+j,c')}}.$$

Now, for a given $c \in \mathbb{R}$, there are infinitely many $h \in \mathbb{R}$ so that $[(h+h'+j,c+c')]$ $(j \leq n)$ has size one. We use (2) to prove the result for all generic $(h,c) \in \mathbb{R}^2$. □

This determinant formula reflects the Verma filtration of $(M(h,c)\otimes L(h',c'))^{[\mu]}$, with the $a_j^{(h',c')}(h,c)$ acting as error terms. In fact, these error terms contain more information about the structure of $(M(h,c)\otimes L(h',c'))^{[\mu]}$. For example, the following proposition shows that the error terms count the dimension of the radical.

Proposition 5. *Let $(h,c),(h',c') \in \mathbb{R}^2$, and suppose (h,c) does not belong to an infinite block. Define*

$$Rad\langle,\rangle_{[\mu]} = \left\{m \in \left(M(h,c)\otimes L(h',c')\right)^{[\mu]} | \langle m,x\rangle = 0 \textit{ for all } x \in M(h,c)\otimes L(h,c)\right\}.$$

Then $\dim\left(Rad\langle,\rangle_{[\mu]}\right)^{(h+h'+n,c+c')}$ *is the number of zeros in*

$$\prod_{\substack{0\leq j\leq n \\ (h+h'+j,c+c')\in[\mu]}} \left(a_j^{(h',c')}(h,c)\right)^{p(n-j)}.$$

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