Combinatorics and Representations of Complex Reflection Groups $G(r, p, n)$ (Extended Abstract)

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Abstract

For every $r, n, p|r$ there is a complex reflection group, denoted $G(r, p, n)$, consisting of all monomial $n \times n$ matrices such that all the nonzero entries are r^{th} roots of the unity and the r/p^{th} power of the product of the nonzero entries is 1. By considering these groups as subgroups of the colored permutation groups, $\mathbb{Z}_r \wr S_n$, we use Clifford theory to define on $G(r, p, n)$ combinatorial parameters and descent representations previously defined on Classical Weyl groups. One of these parameters is the major index which also has an important role in the decomposition of descent representations into irreducibles. We present also a Carlitz identity for these complex reflection groups.

1 Introduction

Let V be a complex vector space of dimension n. A *pseudo-reflection* on V is a linear transformation on V of finite order which fixes a hyperplane in V pointwise. A *complex reflection group* on V is a finite subgroup $W \leq \mathrm{GL}(V)$ generated by pseudo-reflections. Such groups are characterized by the structure of their invariant ring. More precisely, let $\mathbb{C}[V]$ be the symmetric algebra of V and let us denote by $\mathbb{C}[V]^W$ the algebra of invariants of W. Then Shephard-Todd [26] and Chevalley [13] proved that W is generated by pseudo-reflections if and only if $\mathbb{C}[V]^W$ is a polynomial ring.

Irreducible finite complex reflection groups have been classified by Shephard-Todd [26]. In particular there is a single infinite family of groups and exactly 34 other "exceptional" complex reflection groups. The infinite family $G(r, p, n)$ where r, p, n are positive integers numbers with $p|r$, consists of the groups of $n \times n$ matrices such that

1) the entries are either 0 or r^{th} roots of unity;
2) there is exactly one nonzero entry in each row and each column;
3) the $(r/p)^{\text{th}}$ power of the product of the nonzero entries is 1.

In particular the classical Weyl groups appear as special cases: $G(1,1,n) = S_n$ the symmetric group, $G(2,1,n) = B_n$, the Weyl group of type B, and $G(2,2,n) = D_n$ the Weyl group of type D.

Throughout research on complex reflection groups and their braid groups and Hecke algebras, the fact that they behave like Weyl groups has become more and more clear. In particular, it has recently discoverd that complex reflection groups (and not only Weyl groups) play a key role in the structure as well as in the representation theory of finite reductive groups. For more information on these results the reader is advised to consult the survey article of Broué [9], and the handbook of Geck and Malle [11].

One of the aims of this paper is to show that complex reflection groups continue to behave like Weyl groups also from the combinatorial point of view. In a way similar to Coxeter groups, they have presentations in terms of generators and relations, that can be visualized by Dynkin type diagrams (see e.g., [10]). Moreover, their elements can be represented as colored permutations. In fact, the complex reflection group $G(r,p,n)$ can be naturally identified as a normal subgroups of index p of the wreath product $G(r,n) := \mathbb{Z}_r \wr S_n$, where $\mathbb{Z}_r$ is the cyclic group of order r. This makes it possible to handle complex reflection groups by purely combinatorial methods. In Sections 2 and 7 we follow this approach. In particular, we introduce the concept of major index and descent number for complex reflection groups. Their joint distribution over the group is computed, giving rise to a nice identity that relates the two new statistics with the degrees of $G(r,p,n)$.

Then our investigation continues by showing the interplay between these new combinatorial objects and the representation theory of the group. More precisely, if we set $\mathbf{x} = x_1, \ldots, x_n$ as a basis for V, then $\mathbb{C}[V]$ can be identified with the ring of polynomials $\mathbb{C}[\mathbf{x}]$. The ring of invariants $\mathbb{C}[\mathbf{x}]^W$ is then generated by 1 and by a set of n algebraically independent homogeneous polynomials $\{f_1, \ldots, f_n\}$ which are called *basic invariants*. Although these polynomials are not uniquely determined, their degrees $d_1, \ldots, d_n$ are basic numerical invariants of the group, and are called the *degrees of* W. Let us denote by $\mathcal{I}_W$ the ideal generated by the invariants of strictly positive degree. The *module of coinvariants* of W is defined by

$$\mathbb{C}[\mathbf{x}]_W := \mathbb{C}[\mathbf{x}]/\mathcal{I}_W.$$

Since $\mathcal{I}_W$ is W-invariant, the group W acts naturally on $\mathbb{C}[\mathbf{x}]_W$. In fact, it is well known that $\mathbb{C}[\mathbf{x}]_W$ is isomorphic to the *left regular representation* of W. It follows that the dimension of $\mathbb{C}[\mathbf{x}]_W$ as a $\mathbb{C}$-module is equal to the order of the group W. In section 3, by using the combinatorial tools previously introduced, an explicit monomial basis for the module of coinvariants, called colored-descent basis, is provided.

Recently, another basis for $\mathbb{C}[\mathbf{x}]_W$ has been given by Allen [4]. Although both our and Allen's basis coincide with the Garsia-Stanton basis in the case of S_n, in general they are different as can be checked already in the small case of $G(2,2,2)$. It would be interesting to see if Allen's basis leads to an analogous definition of descent representations.

All this machinery leads to a natural definition of a new set of $G(r,p,n)$-modules, that we call *colored-descent representations*. They are generalizations of the descent representations introduced by Adin, Brenti, and Roichman in [3] for the symmetric and hyperoctahedral group, and which refine the descent representations of Solomon [25]. The decomposition into irreducibles of the colored-descent representations is provided. Moreover it turns out that the multiplicity of any irreducible representations is counted by the cardinality of a particular class of standard Young tableaux.

2 Complex Reflection Groups

generators

For our exposition it will be much more convenient to consider wreath products not as groups of complex matrices, but as groups of colored permutations.

For any $n \in \mathbb{P} := \{1, 2, \ldots\}$ we let $[n] := \{1, 2, \ldots, n\}$, and for any $a, b \in \mathbb{N}$ we let $[a, b] := \{a, a+1, \ldots, b\}$. Let S_n be the symmetric group on $[n]$. A permutation $\sigma \in S_n$ will be denoted by $\sigma = \sigma(1) \cdots \sigma(n)$.

Let $r, n \in \mathbb{P}$. Define:

$$G(r,n) := \{((c_1, \ldots, c_n), \sigma) \mid c_i \in [0, r-1], \sigma \in S_n\}. \tag{1}$$

def-grn

Any c_i can be considered as the color of the corresponding entry $\sigma(i)$. This explains the fact that this group is also called the *group of r-colored permutations*. Sometimes we will represent its elements in window notation as

$$g = g(1) \cdots g(n) = \sigma(1)^{c_1} \cdots \sigma(n)^{c_n}.$$

When it is not clear from the context, we will denote c_i by $c_i(g)$. Moreover, if $c_i = 0$, it will be omitted in the window notation of g. We denote by

$$\mathrm{Col}(g) := (c_1, \ldots, c_n) \quad \text{and} \quad \mathrm{col}(g) := \sum_{i=1}^{n} c_i,$$

the *color vector* and the *color weight* of any $\gamma := ((c_1, \ldots, c_n), \sigma) \in G(r,n)$.
For example, for $g = 4^1 3 2^4 1^2 \in G(5,4)$ we have $\mathrm{Col}(g) = (1,0,4,2)$ and $\mathrm{col}(g) = 7$.

Now let $p \in \mathbb{P}$ such that $p | r$. The *complex reflection group* $G(r,p,n)$ is the subgroup of $G(r,n)$ defined by

$$G(r,p,n) := \{g \in G(r,n) : \mathrm{col}(g) \equiv 0 \bmod p\}. \tag{2}$$

def-grpn

In particular we have the following chain of inclusions

$$G(r,r,n) \trianglelefteq G(r,p,n) \trianglelefteq G(r,1,n) = G(r,n),$$

where $\trianglelefteq$ stands for normal subgroup.

3 Colored Descent Basis

ol-des-basis

In order to lighten the notation, we let $G := G(r,n)$, $H := G(r,p,n)$, and $d := r/p$. The wreath product G acts on the ring of polynomials $\mathbb{C}[\mathbf{x}]$ as follows

$$\sigma(1)^{c_1}\cdots\sigma(n)^{c_n}\cdot P(x_1,\ldots,x_n) = P(\zeta^{c_{\sigma(1)}}x_{\sigma(1)},\ldots,\zeta^{c_{\sigma(n)}}x_{\sigma(n)}),$$

where ζ denotes a primitive r^{th} root of unity. A set of fundamental invariants under this actions is given by the *elementary symmetric functions* $e_j(x_1^r,\ldots,x_n^r)$, $1 \le j \le n$. Now, consider the restriction of the previous action on $\mathbb{C}[\mathbf{x}]$ to H. A set of fundamental invariants is given by

$$f_j(x_1,\ldots,x_n) := \begin{cases} e_j(x_1^r,\ldots,x_n^r) & \text{for } j = 1,\ldots,n-1 \\ x_1^d\cdots x_n^d & \text{for } j = n. \end{cases}$$

It follows that the degrees of H are $r, 2r, \ldots, (n-1)r, nd$.

Let $\mathcal{I}_H := (f_1,\ldots,f_n)$, the module of coinvariants $\mathbb{C}[\mathbf{x}]_H := \mathbb{C}[\mathbf{x}]/\mathcal{I}_H$ has dimension equal to $|H|$, that is $\frac{n!r^n}{p}$. In what follows we will associate to any element $h \in H$ an ad-hoc monomial in $\mathbb{C}[\mathbf{x}]$. Those monomials will form a linear basis for the module of coinvariants. In order to do this, we need to introduce various statistics on complex reflection groups.

For any $r,p,n \in \mathbb{P}$, with $p|r$ and $d := r/p$ we define the following subset of $G(r,n)$,

$$\Gamma(r,p,n) = \{\gamma = ((c_1,\ldots,c_n),\sigma) \in G(r,n) \mid c_n < d\}. \qquad (3)$$ def-Gamma

Note that $\Gamma := \Gamma(r,p,n)$ it is not a subgroup of G. Clearly, $|\Gamma| = n!r^{n-1}d$ and so it is in bijection with H. Although it is not fundamental for our purposes, we specify a bijection, in such a way that some of the definitions we will introduce, coincide with the usual ones, once we specialize H to any classical Weyl group. Indeed, one can easily check that the mapping

$$((c_1,\ldots,c_n),\sigma) \mapsto ((c_1,\ldots,\lfloor\frac{c_n}{p}\rfloor),\sigma) \qquad (4)$$ bijection

is a bijection between H and Γ. As usual for any $a \in \mathbb{Q}$, $\lfloor a\rfloor$ denotes the greatest integer $\le a$. In order to make our definitions more natural and clear, from now on, we will work with Γ instead of H. Clearly, via the above bijection every function on Γ can be considered as a function on H and viceversa.

We fix the following order $\prec$ on colored integer numbers

$$1^{r-1} \prec 2^{r-1} \prec \ldots \prec n^{r-1} \prec \ldots \prec 1^1 \prec 2^1 \prec \ldots \prec n^1 \prec 1 \prec 2 \prec \ldots \prec n. \qquad (5)$$ order

The *descent set* of an colored integer sequence $\gamma \in \Gamma$ is defined by $\mathrm{Des}(\gamma) := \{i \in [n-1] : \gamma_i \succ \gamma_{i+1}\}$. Moreover for any $\gamma = ((c_1,\ldots,c_n),\sigma) \in \Gamma$ we let

$$d_i(\gamma) := |\{j \in \mathrm{Des}(\gamma) : j \succeq i\}| \;\text{ and }\; m_i(\gamma) := r\cdot d_i(\gamma) + c_i(\gamma). \qquad (6)$$ def-icomp

For every $\gamma \in \Gamma$ we define the $G(r,p,n)$-*major index* of γ by

$$\mathrm{m}(\gamma) := \sum_{i=1}^{n} m_i(\gamma). \qquad (7) \quad \boxed{\texttt{def-maj}}$$

For example, let $\gamma = 62^5 4^4 3^1 1^6 5^3 \in \Gamma(8,2,6)$. We have $(d_1(\gamma), \ldots, d_n(\gamma)) = (2,1,1,0,0,0)$, $(m_1(\gamma), \ldots, m_n(\gamma)) = (16,13,12,9,6,3)$ and $\mathrm{m}(\gamma) = 59$.

We are ready to associate to every element of Γ a monomial in $\mathbb{C}[\mathbf{x}]$. Let $\gamma = ((c_1, \ldots, c_n), \sigma) \in \Gamma$. We define

$$\mathbf{x}_\gamma := \prod_{i=1}^{n} x_{\sigma(i)}^{m_i(\gamma)}. \qquad (8) \quad \boxed{\texttt{defmonomial}}$$

It is clear that $m_n(\gamma) < d$, hence $\mathbf{x}_\gamma$ is nonzero in $\mathbb{C}[\mathbf{x}]_H$.
For example, if $\gamma = 62^5 4^4 3^1 1^6 5^3 \in \Gamma(8,2,6)$ then $\mathbf{x}_\gamma = x_1^6 x_2^{13} x_3^9 x_4^{12} x_5^3 x_6^{16}$.

We restrict our attention to the quotient $S := \mathbb{C}[\mathbf{x}]/(f_n)$. Hence we consider nonzero monomials $M = \prod_{i=1}^{n} x_i^{a_i}$ such that $a_i < d$ for at least one $i \in [n]$. We associate to M the element $\gamma(M) = ((c_1, \ldots, c_n), \sigma) \in \Gamma$ such that for all $i \in [n]$

i) $a_{\sigma(i)} \geq a_{\sigma(i+1)}$;
ii) $a_{\sigma(i)} = a_{\sigma(i+1)} \Longrightarrow \sigma(i) < \sigma(i+1)$,
iii) $c_i \equiv a_{\sigma(i)} \pmod r$.

We denote by $\lambda(M) := (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ the *exponent partition* of M, and we call $\gamma(M) \in \Gamma$ the *colored index permutation*.

Now, let $M = \prod_{i=1}^{n} x_i^{a_i}$ be a nonzero monomial in S, and let $\gamma := \gamma(M)$ be its colored index permutation. Consider now the monomial $\mathbf{x}_\gamma$ associated to γ.

We associate to M another partition, $\mu(M)$, defined by

$$\mu'(M) := \left(\frac{a_{\sigma(i)} - m_i(\gamma)}{r} \right)_{i=1}^{n-1}, \qquad (9) \quad \boxed{\texttt{def-mu}}$$

where, as usual, μ' denotes the conjugate partition of μ.

Example 3.1. Let $r = 8$, $p = 2$, and $n = 6$ and consider the monomial $M = x_1^6 x_2^{21} x_3^{17} x_4^{20} x_5^3 x_6^{32} \in \mathbb{C}[x_1, \ldots, x_6]/(f_6)$. The exponent partition $\lambda(M) = (32,21,20,17,6,3)$ is obtained by reordering the power of x_i's following the colored index permutation $\gamma(M) = 62^5 4^4 3^1 1^6 5^3 \in \Gamma(8,2,6)$. We have already computed the monomial $\mathbf{x}_{\gamma(M)} = x_1^6 x_2^{13} x_3^9 x_4^{12} x_5^3 x_6^{16}$. It follows that $\mu(M) = (4,1)$.

We now define a partial order on the monomials of the same total degree in S. Let M and M' be nonzero monomials in S with the same total degree and such that the exponents of x_i in M and M' have the same parity $(\mathrm{mod}\ r)$ for every $i \in [n]$. Then we write $M' < M$ if one of the following holds:

1) $\lambda(M') \lhd \lambda(M)$, or

2) $\lambda(M') = \lambda(M)$ and $\mathrm{inv}(\gamma(M')) > \mathrm{inv}(\gamma(M))$.

Here, $\mathrm{inv}(\gamma) := |\{(i,j) \mid i < j \text{ and } \gamma(i) \succ \gamma(j)\}|$, and $\lhd$ denotes the *dominance order* defined on the set partitions of a fixed nonnegative integer n by: $\mu \unlhd \lambda$ if for all $i \geq 1$

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

Theorem 3.2. *The set*

$$\{\mathbf{x}_\gamma + \mathcal{I}_H : \gamma \in \Gamma\}$$

is a basis for $\mathbb{C}[\mathbf{x}]_H$.

Example 3.3. The elements of $\Gamma(6,3,2)$, $d = 2$, are

$$\begin{array}{llllll}
12 & 1^12 & 1^22 & 1^32 & 1^42 & 1^52 \\
12^1 & 1^12^1 & 1^22^1 & 1^32^1 & 1^42^1 & 1^52^1 \\
21 & 2^11 & 2^21 & 2^32 & 2^41 & 2^51 \\
21^1 & 2^11^1 & 2^21^1 & 2^32^1 & 2^41^1 & 2^51^1.
\end{array}$$

The corresponding monomials are

$$\begin{array}{llllll}
1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\
x_1^6x_2 & x_1x_2 & x_1^2x_2 & x_1^3x_2 & x_1^4x_2 & x_1^5x_2 \\
x_2^6 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\
x_2^6x_1 & x_2^7x_1 & x_2^2x_1 & x_2^3x_1 & x_2^4x_1 & x_2^5x_1.
\end{array}$$

It is easy to check that they form a basis for $\mathbb{C}[x_1, x_2]/(x_1^6 + x_2^6, x_1^2x_2^2)$.

4 The Representation Theory of $G(r, p, n)$

In this section we present the representation theory of the group $H := G(r,p,n)$. We follow the exposition of [31], (see also [20]). Since the irreducible representations of H are related to the irreducible representations of G via Clifford Theory, we start this section by presenting the representation theory of G.

Let $g = \sigma(1)^{c_1} \cdots \sigma(n)^{c_n} \in G$. First divide $\sigma \in S_n$ into cycles, and then provide the entries with their original color c_i by obtaining colored cycles. The color of a cycle is simply the sum of all the colors of its entries. For every $i \in [0, r-1]$, let α^i be the partition formed by the lengths of the cycles of g having color i. We may thus associate g with the r-partition $\vec{\alpha} = (\alpha^0, \ldots, \alpha^{r-1})$. Note that $\sum\limits_{i=0}^{r-1} |\alpha^i| = n$. We refer to $\vec{\alpha}$ as the *type* of g. One can prove that two elements of G are conjugate if and only if the have the same type.

It is well known that irreducible representations of G are also indexed by r-tuple of partitions $\vec{\lambda} := (\lambda^0, \dots, \lambda^{r-1})$ with $\sum_{i=0}^{r-1} |\lambda^i| = n$. We denote this set by $\mathcal{P}_{r,n}$.

As mentioned above, the passage to the representation theory of $G(r,p,n)$, is by Clifford theory. The group G/H can be identified with the cyclic group C of order p of the characters δ of G satisfying $H \subset Ker(\delta)$. More precisely, define the linear character δ_0 of G by $\delta_0((c_1, \dots, c_n), \sigma) := \zeta^{c_1+\dots+c_n}$, so that $C =< \delta_0^d >\simeq \mathbb{Z}_p$. The group C acts on the set of irreducible representations of G by

$$V(\vec{\lambda}) \mapsto \delta \otimes V(\vec{\lambda})$$

where $V(\vec{\lambda})$ is the irreducible representation of G indexed by $\vec{\lambda}$, and $\delta \in C$. This action can be explicitly described as follows. Let $\vec{\lambda} = (\lambda^0, \dots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$, we define a *1-shift* of $\vec{\lambda}$ by

$$(\vec{\lambda})^{\circlearrowleft 1} := (\lambda^{r-1}, \lambda^0, \dots, \lambda^{r-2}). \tag{10}$$ shift-op

By applying i-times the *shift operator* we get $(\vec{\lambda})^{\circlearrowleft i}$. Then once can show (see [20, Section 4]) that

$$\delta^i \otimes V(\vec{\lambda}) \simeq V((\vec{\lambda})^{\circlearrowleft i}), \tag{11}$$ action

for every $\delta \in C$.

Now let us denote by $[\vec{\lambda}]$ a C-orbit of the representation $V(\vec{\lambda})$. From (11) we obtain that $[\vec{\lambda}] = \{V(\vec{\mu}) \ : \ \vec{\mu} \sim \vec{\lambda}\}$, where the equivalence relation is defined by

$$\vec{\lambda} \sim \vec{\mu} \ \text{ if and only } \ \vec{\mu} = (\vec{\lambda})^{\circlearrowleft i \cdot d} \ \text{ for some } \ i \in [0, p-1]. \tag{12}$$ orbit

Let us denote $b(\vec{\lambda}) := |[\vec{\lambda}]|$, and set $u(\vec{\lambda}) := \frac{p}{b(\vec{\lambda})}$. Consider the stabilizer of $\vec{\lambda}$, $C_{\vec{\lambda}}$, that is:

$$C_{\vec{\lambda}} := \{\delta \in C \mid V(\vec{\lambda}) = \delta \otimes V(\vec{\lambda})\}.$$

Clearly, $C_{\vec{\lambda}}$ is a subgroup of C generated by $\delta_0^{b(\vec{\lambda}) \cdot d}$ and so $|C_{\vec{\lambda}}| = u(\vec{\lambda})$.

It can be proven that the restriction of the irreducible representation $V(\vec{\lambda})$ of G to H decomposes into $u(\vec{\lambda}) = |C_{\vec{\lambda}}|$ non-isomorphic irreducible H modules. On the other hand, any other G-module in the same orbit $[\vec{\lambda}]$ will give us the same result. Actually, one can prove even more:

repsofgrpn **Theorem 4.1.** *(See [31])*

There is a one to one correspondence between the irreducible representations of H and the ordered pairs $([\vec{\lambda}], \delta)$ where $[\vec{\lambda}]$ is the orbit of the irreducible representation $V(\vec{\lambda})$ of G and $\delta \in C_{\vec{\lambda}}$. Moreover if $\chi^{\vec{\lambda}}$ denotes the character of $V(\vec{\lambda})$ then

i) $\chi^{\vec{\lambda}} = \chi^{\vec{\mu}}$ *for all* $\vec{\mu} \in [\vec{\lambda}]$, *and*

ii) $\chi^{\vec{\lambda}} = \sum_{\delta \in C_{\vec{\lambda}}} \chi^{([\vec{\lambda}], \delta)}$.

Here is a simple but important example. The irreducible representations of B_n ($G(2,1,n)$ in our notation) are indexed by bi-partitions of n. The Coxeter group D_n ($G(2,2,n)$ in our notation), is a subgroup of B_n of index 2. Thus the stabilizer of the action of $B_n/D_n \cong \mathbb{Z}_2$ on a pair of Young diagrams $\vec{\lambda} = (\lambda^1, \lambda^2)$ is either $\mathbb{Z}_2$ if $\lambda^1 = \lambda^2$, or $\{\mathrm{id}\}$ if $\lambda^1 \neq \lambda^2$. In the first case, the irreducible representation of B_n corresponding to $\vec{\lambda}$, when restricted to D_n, splits into two non-isomorphic irreducible representations of D_n. In the second case $\vec{\lambda} = (\lambda^1, \lambda^2)$ and $\vec{\lambda}^T = (\lambda^2, \lambda^1)$ correspond to two isomorphic irreducible representations of D_n.

We conclude this section by giving a new definition of major index on r-tuples of standard Young tableaux. Let $\vec{\lambda} = (\lambda^0, \ldots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$ a r-partition of n. A *Ferrers diagram* of shape $\vec{\lambda}$ is obtained by the union of the Ferrers diagrams of shapes $\lambda^0, \ldots, \lambda^{r-1}$, where the $(i+1)^{\text{th}}$ diagram lies south west of the i^{th}. A *standard Young r-tableau* $T := (T^0, \ldots, T^{r-1})$ of shape $\vec{\lambda}$ is obtained by inserting the integers $1, 2, \ldots, n$ as *entries* in the corresponding Ferrers diagram increasing along rows and down columns of each diagram separately. We denote by $\mathrm{SYT}(\vec{\lambda})$ the set of all r-standard Young tableaux of shape $\vec{\lambda}$.

A *descent* in a r-standard Young tableau T is an entry i such that $i+1$ is strictly below i. We denote the set of descents in T by $\mathrm{Des}(T)$. The *major index* of a tableau T is $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$. We define also $\mathrm{Col}(T) := (\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i = k$ if $i \in T^k$, and $\mathrm{col}(T) := \epsilon_1 + \ldots + \epsilon_n$.

We define the *flag-major index* of an r-tableau by

$$\mathrm{fmaj}(T) := r \cdot \mathrm{maj}(T) + \mathrm{col}(T). \qquad (13)$$

def-fmaj-tab

tableau4

Figure 1: A 4-tableau and its 2-shift

For example, the tableau T in Figure 4 belongs to $\mathrm{SYT}((3,2),(1,1,1),(3,1),(2))$. We have that $\mathrm{Des}(T) = \{2,4,6,7\}$, $\mathrm{maj}(T) = 19$, $\mathrm{col}(T) = 3+8+6 = 17$, and so $\mathrm{fmaj}(T) = 93$.

The following definition is fundamental in our work. Let $\vec{\lambda} = (\lambda^0, \ldots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$. We define a *$C$-standard Young tableau* $T = (T^0, \ldots, T^{r-1})$ of type $[\vec{\lambda}]$ to be a standard Young r-tableau of one of the shapes in $[\vec{\lambda}]$ such that $n \in T^0 \cup \cdots \cup T^{d-1}$.

Let T an r-tableau of shape $\vec{\lambda}$. Then from (**??**), it follows that all the possible tableaux in $[\vec{\lambda}]$, have shapes obtained from that of $\vec{\lambda}$ by applying $i \cdot d$-shifts for $i = 0, \ldots, p-1$.

We denote by $\mathrm{CSYT}[\vec{\lambda}]$ the set of all C-standard r-tableaux of type $\vec{\lambda}$. We define the *$G(r,p,n)$-major index* of a C-standard tableau as the restriction of fmaj to CSYT, denoted

$$\mathrm{m}(T) := r \cdot \mathrm{maj}(T) + \mathrm{col}(T). \tag{14}$$

def-m-tab

5 Colored-descent representations of $G(r,p,n)$

The module of coinvariants $\mathbb{C}[\mathbf{x}]_H$ has a natural grading induced from that of $\mathbb{C}[\mathbf{x}]$. If we denote by R_k its k^{th} homogeneous component, we have

$$\mathbb{C}[\mathbf{x}]_H = \bigoplus_{k \geq 0} R_k.$$

In this section we introduce a set of $G(r,p,n)$-modules $R_{\mathcal{D},\mathcal{C}}$ which decompose R_k. These representations, called colored-descent representations, generalize the descent representations introduced for S_n and B_n by Adin, Brenti and Roichman in [3]. See also [8] for the case of D_n.

If $|\lambda| = k$ then one has:

$$\begin{aligned} J_\lambda^{\trianglelefteq} &:= \mathrm{span}_{\mathbb{C}}\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma,\ \lambda(\mathbf{x}_\gamma) \trianglelefteq \lambda\} \text{ and} \\ J_\lambda^{\triangleleft} &:= \mathrm{span}_{\mathbb{C}}\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma,\ \lambda(\mathbf{x}_\gamma) \triangleleft \lambda\} \end{aligned}$$

are two submodules of R_k. Their quotient is still an H-module, denoted by

$$R_\lambda := \frac{J_\lambda^{\trianglelefteq}}{J_\lambda^{\triangleleft}}.$$

For any $\mathcal{D} \subseteq [n-1]$ we define the partition $\lambda_{\mathcal{D}} := (\lambda_1, \ldots, \lambda_{n-1})$, where $\lambda_i := |\mathcal{D} \cap [i, n-1]|$. For $\mathcal{D} \subseteq [n-1]$ and $\mathcal{C} \in [0, r-1]^n$, we define the vector

$$\lambda_{\mathcal{D},\mathcal{C}} := r \cdot \lambda_{\mathcal{D}} + \mathcal{C},$$

where sum stands for sum of vectors.

From now on we denote $R_{\mathcal{D},\mathcal{C}} := R_{\lambda_{\mathcal{D},\mathcal{C}}}$, and by $\bar{\mathbf{x}}_\gamma$ the image of the colored-descent basis element $\mathbf{x}_\gamma \in J^{\trianglelefteq}_{\lambda_{\mathcal{D},\mathcal{C}}}$ in the quotient $R_{\mathcal{D},\mathcal{C}}$.

barra **Proposition 5.1.** *For any $\mathcal{D} \subseteq [n-1]$ and $\mathcal{C} \in [0, r-1]^n$, the set*

$$\{\bar{\mathbf{x}}_\gamma \ : \ \gamma \in \Gamma, \ \mathrm{Des}(\gamma) = \mathcal{D} \text{ and } \mathrm{Col}(\gamma) = \mathcal{C}\}$$

is a basis of $R_{\mathcal{D},\mathcal{C}}$. □

The H-modules $R_{\mathcal{D},\mathcal{C}}$ are called *colored-descent representation* in analogy with [3, Section 3.5]. They decompose the k^{th} component of $\mathbb{C}[\mathbf{x}]_H$ as follows.

decom **Theorem 5.2.** *For every $0 \leq k \leq r\binom{n}{2} + n(d-1)$,*

$$R_k \cong \bigoplus_{\mathcal{D},\mathcal{C}} R_{\mathcal{D},\mathcal{C}}$$

as H-modules, where the sum is over all $\mathcal{D} \subseteq [n-1], \mathcal{C} \in [0, r-1]^n$ such that

$$r \cdot \sum_{i \in \mathcal{D}} i + \sum_{j \in \mathcal{C}} j = k.$$

6 Decomposition of $R_{\mathcal{D},\mathcal{C}}$

In this section we prove a simple combinatorial description of the multiplicities of the irreducible representations of H in $R_{\mathcal{D},\mathcal{C}}$.

Theorem 6.1. *For every $\mathcal{D} \subseteq [n-1]$ and $\mathcal{C} \subseteq [0, r-1]^n$, $\vec{\lambda} \in \mathcal{P}_{r,n}$ and $\delta \in C_{\vec{\lambda}}$, the multiplicity of the irreducible representation of $G(r,p,n)$ corresponding to the pair $([\vec{\lambda}], \delta)$ in $R_{\mathcal{D},\mathcal{C}}$ is*

$$|\{T \in \mathrm{CSYT}[\vec{\lambda}] \mid \mathrm{Des}(T) = \mathcal{D}, \mathrm{Col}(T) = \mathcal{C}\}|.$$

As a corollary of this and of Theorem 5.2 we obtain the following result that is a generalization of a well known theorem on the decomposition of the coinvariant algebra of the symmetric group, (see e.g., [18] and [29]).

stembrH **Theorem 6.2.** *For $0 \leq k \leq r\binom{n}{2} + n(d-1)$, the representation R_k is isomorphic to the direct sum $\oplus m_{k,(\lambda,\delta)} V^{([\vec{\lambda}],\delta)}$, where $V^{([\vec{\lambda}],\delta)}$ is the irreducible representation of H labeled by $([\vec{\lambda}], \delta)$, and*

$$m_{k,([\vec{\lambda}],\delta)} :=| \{T \in \mathrm{CSYT}[\vec{\lambda}] \ : \ \mathrm{m}(T) = k\} | .$$

7 Combinatorial Identities

identities

In the case of classical Weyl groups and wreath products, any major statistic is associated with a descent statistic and their joint distribution is given by a nice closed formula, called *Carlitz identity*. In this last section we show that this is the case also for the complex reflection groups $G(r,p,n)$.

The following theorem presents the joint distribution of fdes and fmaj over $G(r,n)$.

Ca-G **Theorem 7.1 (Carlitz identity for G).** *Let $n \in \mathbb{N}$. Then*

$$\sum_{k\geq 0}[k+1]_q^n t^k = \frac{\sum_{g\in G(r,n)} t^{\mathrm{fdes}(g)} q^{\mathrm{fmaj}(g)}}{(1-t)(1-t^r q^r)(1-t^r q^{2r})\cdots(1-t^r q^{nr})}.$$

Using the above theorem and a specific decomposition of $G(r,n)$ into subsets which are in a bijection with $G(r,p,n)$, we get the following:

Ca-H **Theorem 7.2 (Carlitz identity for H).** *Let $n \in \mathbb{N}$. Then*

$$\sum_{k\geq 0}[k+1]_q^n t^k = \frac{\sum_{h\in G(r,n,p)} t^{\mathrm{d}(h)} q^{\mathrm{m}(h)}}{(1-t)(1-t^r q^r)(1-t^r q^{2r})\cdots(1-t^r q^{(n-1)r})(1-t^d q^{nd})}.$$

We refer to Theorem 7.1 and 7.2 as the Carlitz identities for G and H, respectively. It is worth to note that the powers of the q's in the denominators of the two formulas, $r, 2r, \ldots, nr$, and $r, 2r, \ldots, (n-1)r, nd$ are actually the degrees of $G(r,n)$ and $G(r,p,n)$.

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