# BORELORBITS OF $X^2=0$ IN $gl_n$

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ABSTRACT. We analyzes the structure of Borel orbits in the subvariety of $gl_n$ defined by $X^2 = 0$. The number of Borel orbits is finite, and is in one to one correspondence with certain partial permutation matrices. Equations are found up to radical for the Zariski closure of each orbit and these equations are shown to be generically reduced. The orbits are given a poset structure, which can also be described in terms of certain words. The Zariski closure of an orbit can be determined from the poset. The dimension of an orbit (as an algebraic variety) is given by a rank function for the poset, which is defined in terms of a statistic of the word of an orbit. An algorithm for calculating the degree of the Zariski closure of a given orbit is discussed.

## Section 1. Orbital Varieties

Fix a positive integer $n$ and a an algebraically closed field $K$ (but we make no restrictions on the characteristic). Let $B_n(K)$ (or simply $B_n$) be the set of upper triangular $n \times n$ matrices. Let $X$ be a nilpotent $n \times n$ matrix, that is a matrix satisfying $X^m = 0$ for some positive integer $m$. Notice that the eigenvalues of $X$ are all 0, so that $X$ is determined up to conjugacy by its Jordan canonical form. In turn, these canonical forms are parameterized by partitions of $n$ boxes (with column lengths corresponding to sizes of the Jordan blocks).

The set of all nilpotent matrices in $gl_n$ corresponding to a given partition is a natural object of study. Let $K^n$ be the $n$ dimensional $K$ vector space with basis $e_1, \ldots, e_n$. Recall that $gl_n$ acts on $K^n$ by left multiplication. (Concretely, matrices act on column vectors). By a result of Gerstenhaber (see [3]), the Zariski closure of a conjugacy class of nilpotent matrices is defined by power-rank conditions, which are equations coming from conditions of the form $\dim_k(X^m K^n) \leq a_m$ for various nonnegative integers $a_m$. (However, this set of equations is generally not reduced).

A natural object to consider is the intersection of a conjugacy class of a nilpotent matrix with the set $B_n$ of upper triangular matrices. These varieties arise in the study of Steinberg's triple variety [10] and their irreducible components classify the irreducible components of Springer fibers arising from the resolution of the flag variety. Understanding the irreducible components will also help in quantizing nilpotent conjugacy classes.

As the previous paragraph suggests, the intersection of a conjugacy class with the upper triangular matrices is not usually irreducible as an algebraic variety. The simplest example is the set of all $3 \times 3$ matrices corresponding to the partition 21, which has two components. An **orbital variety** is an irreducible component of the intersection of a nilpotent conjugacy class of $gl_n$ and the upper triangular matrices $B_n$.

**Example**

For the partition 21, there are two orbital varieties corresponding to the following linear subspaces of $B_n$.

1. $\begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix}$

By the work of Spaltenstein [7] and Springer [10], the orbital varieties arising from a partition $\lambda$ are in bijective correspondence with the standard Young tableau of shape $\lambda$ and content $1^n$. Given an orbital variety, one can obtain a tableau in the following manner. Take a generic matrix in the orbital variety. Then for each $1 \leq i \leq n$, the upper left square $i \times i$ submatrix is nilpotent, so one gets a sequence of partitions $\lambda_1, \ldots, \lambda_n = \lambda$, with $\lambda_i \subset \lambda_{i+1}$ for all $i$. Now one obtains a tableau of the appropriate shape and content by placing $i$ in the unique box of $\lambda_i/\lambda_{i-1}$.

The inverse map from tableau (of shape $\lambda$ and content $1^n$) to orbital varieties can be described in terms of RSK. Given a tableau $T$, one obtains an element $w$ of the Weyl group by applying RSK to the pair $(T, T)$ (in fact, one gets an involution). Then the corresponding orbital variety is $O_\lambda \cap \overline{n_+ \cap (w \cdot n_+)}$, where $O_\lambda$ is the corresponding nilpotent orbit and $n_+$ is the set of strictly upper triangular matrices.

The construction of the previous paragraph allows one to determine nice geometric information about orbital varieties. For example, for a given $\lambda$ any orbital variety corresponding to $\lambda$ has dimension $\frac{1}{2}\dim O_\lambda$. However, this construction gives little algebraic information. No algorithm is known for the equations of an orbital variety, and similarly it is difficult to determine if one orbital variety is contained in the closure of a second orbital variety.

One nice properties of orbital varieties is that they are stable under the conjugation action of the Borel group of invertible upper triangular matrices. Our general philosophy is to try to find and understand nice Borel stable subvarieties of the nilpotent cone. The natural inclination is to understand all Borel orbits; however there are infinitely many orbits, and moreover there exist continuous families of orbits. The right idea seems to be to use Borel invariants to define nice sets of Borel stable varieties of the nilpotent cone. However, the goal of our paper is more modest; we only examine the action of the Borel group in the set of matrices $X$ with $X^2 = 0$.

## Section 2. Borel orbits in $X^2 = 0$

**Remark**

Much of the work found in this paper was done independently by Melnikov ([5], [6]), but with the extra limitations that the matrices to be considered are upper triangular. Our work does not have this restriction. Also, Melnikov states her description of the Borel orbit poset in terms of involutions; our use of words to describe the Borel orbit poset make certain results easier to describe.

Let $M_n = gl_n$ be the set of all $n \times n$ matrices. Then $M_n$ forms a variety with coordinate ring $K[x_{ij}]$ (where $x_{ij}$ corresponds to the entry in the ith column and jth row of the generic matrix). Let $V_n$ be the subvariety of $M_n$ consisting of all $n \times n$ matrices with $X^2 = 0$. Notice that $V_n$ can be described set theoretically by the ideal $I_n$ generated by polynomials $p_{ij} = \Sigma_{k=1}^{k=n} x_{ik}x_{kj}$ for each ordered pair $(i, j)$. (These equations do *not* generate $V_n$ scheme theoretically, see section 12).

Let $B_n$ be the $n \times n$ Borel group of invertible upper triangular matrices. $B_n$ acts on $V_n$ by $b(X) = bXb^{-1}$ for all $b \in B$, $X \in V_n$. We wish to study the Borel orbits of $V_n$. One way such orbits arise is from certain partial permutation matrices.

**Definition**

A partial permutation matrix is a matrix such that all entries are 0 or 1and such that each row and each column has at most one nonzero entry.

Notice that the concept of a partial permutation matrix is a generalization of a permutation matrix. Recall that a permutation matrix is determined uniquely by a word of length $n$ in the letters $1, \ldots, n$. By generalizing this definition, a partial permutation can be described by a word of length $n$, but now using the alphabet $0, 1, \ldots, n$. This construction will also allow us to define a useful statistic later on.

**Definition**
Given a partial permutation matrix $P$, define the word of $P$ to be $W_P = w_1, \ldots, w_n$, where $w_j = i$ if $Pe_j = e_i$ and $w_j = 0$ if $Pe_j = 0$.

**Comment.** Notice that the word of a partial permutation may have multiple zeroes, but each nonzero term occurs at most once.

Obviously not all partial permutation matrices give rise to orbits in $V_n$. We now classify all such matrices in terms of their words.

**Definition**

A word $W$ is a valid $X^2$ word if for some partial permutation matrix $P$ with $P^2 = 0$, $W = W_P$. Alternatively, a word $W$ is a valid $X^2$ word if and only if the following two conditions hold:

1. No nonzero number of $W$ appears more than once.

2. If $w_j = i > 0$, then $w_i = 0$.

The second condition suggest the following important definition that will be used later.

**Definition**

Let $W = w_1, \ldots, w_n$ be a valid $X^2$ word. Suppose $w_i = 0$. We call $w_i$ a bound zero (or just bound) if for some $j$, $w_j = i$. A letter $w_i$ is said to be free if it is not a bound letter (in particular all nonzero letters are free).

**Example.**
The word 0103 is a valid $X^2$ word assigned to the partial permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that it is easy to count the number of valid $X^2$ words. They are in obvious bijection with the set of directed partial matchings on $n$ labeled points. (Given a valid $X^2$ word $W$, construct the graph with edges pointing from $i$ to $j$ if and only if $w_i = j$). Moreover, the exponential generating function for these words is $e^{x^2+x}$.

In general not all Borel orbits contain a partial permutation matrix. But in the case of $X^2 = 0$, we have the following theorem.

**Theorem 1** Let $X \in V_n$. The orbit $B \cdot X$ contains a unique partial permutation matrix $P_X$, and this partial permutation matrix is given by a valid $X^2$ word. In particular, there are finitely many orbits, indexed by the valid $X^2$ words.

The idea behind the proof of this theorem is to construct Borel invariants. Given a matrix $X$, one can then use these invariants to determine a partial permutation matrix $P_X$. To finish the proof, one uses the fact that $X^2 = 0$ to inductively show that $X$ and $P_X$ are conjugate.

**Section 3. Flags and Borel invariants**

In order to construct the Borel invariants, we will have to recall the definition of a complete flag.

**Definition.** A complete flag of the vector space $K^n$ is a sequence $V_0 \subset V_1 \subset \cdots \subset V_n = K^n$ of vector subspaces of $K^n$ such that $V_i$ has dimension $i$.

**Example.** An example of a complete flag is the standard complete flag, where $V_i = K^i$, the $i$ dimensional vector space with basis $e_1, \ldots, e_i$. (The vector spaces $K^i$ will also be called the standard $i$ flag.) Notice that a matrix $b$ is an invertible upper triangular matrix if and only if $bK^i = K^i$ for all $1 \leq i \leq n$, that is if and only if it preserves the standard complete flag.

Now we can define a collection of Borel invariants that will allow us to classify all the orbits of $V_n$.

**Definition.** For any $0 \leq i, j \leq n$, let $r_{i,j}(X) = \dim(K^i + XK^j) - i$. Equivalently, $r_{i,j}(X)$ is the rank of the lower left $n - i \times j$ submatrix of $X$.

Notice that the first definition of the $r_{i,j}(X)$'s shows that they are invariant under the action of the Borel group, that is for all $b \in B$, we have $r_{i,j}(X) = r_{i,j}(bXb^{-1})$. One can see this by noting that the action of $b$ just corresponds to a change of basis that fixes the standard flag.

For a partial permutation matrix $P$, $r_{i,j}(P)$ equals the number of ones in the lower left $n - i \times j$ submatrix of $P$. Alternatively, $r_{i,j}(P)$ can be calculated from $W_P$ as the number of elements of $w_1, \ldots, w_j$ that are greater than $i$.

**Example.**
For the word 0103, one gets the following matrix of $r_{i,j}$'s.

$$\begin{array}{cc} & \begin{array}{cccc} j=1 & j=2 & j=3 & j=4 \end{array} \\ \begin{array}{c} i=0 \\ i=1 \\ i=2 \\ i=3 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

We have seen that the word $W_P$ determines the $r_{i,j}$'s. However, the converse is also true.

**Lemma**
Given $P$ a partial permutation matrix with word $w_1, \ldots, w_n$, we have $w_j = i > 0$ if and only if $r_{i,j}(P) = r_{i,j-1}(P) = r_{i-1,j}(P) = r_{i-1,j-1}(P) + 1$. For a given $j$, $w_j = 0$ if and only if there is no $i$ such that the condition in the previous sentence holds.

In particular, the Borel invariants $r_{i,j}(P)$ distinguish between any two partial permutation matrices.

Suppose that we have $X^2 = 0$. If $X$ was conjugate to a partial permutation matrix $P$, then we could determine the word of $P$ (and thus $P$ itself) by noting that $r_{i,j}(P) = r_{i,j}(X)$ for all $i$ and $j$. By mimicking this procedure, we can assign a potential partial permutation $P_X$ to $X$.

The final step of the proof is to show that $X$ and $P_X$ are actually conjugate. Note that we must use the fact that $X^2 = 0$ here; for example the matrices

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

have the same $r_{i,j}$'s, but are not conjugate. In this case the first matrix has $X^3 = 0$ but not $X^2 = 0$.

Now we use the word $W_{P_X}$ of $P_X$ and the fact that $X^2 = 0$ to find an upper triangular change of coordinates where $K^n$ decomposes into a direct sum of an $X$ stable two dimensional vector space and an $X$ stable $n - 2$ dimensional vector space. (The two dimensional vector space will be spanned by $e_i, e_{w_i}$, where $w_i$ is the first nonzero letter of $W_{P_X}$, while the $n-2$ is spanned by the other vectors). By induction, we see that $X$ is conjugate to $P_X$.

**Section 4. Algebraic considerations and the combinatorial Borel poset**

Because of the previous theorem, we can identify a Borel orbit of $V_n$ with either the unique partial permutation of that orbit or with the valid $X^2$ word of that partial permutation matrix. Notice that each Borel orbit is a locally closed algebraic set, since the condition $r_{i,j}(X) = r_{ij}$ is defined the vanishing and non vanishing of certain minors. One would like to determine the Zariski closure of any Borel orbit, and also certain geometric information such as the dimension of any orbit.

First we attempt to determine the Zariski closure of any orbit. Here we have an obvious candidate. Suppose that an orbit associated to $P$ is defined by the equations $r_{i,j}(X) = r_{i,j}(P)$ and $X^2 = 0$. Notice that the condition $r_{i,j}(X) \leq r_{i,j}(P)$ is an algebraic condition, defined by the vanishing of all $r_{ij}$ of the lower left $n - i \times j$ submatrix. Now we can conjecture that the Zariski closure of the orbit should be the variety $Cl(P)$ defined by $r_{i,j}(X) \leq r_{i,j}(P)$ and $X^2 = 0$. Clearly, this variety contains the Zariski closure of $P$.

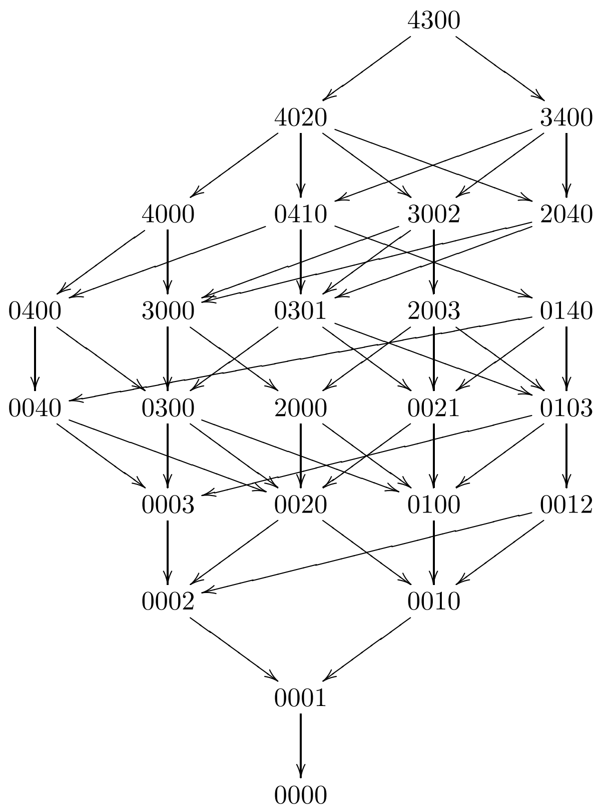
Now, the Zariski closure of a Borel orbit of $V_n$ is a union of Borel orbits of $V_n$. So we need to determine which partial permutation matrices $Q$ are contained the Zariski closure of $P$. Our approach will be to consider the set of all $Q$ such that $r_{i,j}(Q) \leq r_{i,j}(P)$ for all $i$ and $j$, and then to show that all such $Q$ are contained in the Zariski closure of $P$. This conjecture for the Zariski closure of an orbit inspires us to define the following poset.

**Definition.** The **combinatorial Borel orbit poset** is a poset on the set of $B$-orbits in $V_n$, with the relation that $Q \leq P$ if and only if $Q \subset Cl(P)$. Equivalently, $Q \leq P$ if and only if $r_{i,j}(Q) \leq r_{i,j}(P)$ for all $i$ and $j$.

The condition that $Q \leq P$ can also be interpreted as a combinatorial condition on the words $W_Q = v_1, \ldots, v_n$ and $W_P = w_1, \ldots, w_n$. Namely $Q \leq P$ if and only if for each $1 \leq l \leq n$, the elements $v_1, \ldots, v_l$ of the initial $l$ subword of $W_Q$ cover the elements $w_1, \ldots, w_l$ of the initial $l$ subword of $W_P$, in the sense that there exists a permutation $\sigma_l \in S_l$, such that $w_i \leq v_{\sigma_l(i)}$ for all $1 \leq i \leq l$.

**Example**

We give the Hasse diagram for the combinatorial poset of valid $X^2$ words of length $n = 4$.




**Section 5. Standard Moves and finding the Zariski closure**

As part of the proof that $Cl(P)$ is the Zariski closure of $P$, we show that the combinatorial Borel poset is the transitive closure of a finite set of standard moves. Recall that $w_i = 0$ is bound if $w_j = i$ for some $j$, and $w_i$ is free otherwise.

**Definition.** A standard move in the combinatorial Borel poset replaces a valid $X^2$ word $W$ by a smaller valid $X^2$ word $V$ in one of the four following ways.

1. Let $w_i$ be any nonzero letter of $W$. Then a standard move of type 1 has $V = w_1, \ldots, w_{i-1}, w_i^*, w_{i+1}, \ldots, w_n$ where $w_i^*$ is any value strictly less than $w_i$ such that $V$ is a valid $X^2$ word. In other words, $V$ is obtained from $W$ by decreasing the value of one particular nonzero letter. Note that for any nonzero letter $w_i$ of $W$, there exists a word $V$ obtained by from $W$ by decreasing the letter $w_i$, since we can always replace $w_i$ by 0 to get a valid $X^2$ word.

2. Let $w_i > w_j$ be any free letters with $i < j$. (Recall that a letter $w_k$ is free unless for some $w_l = k$. In particular, any nonzero letter is free). Then a standard move of type 2 has $V = w_1, \ldots, w_{i-1}, w_j, w_{i+1}, \ldots, w_{j-1}, w_i, w_{j+1}, \ldots, w_n$. In other words, $V$ is obtained from $W$ by switching the free letters $w_i$ and $w_j$. Notice that $w_j$ may equal 0 if the zero is free, but $w_i$ is never zero.

3. Let $w_i = k$ be any free letter and let $w_j = 0$ be any bound zero with $i < j$. Recall that since $w_j = 0$ is bound, then for some $l$, $w_l = j$. Then a standard move of type 3 has $V = w_1, \ldots, w_{i-1}, 0, w_{i+1}, \ldots, w_{j-1}, w_i, w_{j+1}, \ldots, w_{l-1}, i, w_{l+1}, \ldots, w_n$. In other words, $V$ is obtained from $W$ by switching the nonzero letter $w_i$ and the bound zero $w_j = 0$ and replacing $w_l = j$ with $i$. (In general, we do not assume either $i < l$ or $j < l$ for a standard move of type 3).

4. Let $w_i = j$ be any nonzero letter such that $j > i$ (note $w_j = 0$). Then a standard move of type 4 has $V = w_1, \ldots, w_{j-1}, 0, w_{j+1}, \ldots, w_{i-1}, j, w_{i+1}, \ldots, w_n$. In other words, $V$ is obtained from $W$ by replacing $w_i$ with 0 and replacing $w_j$ with $i$.

We give several examples of standard moves, along with a sequence of permutations that show that the larger word covers the smaller word.

**Examples**

1a. $00 < 01$. This a standard move of type 1, with $i = 2$ and $w_i^* = 0$. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = id_2$.

1b. $001 < 002$. This is a standard move of type 1, with $i = 3$ and $w_i^* = 1$. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = id_2$, $\sigma_3 = id_3$.

2a. $0012 < 0021$. This is a standard move of type 2, with $i = 3$ and $j = 4$. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = id_2$, $\sigma_3 = id_3$, $\sigma_4 = (34)$.

2b. $001 < 010$. This is a standard move of type 2, with $i = 2$ and $j = 3$. Notice $w_j$ is a free zero. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = id_2$, $\sigma_3 = (1)(23)$.

3. $0012 < 0103$. This is a standard move of type 3, with $i = 2$, $j = 3$ and $l = 4$. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = id_2$, $\sigma_3 = (23)$, $\sigma_4 = (23)$.

4. $01 < 20$. This is a standard move of type 4, with $i = 1$ and $j = 2$. A covering sequence is given by $\sigma_1 = id_1$, $\sigma_2 = (12)$.

One can generalize the covering sequences given above to show that any standard move gives rise to a relation in the combinatorial Borel poset. The following theorem shows that these moves generate the poset.

**Theorem**
The combinatorial Borel orbit poset relation $\leq$ is the transitive closure of the standard moves.

The proof of this theorem involves some difficult combinatorics, and the construction of a nice algorithm to construct a covering. As a consequence of this theorem, in order to show that $Cl(P)$ is the Zariski closure of $P$, it suffices to show that for any $Q$ obtained from $P$ by a standard move, $Q$ is in the Zariski closure of $P$. One can now use geometric methods to finish the proof; namely one constructs an affine line such that the general point lies in $P$ but a special point lies in $Q$.

**Corollary**

The Zariski closure of $P$ is $Cl(P)$.

**Section 6. A dimension statistic**

The Hasse diagram from the $n = 4$ case suggests that the combinatorial Borel poset is ranked. In fact, that is the case. First, we construct the rank statistic. Recall that in a valid $X^2$ word, a letter $w_i = 0$ is a bound zero if for some $j$, $w_j = i$. A letter is free if it is not bound.

**Definition.** Let $W = w_1, \ldots, w_n$ be a valid $X^2$ word. A free inversion of $W$ is a pair $(i, j)$ with $1 \leq i < j \leq n$ such that $w_i$ and $w_j$ are both free letters and with $w_i > w_j$. We define $FI(W)$ to be the number of free inversions of $W$ and we define $\pi(W) = FI(W) + \Sigma_{i=1}^{n} w_i$.

**Theorem**

The combinatorial Borel poset is a ranked poset, with rank function $\pi(W)$. Also, $\pi(W)$ is the Krull dimension of the Zariski closure the orbit associated to $W$.

We have two methods of proof for the second statement. One method just involves analyzing the conditions for $P$ to cover $Q$, and showing $\pi(P) = \pi(Q) + 1$ in this case. The alternative method is to compute the Borel stabilizer of the partial permutation matrix.

**Section 7. Hyperplanes and algebraic considerations**

We know that the Zariski closure of an orbit corresponding to a partial permutation matrix $P$ is defined set-theoretically by the conditions $X^2 = 0$ and $r_{i,j}(X) \leq r_{i,j}(P)$ for all $i, j$. One wants to prove that this set of equations is reduced, or to compute the radical if it is not reduced. In fact, for nonupper triangular orbits, the radical contains additional traces arising from the Levi factor of the appropriate parabolic subgroup.

Similarly, one would like to know the degree of an orbit closure as an algebraic variety. Also, these orbits seem to be Cohen-Macaulay in general.

Our general strategy to attack these questions is to look for Borel invariant hyperplane sections and try to set up an induction. Suppose $i = w_j$ is the largest element of the valid $X^2$ word $W$. Then $x_{i,j} = 0$ is such a Borel invariant hyperplane section, corresponding to the condition the $r(i-1, j) = 0$.

Using this method, one can show inductively by the method of principal radical systems that the ideals we have constructed are reduced for certain upper triangular orbits. (For nonupper triangular orbits, one must consider orbits of partial permutation matrices under the action of certain linear subgroups of the Borel group.)

As a corollary, one can show that any unions of matrix Schubert varieties coming from a rank table is reduced.

Once one can show that the hyperplane sections are reduced, it is relatively easy to show that the upper triangular orbits are Cohen-Macaulay.

Finally, the hyperplane sections also give us an inductive formula for computing degree.

**Theorem**

Let $W$ be a valid $X^2$ word. Let $i = w_j$ be the largest nonzero letter of $W$.
a. If $W = 0, 0, \ldots, 0$, $\deg(Cl(O_W)) = 1$.
b. Otherwise,

$$\deg(Cl(O_W)) = \sum_{V \leq W, \pi(V)+1=\pi(W), v_i < w_i} m_V \deg(Cl(O_V)),$$

where $m_V = 1$ if $V$ is obtained from $W$ by a standard move of type 1, 2, or 3, and $m_V = 2$ if $V$ is obtained from $W$ by a standard move of type 4.

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