PERMUTATION REPRESENTATIONS ON INVERTIBLE MATRICES

YONA CHERNIAVSKY , ELI BAGNO

(Extended abstract)

1. Introduction

The (0,1)-matrices have a wide variety of applications in combinatorics as well as in computer science. A lot of research had been devoted to this area. By considering the set of $n \times n$ (0,1)-matrices as a boolean monoid and relating them to posets, one can get interesting representations of S_n. Note that $S_n \times S_n$ acts on matrices by permuting rows and columns. Some aspects of the corresponding equivalence relation are treated in [I] and [Li]. A simultaneous lexicographic ordering of the rows and the columns using this action is shown in [MM].

The above action of $S_n \times S_n$ gives rise to a permutation representation of $S_n \times S_n$ on (0,1)-matrices. If we diagonally embed S_n in $S_n \times S_n$ we get a generalization of the conjugacy representation of S_n.

Adin and Frumkin [AF] showed that the conjugacy character of the symmetric group is close, in some sense, to the regular character of S_n. More precisely, the quotient of the norms of the regular character and the conjugacy character as well as the cosine of the angle between them tend to 1 when n tends to infinity. This implies that these representations have essentially the same decompositions.

Roichman [R] further points out a wide family of irreducible representations of S_n whose multiplicity in the conjugacy representation is asymptotically equal to their dimension, i.e. their multiplicity in the regular representation.

In this paper we use the action of $S_n \times S_n$ on the (0,1)- matrices to define two families of representations on a family of orbits of this action. The first family forms an interpolation between the regular representation of $S_n \times S_n$ and the 'diagonal sum' of the irreducible representations of S_n: $\bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$. The other family is a generalization of the conjugacy representation of S_n. In both cases we calculate characters and present the decomposition of these representations into irreducibles. The second family of representations can be seen as an extension of the results of [AF] and [R].

2. Preliminaries

2.1. **Symmetric Groups.** S_n is the group of all bijections from the set $\{1...n\}$ to itself. Every $\pi \in S_n$ may be written in *disjoint cycle form* usually omitting the 1-cycles of π. For example, $\pi = 365492187$ may also be written as $\pi = (9,7,1,3,5)(2,6)$. Given $\pi, \tau \in S_n$ let $\pi\tau := \pi \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$. Note that two permutations are conjugate in S_n if and only if they have the same cycle structure. In this paper we write $\pi \sim \sigma$ if the permutations π and σ are conjugate in S_n. We denote by $\hat{S}_n$

the set of conjugacy classes of S_n and by $C_\pi \leq S_n$ the centralizer subgroup of the element $\pi \in S_n$. Let $C(\pi) \subseteq S_n$ denote the conjugacy class of the element $\pi \in S_n$. By $supp(\pi)$ we mean the set of digits which are not fixed by π. An element $\pi \in S_n$ with $|supp(\pi)| = t$ can be considered as an element of S_t and then C_π^t denotes the centralizer subgroup of the element π in S_t while $C^t(\pi)$ denotes the conjugacy class of the element π in S_t. $\pi_k\pi_{n-k}$ denotes an element of $S_k \times S_{n-k}$ where $\pi_k \in S_k$ and $\pi_{n-k} \in S_{n-k}$.
$C^{k\times(n-k)}(\pi_k\pi_{n-k})$ denotes the conjugacy class of the element $\pi_k\pi_{n-k}$ in $S_k \times S_{n-k}$.

There is an obvious embedding of S_n in $GL_n(\mathbb{F})$ where is $\mathbb{F}$ is any field. Just think about a permutation $\pi \in S_n$ as an $n \times n$ matrix obtained from the identity matrix by permutations of the rows. More explicitly: for every permutation $\pi \in S_n$ we identify π with the matrix:

$$[\pi]_{i,j} = \begin{cases} 1 & i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

Further we identify a permutation with the corresponding permutation matrix.

2.2. **Color permutation groups.** For later use, we define here the color permutation groups. For $r, n \in \mathbb{N}$, let $G_{r,n}$ denote the group of all n by n monomial matrices whose non-zero entries are complex r-th roots of unity. This group can also be described as the wreath product $C_r \wr S_n$ which is the semi-direct product $C_r^n \rtimes S_n$, where C_r^n is taken as the subgroup of all diagonal matrices in $G_{r,n}$. For $r = 1$, $G_{r,n}$ is just S_n while for $r = 2$, $G_{r,n} = B_n$, the Weyl group of type B.

2.3. **Representations.**

2.3.1. *Permutation representations.* In this work we deal mainly with permutation representations. Given an action of a group G on a set M, the appropriate representation space is the space spanned by the elements of M on which G acts by linear extension. We list two well known facts about permutation representations.

Fact 2.1. *The character of the permutation representation calculated at some $g \in G$ equals to the number of fixed points under g.*

Fact 2.2. *The multiplicity of the trivial representation in a given permutation representation is equal to the number of orbits under the corresponding action.*

An important example we will use extensively in this work is the conjugacy representation which is the permutation representation obtained by the action of the group on itself by conjugation.

2.3.2. *Representations of S_n.* Let n be a nonnegative integer. A *partition* of n is an infinite sequence of nonnegative integers with finitely many nonzero terms $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum_{i=1}^{\infty} \lambda_i = n$.

The sum $\sum \lambda_i = n$ is called the *size* of λ, denoted $|\lambda|$; write also $\lambda \vdash n$. The number of parts of λ, $\ell(\lambda)$, is the maximal j for which $\lambda_j > 0$. The unique partition of $n = 0$ is the *empty partition* $\emptyset = (0, 0, \ldots)$, which has length $\ell(\emptyset) := 0$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k, \ldots)$ define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_i, \ldots)$ by letting λ'_i be the number of parts of λ that are $\geq i$ $(\forall i \geq 1)$.

A partition $\lambda = (\lambda_1, ..., \lambda_k)$ may be viewed as the subset

$$\{(i,j) \mid 1 \le i \le k, 1 \le j \le \lambda_i\} \subseteq \mathbb{Z}^2,$$

the corresponding *Young diagram*. Using this interpretation we may speak of the intersection $\lambda \cap \mu$, the set difference $\lambda \setminus \mu$ and the symmetric set difference $\lambda \triangle \mu$ of any two partitions. Note that $|\lambda \triangle \mu| = \sum_{k=1}^{\infty} |\lambda_k - \mu_k|$.

It is well known that the irreducible representations of S_n are indexed by partitions of n (See for example [Sa]) and the representations of $S_n \times S_n$ are indexed by pairs of partitions (λ, μ) where $\lambda, \mu \vdash n$. For every two representations of S_n, λ and ρ, we denote by $m(\lambda, \rho)$ the multiplicity of λ in ρ. If we denote by $\langle \, , \rangle$ the standard scalar product of characters of a finite group G i.e.

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{\pi \in G} \chi_1(\pi) \overline{\chi_2(\pi)}$$

then $m(\lambda, \rho) = \langle \chi_\lambda, \chi_\rho \rangle$.

Similarly, $m\big((\lambda, \mu), \varphi\big)$ denotes the multiplicity of the representation of $S_n \times S_n$ corresponding to the pair of partitions (λ, μ) , $\lambda \vdash n$, $\mu \vdash n$ in the decomposition of φ, where φ is any representation of $S_n \times S_n$.

We cite here for later use the branching rule for the representations of S_n. We start with a definition needed to state the branching rule.

Definition 2.3. *Let $\lambda \vdash n$ be a Young diagram. Then a corner of λ is a cell $(i,j) \in \lambda$ such whose removal leaves leaves the Young diagram of a partition. Any partition obtained by such a removal is denoted by λ^-.*

Proposition 2.4. [Sa] *If $\lambda \vdash n$ then*

$$S^\lambda \downarrow_{S_{n-1}}^{S_n} \cong \bigoplus_{\lambda^-} S^{\lambda^-}.$$

3. The action of $S_n \times S_n$ on invertible matrices

Definition 3.1. *Let G be a subgroup of $S_n \times S_n$ and let $\mathbb{F}$ be any field. We define an action of G on the group $GL_n(\mathbb{F})$ by*

$$(\pi, \sigma) \bullet A = \pi A \sigma^{-1} \ \ \textit{where } (\pi, \sigma) \in G \textit{ and } A \in GL_n(\mathbb{F}) \tag{1}$$

It is easy to see that this really defines a group action.

In this work we deal only with the cases: $G = S_n \times S_n$ and $G = (S_k \times S_{n-k}) \times (S_k \times S_{n-k})$.

Definition 3.2. *Let M be a finite subset of $GL_n(\mathbb{F})$, invariant under the action of $S_n \times S_n$ defined above. We denote by α_M the permutation representation of G obtained from the action* (1) . *In the sequel we identify the action* (1) *with the permutation representation α_M associated with it.*

3.1. **A generalization of the conjugacy representation of S_n.**

In this section we present a conjugacy representation of S_n on a subset M of $GL_n(\mathbb{F})$.

Definition 3.3. *Denote by β the permutation representation of S_n obtained by the following action on M.*

$$\pi \circ A = (\pi, \pi) \bullet A = \pi A \pi^{-1} \tag{2}$$

The connection between α_M and β_M is given by the following easily seen claim:

Claim 3.4. *Consider the diagonal embedding of S_n into $S_n \times S_n$. Then*

$$\beta_M = \alpha_M \downarrow^{S_n \times S_n}_{S_n} . \qquad \square$$

Theorem 3.5. *For every finite set $M \subseteq GL_n(\mathbb{F})$ invariant under the action* (1) *of $S_n \times S_n$ defined above:*
If π and σ are conjugate in S_n then

$$\chi_{\alpha_M}((\pi, \sigma)) = \chi_{\alpha_M}((\pi, \pi)) = \chi_{\beta_M}(\pi) = \#\{A \in M \mid \pi A = A\pi\} .$$

If π is not conjugate to σ in S_n then

$$\chi_{\alpha_M}((\pi, \sigma)) = 0 .$$

Proof. See Theorem 4.5 in [CS]. $\square$

4. The action of $S_n \times S_n$ on (0,1)-matrices

In this section we specialize the action (1) of $S_n \times S_n$ defined in Section 3 to (0,1)-matrices. Consider the group $G = GL_n(\mathbb{Z}_2)$. For every $A \in G$ denote by $o(A)$ the number of nonzero entries in A. One can associate with A a pair of partitions of $o(A)$ with n parts $(\eta(A), \theta(A))$ where $\eta(A)$ describes the distribution of nonzero entries in the rows of A and $\theta(A)$ describes the same distribution for columns. For example, if:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

then $\eta(A) = (4, 3, 1, 1) \vdash 9$ and $\theta(A) = (3, 3, 2, 1) \vdash 9$.

If we fix a pair of partitions (η, θ) then the set of matrices corresponding to (η, θ) is closed under the action (1), but this action is not necessarily transitive on such a set, i.e. it can be decomposed into a union of several orbits.

We present now a family of subsets of $GL_n(\mathbb{Z}_2)$ which will be proven shortly to be orbits of our action:

Definition 4.1.

$$H_n^0 = \{A \in G \mid \eta(A) = \theta(A) = (1, 1, 1, \ldots, 1) = 1^n\}$$

$$H_n^1 = \{A \in G \mid \eta(A) = (n, 1, 1, \ldots, 1), \theta(A) = (2, 2, \ldots, 2, 1) = 2^{n-1}\, 1\}$$

$$H_n^2 = \{A \in G \mid \eta(A) = (n, n-1, 1, \ldots, 1), \theta(A) = (3, 3, \ldots, 3, 2, 1) = 3^{n-2}\, 2\, 1\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned} H_n^k = \{A \in G \mid & \eta(A) = (n, n-1, \ldots, n-(k-1), 1, \ldots, 1), \\ & \theta(A) = (k+1, k+1, \ldots, k+1, k, k-1, \ldots, 2, 1) = (k+1)^{n-k}\, k\, (k-1)\, \ldots\, 2\, 1\} \end{aligned}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$H_n^n = \{A \in \ \mid \eta(A) = \theta(A) = (n, n-1, n-2, \ldots, n-(k-1), \ldots, 3, 2, 1)\}$$

Note that in the above example $A \in H_4^2$.
A few remarks on the sets H_n^k are in order: First, note that $|H_n^k| = n!(n)_k$. Secondly, note that H_n^0 is S_n, embedded as permutation matrices. Also note that the set $H_n^0 \cup H_n^1$ is closed under matrix multiplication and matrix inversion and is actually isomorphic to the group S_{n+1}. Another simple observation is that $H_n^n = H_n^{n-1}$.

In order to prove that the sets H_n^k are transitive under the action we need the following definition:

Definition 4.2. *Denote by $U_{n,k}$ the following binary $n \times n$ matrix : the upper left $k \times k$ block is upper triangular with the upper triangle filled by ones, the upper right $k \times (n-k)$ block is filled by ones, the lower left $(n-k) \times k$ block is the zero matrix and the lower right $(n-k) \times (n-k)$ block is the identity matrix I_{n-k}.*

Proposition 4.3. *Each set H_n^k is transitive under the action α of $S_n \times S_n$. More explicitly, $H_n^k = \{\pi U_{n,k} \sigma \,|\, \pi, \sigma \in S_n\}$*

For the case $k = n$ the permutation representation $\alpha_{H_n^k}$ can be easily described:

Proposition 4.4. *The representation $\alpha_{H_n^n}$ is isomorphic to the regular representation of $S_n \times S_n$.*

4.1. **A natural mapping from H_n^k onto S_n.** In this section we present an epimorphism between the representation of $S_n \times S_n$ on H_n^k to the representation of $S_n \times S_n$ on S_n. We will use this mapping later when we decompose the permutation representation α into irreducibles representations.

Definition 4.5. *Define the mapping $T_{n,k} : H_n^k \longrightarrow S_n$ by $T_{n,k}(\pi U_{n,k} \sigma) = \pi\sigma$.*

Proposition 4.6. *The mapping $T_{n,k}$ preserves the action α of $S_n \times S_n$ on H_n^k, i.e.*

$$T_{n,k}(\pi A \sigma) = \pi T_{n,k}(A) \sigma \text{ for any } A \in H_n^k.$$

It is also clear from the definition that $T_{n,k}$ is onto and it is easy to see that $|T_{n,k}^{-1}(\pi)| = k!\binom{n}{k} = (n)_k$.

5. The representation β_M for $M = H_n^k$.

In [F] it was proven that the conjugacy representation of S_n contains every irreducible representation of S_n as a constituent. The representation β defined in Section 3.1 is a type of a conjugacy representation of S_n on H_n^k.

Proposition 5.1. *Denote the conjugacy representation of S_n by ψ. Then every irreducible representation of S_n is a constituent in $\beta_{H_n^k}$. In other words*

$$m\left(\lambda, \beta_{H_n^k}\right) > 0 \qquad \text{for any} \quad \lambda \vdash n.$$

where $m(\lambda, \beta_{H_n^k})$ denotes the multiplicity of the irreducible representation corresponding to λ in $\beta_{H_n^k}$.

We turn now to the calculation of the character of $\beta_{H_n^k}$. By the definition, we have:

$$\chi_{\beta_{H_n^k}}(\pi) \left(= \chi_{\alpha_{H_n^k}}(\pi, \pi)\right) = \#\{A \in H_n^k \,|\, \pi A = A \pi\}$$

but we can achieve much more than that:

Proposition 5.2.

$$\chi_{\beta_{H_n^k}}(\pi) = |C_\pi|(n - |supp(\pi)|)_k = (n - |supp(\pi)|)_k \chi_{Conj}(\pi)$$

where χ_{Conj} is the conjugacy character of S_n.

We turn now to the calculation of the multiplicity of every irreducible representation of S_n in $\beta_{H_n^k}$.

Proposition 5.3. *Let $\lambda \vdash n$.*

$$m\big(\lambda, \beta_{H_n^k}\big) = \sum_{C \in \hat{S}_n} \chi_\lambda(C)(n - |supp(C)|)_k$$

where $\hat{S}_n$ denotes the set of conjugacy classes of S_n. □

6. Asymptotic behavior of the representation $\beta_{H_n^k}$.

In this section we generalize the results of Roichman [R], Adin, and Frumkin [AF] concerning the asymptotic behavior of the conjugacy representation of S_n. These two results imply that the conjugacy representation and the regular representation of S_n have essentially the same decomposition. In our case, as we prove in this section, the representation $\beta_{H_n^k}$ is essentially $(n)_k$ times the regular representation of S_n. We start by citing the result from [R].

Theorem R1 Let $m(\lambda)$ be the multiplicity of the irreducible representation S^λ in the conjugacy representation of S_n, and let f^λ be the multiplicity of S^λ in the regular representation of S_n. Then for any $0 < \varepsilon < 1$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda'_1}{n}\} \leq \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{m(\lambda)}{f^\lambda} < 1 + \varepsilon.$$

The following generalization of this theorem is straightforward:

Proposition 6.1. *For any $0 < \varepsilon < 1$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with max$\{\frac{\lambda_1}{n}, \frac{\lambda'_1}{n}\} \leq \delta(\varepsilon)$,and for any $k \leq n$*

$$1 - \varepsilon < \frac{m(\lambda, \beta_{H_n^k})}{(n)_k f^\lambda} < 1 + \varepsilon.$$

The following asymptotic result from [AF] can also be generalized for the characters $\chi_{\beta_{H_n^k}}$.

Theorem AF Let $\chi_R^{(n)}$ and $\chi_{Conj}^{(n)}$ be the regular and the conjugacy characters of S_n respectively. Then

$$\lim_{n\to\infty} \frac{\|\chi_R^{(n)}\|}{\|\chi_{Conj}^{(n)}\|} = 1\,,$$

$$\lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{Conj}^{(n)} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{Conj}^{(n)}\|} = 1$$

where $\| * \|$ denotes the norm with respect to the standard scalar product of characters.

Our generalization looks as follows:

Proposition 6.2. *In the notations of Theorem AF*

$$\lim_{n\to\infty}\frac{\|(n)_k\chi_R^{(n)}\|}{\|\chi_{\beta_{H_n^k}}\|}=1\,,$$

$$\lim_{n\to\infty}\frac{\langle (n)_k\chi_R^{(n)},\chi_{\beta_{H_n^k}}\rangle}{\|(n)_k\chi_R^{(n)}\|\cdot\|\chi_{\beta_{H_n^k}}\|}=\lim_{n\to\infty}\frac{\langle \chi_R^{(n)},\chi_{\beta_{H_n^k}}\rangle}{\|\chi_R^{(n)}\|\cdot\|\chi_{\beta_{H_n^k}}\|}=1\,,$$

where k is bounded or tends to infinity remaining less than n.

7. The representations α_M for $M=H_n^k$

In this section we deal with the representations $\alpha_{H_n^k}$ defined in Section 3. We use the branching rule and the Frobenious reciprocity to decompose these representations into irreducible representations of $S_n\times S_n$. As we have already seen in example **??**, $\alpha_{H_n^0}\cong\bigoplus_{\lambda\vdash n}S^\lambda\otimes S^\lambda$ while $\alpha_{H_n^n}$ is the regular representation of $S_n\times S_n\cong\bigoplus_{\lambda,\rho\vdash n}f^\lambda f^\rho S^\lambda\otimes S^\rho$ and thus $\alpha_{H_n^k}$ can be seen as a type of an interpolation between these two representations.

First, concerning the character of $\alpha_{H_n^k}$, by combining Proposition 5.2 and Theorem 3.5 together we get:

$$\chi_{\alpha_{H_n^k}}(\pi,\sigma)=\begin{cases}|C_\pi|(n-|supp(\pi)|)_k\ , & \pi \text{ and } \sigma \text{ are conjugate in } S_n\\ 0\ , & \text{otherwise}\end{cases}$$

We turn now to the lation of the multiplicity of an irreducible representation of $S_n\times S_n$ in $\alpha_{H_n^k}$.

Proposition 7.1. *For any n and any $0\le k\le n$*

$$m\left((\lambda,\mu),\alpha_{H_n^k}\right)=\frac{1}{n!}\sum_{\pi\in S_n}\chi_\lambda(\pi)\chi_\mu(\pi)(n-|supp(\pi)|)_k \qquad \square$$

The boundary cases $k=0$ and $k=n$ are discussed in Example **??** and Proposition 4.4 respectively.

7.1. A combinatorial view of $\alpha_{H_n^k}$.

In this section we present another approach to the representation $\alpha_{H_n^k}$. This approach will give us a combinatorial view on the multiplicity formulas we calculated in the last section.

Definition 7.2. *Define the following subset of H_n^k:*

$$W_n^k=\{\pi_k\pi_{n-k}U_{n,k}\sigma_k\sigma_{n-k}\,|\,\pi_k,\sigma_k\in S_k \text{ and } \pi_{n-k},\sigma_{n-k}\in S_{n-k}\}\,.$$

The set W_n^k is the orbit of the matrix $U_{n,k}$ under the action α restricted to the subgroup $(S_k\times S_{n-k})\times(S_k\times S_{n-k})$.

Definition 7.3. *Denote by $\omega_{n,k}$ the permutation representation of the group $(S_k\times S_{n-k})\times(S_k\times S_{n-k})$ on W_n^k corresponding to the action α.*

Claim 7.4.

$$\omega_{n,k}\cong R_k\otimes\left(\bigoplus_{\rho\vdash n-k}S^\rho\otimes S^\rho\right)$$

where R_k is the regular representation of $S_k\times S_k$.

This implies the following:

Claim 7.5.

$$\chi_{\omega_{n,k}}(\pi_k\pi_{n-k},\sigma_k\sigma_{n-k}) = \begin{cases} 0 & \text{when } \pi_k \neq e \text{ or } \sigma_k \neq e \\ 0 & \text{when } \pi_{n-k} \text{ is not conjugate to } \sigma_{n-k} \text{ in } S_{n-k} \\ (k!)^2|C^{n-k}_{\pi_{n-k}}| & \text{when } \pi_k = \sigma_k = e \text{ and } \pi_{n-k} \sim \sigma_{n-k} \text{ in } S_{n-k} \end{cases}$$

□

We can use $\omega_{n,k}$ to get information of $\alpha_{n,k}$.

Proposition 7.6.

$$\alpha_{H_n^k} = \omega_{n,k} \uparrow^{S_n \times S_n}_{(S_k \times S_{n-k}) \times (S_k \times S_{n-k})}$$

We use now the Frobenius reciprocity to obtain the multiplicity of any irreducible representation of $S_n \times S_n$ in $\alpha_{H_n^k}$.

Proposition 7.7. *Let $0 \leq k \leq n$ and let λ, μ be partitions of n. Then*

$$m\left((\lambda,\mu), \alpha_{H_n^k}\right) = \langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle$$

or in other words:

$$\alpha_{H_n^k} = \bigoplus_{\lambda,\mu \vdash n} \langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle S^\lambda \otimes S^\mu \qquad .$$

The number $\langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle$ has a very nice combinatorial interpretation. It follows from the branching rule that this is just the number of ways to delete k boundary cells from the diagrams corresponding to the partitions λ and μ to get the same Young diagram of $n-k$ cells. By the branching rule (see Proposition 2.4) we have thus:

Claim 7.8.

$$\langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle = 0 \quad \textit{when} \quad |\lambda \bigtriangleup \mu| > 2k$$

and it does not vanish otherwise.

Corollary 7.9.

$$m\left((\lambda,\mu), \alpha_{H_n^k}\right) = 0 \quad \textit{when} \quad |\lambda \bigtriangleup \mu| > 2k$$

and

$$m\left((\lambda,\mu), \alpha_{H_n^k}\right) \neq 0 \quad \textit{when} \quad |\lambda \bigtriangleup \mu| \leq 2k. \qquad \square$$

8. The actions α and β on colored permutations

In this section we introduce actions of S_n and $S_n \times S_n$ on another family of sets, namely the colored permutation groups. We start with the actions on $B_n = C_2 \wr S_n$.

8.1. **The action α of $S_n \times S_n$ on signed permutations.** Consider the action α of $S_n \times S_n$ on B_n. We start by describing the orbits of this action.

Definition 8.1. *For every $0 \leq k \leq n$ define*

$$X_n^k = \{A \in B_n \mid A \textit{ has exactly } k \textit{ minuses}\}.$$

For example

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in X_4^2.$$

It is easy to see that the sets X_n^k form a partition of B_n. Also, note that $|X_n^k| = n!\binom{n}{k}$.

Claim 8.2. *Each set X_n^k is an orbit under the action α of $S_n \times S_n$ on B_n, i.e.*

$$X_n^k = \{\pi \tilde{U}_{n,k} \sigma \mid \pi, \sigma \in S_n\},$$

where

$$\tilde{U}_{n,k} = \begin{pmatrix} -I_k & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & I_{n-k} \end{pmatrix}$$

and I_t is the identity $t \times t$ matrix.

We decompose now the representations $\alpha_{X_n^k}$ into irreducible representations just as we did in the previous section.

Definition 8.3. *Define the following subset of X_n^k:*

$$\tilde{W}_n^k = \left\{ \pi_k \pi_{n-k} \tilde{U}_{n,k} \sigma_k \sigma_{n-k} \mid \pi_k, \sigma_k \in S_k \text{ and } \pi_{n-k}, \sigma_{n-k} \in S_{n-k} \right\}.$$

The set $\tilde{W}_n^k$ is the orbit of the matrix $\tilde{U}_{n,k}$ under the action α by the group $(S_k \times S_{n-k}) \times (S_k \times S_{n-k})$.

Definition 8.4. *Denote $\tilde{\omega}_{n,k}$ the permutation representation of the group $(S_k \times S_{n-k}) \times (S_k \times S_{n-k})$ which is obtained from the action α of this group on the set $\tilde{W}_n^k$.*

Claim 8.5.

$$\tilde{\omega}_{n,k} \cong \left(\bigoplus_{\rho \vdash k} S^\rho \otimes S^\rho \right) \otimes \left(\bigoplus_{\rho \vdash n-k} S^\rho \otimes S^\rho \right)$$

$$\chi_{\tilde{\omega}_{n,k}}(\pi_k \pi_{n-k}, \sigma_k \sigma_{n-k}) = \begin{cases} |C^k_{\pi_k}||C^{n-k}_{\pi_{n-k}}| & \text{when } \pi_k \pi_{n-k} \sim \sigma_k \sigma_{n-k} \text{ in } S_k \times S_{n-k} \\ 0 & \text{otherwise} \end{cases}$$

□

Proposition 8.6.

$$\alpha_{X_n^k} = \tilde{\omega}_{n,k} \uparrow^{S_n \times S_n}_{(S_k \times S_{n-k}) \times (S_k \times S_{n-k})}$$

Recall from [Sa] the definition of $c^\lambda_{\rho\nu}$ – the Littlewood-Richardson coefficients defined by the following formula:

$$(S^\rho \otimes S^\nu) \uparrow^{S_n}_{S_k \times S_{n-k}} = \bigoplus_{\lambda \vdash n} c^\lambda_{\rho\nu} S^\lambda,$$

where $\rho \vdash k$ and $\nu \vdash n-k$. Using the Frobenius reciprocity formula we have for every $\lambda \vdash n$:

Claim 8.7.
$$S^\lambda \downarrow^{S_n}_{S_k \times S_{n-k}} = \bigoplus_{\rho \vdash k, \nu \vdash n-k} c^\lambda_{\rho\nu} \left(S^\rho \otimes S^\nu\right).$$

$$\chi_\lambda \downarrow^{S_n}_{S_k \times S_{n-k}} = \sum_{\rho \vdash k, \nu \vdash n-k} c^\lambda_{\rho\nu} \chi_{(\rho,\nu)}.$$

□

We use now the Frobenius reciprocity to obtain the multiplicity of any irreducible representation of $S_n \times S_n$ in $\alpha_{X^k_n}$.

Proposition 8.8. *Let* $0 \le k \le n$ *and* $\lambda, \mu \vdash n$.

$$\begin{aligned} m\big((\lambda,\mu), \alpha_{X^k_n}\big) &= \\ &= \langle \chi_\lambda \downarrow^{S_n}_{S_k \times S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_k \times S_{n-k}} \rangle = \\ &= \sum_{\rho \vdash k, \nu \vdash n-k} c^\lambda_{\rho\nu} c^\mu_{\rho\nu} \end{aligned}$$

By the definition of X^k_n we have $\alpha_{B_n} = \bigoplus_{k=0}^n \alpha_{X^k_n}$ and thus:

Corollary 8.9.
$$m\left((\lambda,\mu), \alpha_{B_n}\right) = \sum_{k=0}^{n} \sum_{\rho \vdash k, \nu \vdash n-k} c^\lambda_{\rho\nu} c^\mu_{\rho\nu}.$$

□

There is a natural mapping between the sets H^k_n and X^k_n defined by:

$$H^k_n \ni \pi U_{n,k} \sigma \overset{\tilde{T}_{n,k}}{\longmapsto} \pi \tilde{U}_{n,k} \sigma \in X^k_n$$

One can verify that $\tilde{T}_{n,k}$ is well defined. Moreover, $\tilde{T}_{n,k}$ commutes with the action α of $S_n \times S_n$ on X^k_n, i.e.:

$$\tilde{T}_{n,k}(\pi A \sigma) = \pi \tilde{T}_{n,k}(A) \sigma \text{ for any } A \in X^n_k.$$

It is easy to see that $\tilde{T}_{n,k}$ is also surjective and thus it induces epimorphisms of modules from the $S_n \times S_n$-module $\alpha_{H^k_n}$ to the $S_n \times S_n$- module $\alpha_{X^k_n}$ and from the S_n-module $\beta_{H^k_n}$ to the S_n- module $\beta_{X^k_n}$. Note also that for $k = 0$ this mapping is the identity mapping since $H^0_n = X^0_n = S_n$ and for $k = 1$ this mapping is bijective. We conclude:

Claim 8.10.
$$m\left((\lambda,\mu), \alpha_{H^k_n}\right) \ge m\left((\lambda,\mu), \alpha_{X^k_n}\right)$$

□

This implies that if
$$\sum_{\rho \vdash k, \nu \vdash n-k} c^\lambda_{\rho\nu} c^\mu_{\rho\nu} \ne 0$$
then $|\lambda \triangle \mu| \le 2k$. This can also be seen by the combinatorial interpretation of the Littlewood-Richardson coefficients.

8.2. **The action β on colored permutations.** Recall that every matrix $B \in B_n$ can be written uniquely in the form $B = Z\pi$ for some $\pi \in S_n$ and some $Z \in C_2^n$. There exists a natural epimorphism from B_n onto S_n defined by omitting the minuses:

$$p : B_n \longrightarrow S_n \qquad\qquad p(Z\pi) = \pi.$$

If we restrict p to X_n^k we obtain a surjective mapping from X_n^k onto S_n which commutes with the action α of $S_n \times S_n$ on X_n^k (and clearly also commutes with the action β of S_n on X_n^k by conjugation). It gives us a surjective homomorphism from the representation $\beta_{X_n^k}$ onto the conjugacy representation representation of S_n denoted by ψ. Therefore, using the result of [F], we have

$$m\left(\lambda, \beta_{X_n^k}\right) \geq m\left(\lambda, \psi\right) > 0 \quad \text{for any} \quad \lambda \vdash n,$$

$$m\left(\lambda, \beta_{B_n}\right) = \sum_{k=0}^{n} m\left(\lambda, \beta_{X_n^k}\right) > 0 \quad \text{for any} \quad \lambda \vdash n.$$

Although the calculation of $\chi_{\beta_{X_n^k}}$ is rather involved, the asymptotic results of [R] and [AF] can be generalized for the representations $\beta_{X_n^k}$ and β_{B_n}. We start by presenting the generalization of Theorem R1:

Proposition 8.11. *For any $0 < \varepsilon < 1$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with max$\{\frac{\lambda_1}{n}, \frac{\lambda'_1}{n}\} \leq \delta(\varepsilon)$,*

$$1 - \varepsilon < \frac{m(\lambda, \beta_{X_n^k})}{\binom{n}{k} f^\lambda} < 1 + \varepsilon \ ,$$

$$1 - \varepsilon < \frac{m(\lambda, \beta_{B_n})}{2^n f^\lambda} < 1 + \varepsilon \ .$$

The generalization of Theorem [AF] is as follows and can be proved by using the inequality $\chi_{\beta_{X_n^k}}(\pi) \leq \binom{n}{k}|C_\pi|$:

Proposition 8.12. *In the notations of Theorem* [AF]

$$\lim_{n\to\infty} \frac{\|\binom{n}{k}\chi_R^{(n)}\|}{\|\chi_{\beta_{X_n^k}}\|} = \lim_{n\to\infty} \frac{\|2^n \chi_R^{(n)}\|}{\|\chi_{\beta_{B_n}}\|} = 1 \, ,$$

$$\lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{\beta_{X_n^k}} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{\beta_{X_n^k}}\|} = \lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{\beta_{B_n}} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{\beta_{B_n}}\|} = 1 \, ,$$

where k is bounded or tends to infinity remaining less than n.

These asymptotic results can be also obtained for the action β (conjugation by permutations) on the group $C_r \wr S_n$. Similarly to $X_n^k \subset B_n$ define the sets $Y_n^k \subset C_r \wr S_n$:

Definition 8.13.

$$Y_n^k = \{A \in C_r \wr S_n \mid A \textit{ has exactly } k \textit{ entries} \neq 0\, ,\, 1\}.$$

Note that the sets Y_n^k form a partition of $C_r \wr S_n$ and $Y_n^k = n!\binom{n}{k}(r-1)^k$. The sets Y_n^k are closed under the action α of $S_n \times S_n$ but they are not transitive under this action.

Consider C_r^n as the group of diagonal matrices with the entries of the form ω^ℓ (where $\omega = \exp\frac{2\pi i}{r}$ – the primitive r-th root of unity and $0 \leq \ell < r$) on the diagonal. Then each matrix $A \in C_r \wr S_n$ can be uniquely written as $A = Z\sigma$ for some $\sigma \in S_n$ and some $Z \in C_r^n$. Just as in the case of B_n, we consider the epimorphism $p : B_n \longrightarrow S_n$ defined by: $p(Z\sigma) = \sigma$.

p induces an epimorphism of modules between $\beta_{Y_n^k}$ and the conjugacy representation of S_n.

We conclude, using the result of [F]:

$$m\left(\lambda, \beta_{Y_n^k}\right) \geq m\left(\lambda, \psi\right) > 0 \quad \text{for any} \quad \lambda \vdash n,$$

$$m\left(\lambda, \beta_{C_r \wr S_n}\right) = \sum_{k=0}^{n} m\left(\lambda, \beta_{Y_n^k}\right) > 0 \quad \text{for any} \quad \lambda \vdash n.$$

The Theorems R1 and AF are obtained in a way similar to the one we used for B_n:

Proposition 8.14. *In the conditions and notations of Theorem R1*

$$1 - \varepsilon < \frac{m(\lambda, \beta_{Y_n^k})}{\binom{n}{k}(r-1)^k f^\lambda} < 1 + \varepsilon,$$

$$1 - \varepsilon < \frac{m(\lambda, \beta_{C_r \wr S_n})}{r^n f^\lambda} < 1 + \varepsilon.$$

In the notations of Theorem AF

$$\lim_{n\to\infty} \frac{\|\binom{n}{k}(r-1)^k \chi_R^{(n)}\|}{\|\chi_{\beta_{Y_n^k}}\|} = \lim_{n\to\infty} \frac{\|r^n \chi_R^{(n)}\|}{\|\chi_{\beta_{C_r \wr S_n}}\|} = 1\,,$$

$$\lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{\beta_{Y_n^k}} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{\beta_{Y_n^k}}\|} = \lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{\beta_{C_r \wr S_n}} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{\beta_{C_r \wr S_n}}\|} = 1\,,$$

where k is bounded or tends to infinity remaining less than n.

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Department of Mathematics and Statistics Bar-Ilan University Ramat-Gan, Israel 52900

E-mail address: cherniy@math.biu.ac.il , bagnoe@math.biu.ac.il