WHEN IS A SCHUBERT VARIETY GORENSTEIN?

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ABSTRACT. A variety is Gorenstein if it is Cohen-Macualay and its canonical sheaf is a line bundle. This property implies a variety behaves like a smooth one for various algebro-geometric purposes. We introduce a new notion of pattern avoidance involving Bruhat order and use it to characterize which Schubert varieties are Gorenstein. We also give an explicit description as a line bundle of the canonical sheaf of a Gorenstein Schubert variety.

1. INTRODUCTION

This extended abstract is a shortened version of the paper [31], with details of the proofs omitted. The main goal of the paper is to give an explicit combinatorial characterization of which Schubert varieties in the complete flag variety are Gorenstein.

Let $\mathrm{Flags}(\mathbb{C}^n)$ denote the variety of complete flags $F_\bullet : \langle 0\rangle \subseteq F_1 \subseteq \ldots \subseteq F_n = \mathbb{C}^n$. Fix a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{C}^n$ and let $E_\bullet$ be the *anti*-canonical reference flag $E_\bullet$, that is, the flag where $E_i = \langle e_{n-i+1}, e_{n-i+2}, \ldots, e_n\rangle$. For every permutation w in the symmetric group S_n, there is the **Schubert variety**

$$X_w = \big\{F_\bullet \mid \dim(E_i \cap F_j) \geq \#\{k \geq n-i+1, w(k) \leq j\}\big\}.$$

These conventions have been arranged so that the codimension of X_w is $\ell(w)$, that is, the length of any expression for w as a product of simple reflections $s_i = (i \leftrightarrow i+1)$.

Gorensteinness is a well-known technical condition which implies that a variety behaves like a smooth one for various algebro-geometric purposes. In particular, smooth varieties are Gorenstein and Gorenstein varieties are by definition Cohen-Macaulay. A variety is **Gorenstein** if it is Cohen-Macaulay and its canonical sheaf is a line bundle. (Throughout this paper we freely identify vector bundles and their sheaves of sections for convenience.) Recall that on a *smooth* variety X, the **canonical sheaf**, denoted ω_X is $\wedge^{\dim(X)}\Omega_X$, where Ω_X is the cotangent bundle of X. For a possibly singular but normal variety, it is the pushforward of the canonical sheaf $\omega_{X_{\mathrm{smooth}}}$ of the smooth part X_{smooth} of X under the inclusion map. In fact, every Schubert variety is normal [10, 26] and Cohen-Macaulay [27], therefore, the above remarks actually suffice to give a complete definition of Gorensteinness for our purposes.

Gorensteinness can also be determined locally using a free resolution. Examples of Gorenstein varieties include all (normal) hypersurfaces. The variety of $m \times n$ matrices of rank at most r is Gorenstein iff $m = n$ (or, trivially, $r = 0$ or $r = \min(m, n)$); this follows either from the characterization of Gorenstein Schubert varieties on the Grassmannian, originally due to Svanes [30] and recoverable from our results here, or in characteristic 0 from the construction of a free resolution due originally to Lascoux [19].

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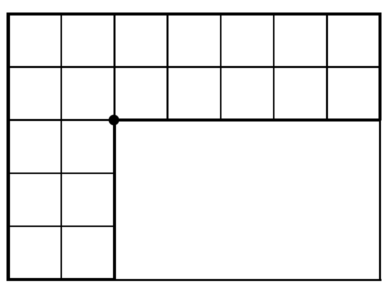
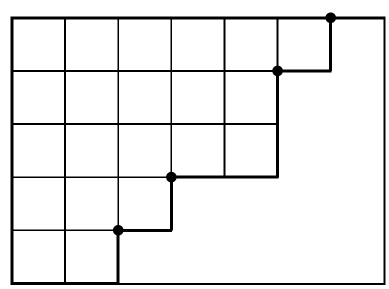
Smoothness and Cohen-Macaulayness of Schubert varieties have been extensively studied in the literature; see, for example, [3, 27] and the references therein. While all Schubert varieties are Cohen-Macaulay, actually very few Schubert varieties are smooth. (See the table near the end of this extended abstract.) Explicitly, X_w is smooth if and only if w is "1324-pattern avoiding" and "2143-pattern avoiding" [20].

Our main result (Theorem 1) gives an explicit combinatorial characterization of Gorensteinness similar to the above smoothness criteria. This answers a question raised by M. Brion and S. Kumar, and that was passed along to us by A. Knutson; see also [28, p. 88]. Our answer uses a generalized notion of pattern avoidance that we will define below.

2. The Grassmannian Case

First let us present the answer for Schubert varieties on the Grassmannian, due originally to T. Svanes in [30]; this case will illustrate one of the conditions given in our main theorem, and can also be derived as a special case of it. On the Grassmannian, Schubert varieties X_λ are indexed by partitions λ sitting inside an $\ell \times (n-\ell)$ rectangle; we use the convention that $|\lambda|$ is the codimension of X_λ in $Gr(\ell, n)$. The smooth Schubert varieties are those indexed by partitions λ whose complement in $\ell \times (n-\ell)$ is a rectangle, see, for example, [3] and the references therein. For example, $\lambda = (7,7,2,2,2)$ indexes a smooth Schubert variety in $\mathrm{Gr}(5,12)$.

$\lambda =$  $\mu =$ 

Alternatively, smooth Schubert varieties are those with exactly one inner corner. View the lower border of partition as a lattice path from the lower left-hand corner to the upper right-hand corner of $\ell \times (n-\ell)$; then an **inner corner** is a lattice point that has a lattice point of the path both directly below and directly to the right of it. The inner corners for the partitions λ and μ above are marked by "dots".

Therefore, the partition $\mu = (6,5,5,3,2)$ above does not index a smooth Schubert variety. However, it does index a Gorenstein Schubert variety; in general, a Grassmannian Schubert variety X_μ is Gorenstein if and only if all of the inner corners of μ sit on the same antidiagonal.

3. Main Definitions

In order to state our main results for a Schubert variety X_w of the flag variety $\mathrm{Flags}(\mathbb{C}^n)$, we will need some preliminary definitions. First we need to associate a partition to each descent of w, and define the associated inner corner distance. Secondly we need to define a new notion of pattern avoidance which we call Bruhat-restricted pattern avoidance.

Let d be a **descent** of w, that is, an index where $w(d) > w(d+1)$. Now write w in one-line notation as $w(1)w(2)\cdots w(n)$, and construct a subword $v_d(w)$ of w by concatenating the right-to-left minima of the segment strictly to the left of $d+1$ with the left-to-right maxima of the segment strictly to the right of d. In particular, $v_d(w)$ will necessarily include $w(d)$ and $w(d+1)$. Let $\widetilde{v}_d(w)$ denote the **flattening** of $v_d(w)$, that is, the unique permutation whose relative position of its entries matches that of $v_d(w)$.

Example 1. Let $w = 314972658 \in S_9$. This permutation has descents at positions 1, 4, 5 and 7. We see that $v_1(w) = 3149$, $v_4(w) = 14978$, $v_5(w) = 147268$, and $v_7(w) = 12658$, so therefore $\widetilde{v}_1(w) = 2134$, $\widetilde{v}_4(w) = 12534$, $\widetilde{v}_5(w) = 135246$, and $\widetilde{v}_7(w) = 12435$.

By construction, $\widetilde{v}_d(w) \in S_m$ is a **Grassmannian permutation**, that is, it has a unique descent at, say, position e. For any Grassmannian permutation $w \in S_m$ with its unique descent at e, let $\lambda(w) \subseteq e \times (m-e)$ denote the associated partition. This is obtained by drawing a lattice path starting from the lower left-hand corner of $e \times (m-e)$ and drawing a unit horizontal line segment at step $i = 1, 2, \ldots, m$ if i appears strictly *after* position e, and a unit vertical line segment otherwise. For example, the Grassmannian permutation $w = 3589\,11 \mid 12467\,10\,12$ corresponds to the partition $\lambda(w) = \mu = (6, 5, 5, 3, 2)$ depicted above. Now, given an inner corner of a partition $\lambda(w)$, let its **inner corner distance** be the sum of the distances from the inner corner to the top and left edges of the rectangle $e \times (m-e)$. Furthermore, suppose that $\lambda(w)$ has all its inner corners on the same antidiagonal; this is equivalent to requiring that the inner corner distance be the same for all inner corners. In this case we call this common inner corner distance $\mathfrak{I}(w)$; if there are no inner corners, we set $\mathfrak{I}(w) = 0$ by convention. For example, in μ above, all the inner corner distances equal 6.

Now we define Bruhat-restricted pattern avoidance. First we recall the classical notion of pattern avoidance and the Bruhat order on S_n. For $v \in S_\ell$ and $w \in S_n$, with $\ell \leq n$, an **embedding of** v **into** w is a sequence of indices $i_1 < i_2 < \cdots < i_\ell$ such that, for all $1 \leq a < b \leq \ell$, $w(i_a) > w(i_b)$ if and only if $v(a) > v(b)$. Then w **pattern avoids** v if there are no embeddings of v into w.

The **Bruhat order** on S_n, which we will denote by $\succ$, is defined as follows. First we say that $w(i \leftrightarrow j)$ **covers** w if $i < j$, $w(i) < w(j)$, and, for each k with $i < k < j$, either $w(k) < w(i)$ or $w(k) > w(j)$; then the Bruhat order is the transitive closure of this covering relation. The Bruhat order is graded by the length of a permutation, and one can check that v can cover w only if $\ell(v) = \ell(w) + 1$.

Now let $\mathcal{T}_v = \{(m_1 \leftrightarrow n_1), \ldots, (m_k \leftrightarrow n_k)\}$ be a set of **Bruhat transpositions** for v, where a Bruhat transposition $(m_j \leftrightarrow n_j)$ is one such that $v \cdot (m_j \leftrightarrow n_j)$ covers v in the Bruhat order. A **$\mathcal{T}_v$-restricted embedding** of v into w is an embedding of v into w such that $w \cdot (i_{m_j} \leftrightarrow i_{n_j})$ covers w (in the Bruhat order) for all $(m_j \leftrightarrow n_j) \in \mathcal{T}_v$. Then w **pattern avoids** v **with Bruhat restrictions** $\mathcal{T}_v$ if there are no $\mathcal{T}_v$-restricted embeddings of v into w. For example, the Bruhat transpositions for 31524 are $(1 \leftrightarrow 3)$, $(1 \leftrightarrow 5)$, $(2 \leftrightarrow 3)$, $(2 \leftrightarrow 4)$, and $(4 \leftrightarrow 5)$. This can be indicated by brackets drawn under the permutation as in Figure 1; the emptiness of the shaded rectangles in the graph of the permutation shows that $(1 \leftrightarrow 5)$ and $(2 \leftrightarrow 3)$ are indeed Bruhat transpositions. Using these "bracket diagrams", being a $\mathcal{T}_v$-restricted embedding means that the brackets associated to the transpositions in $\mathcal{T}_v$ are present in the "bracket diagram" for w. For example, there is no $\{(2 \leftrightarrow 4)\}$-restricted embedding of 2143 into 31524 since such an embedding would require a bracket between the 1 and the 4.

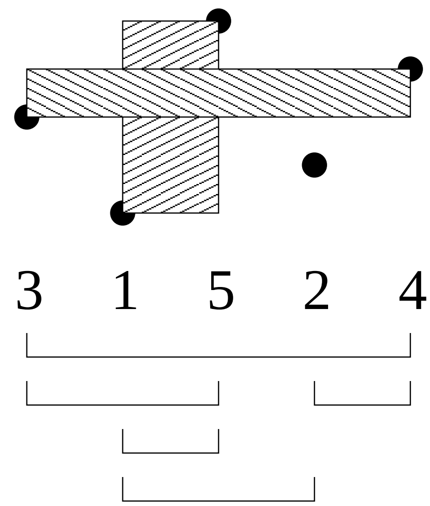


FIGURE 1. Bruhat transpositions for $w = 31524$

4. MAIN THEOREMS

Now we are ready to state our combinatorial characterization of Gorensteinness for Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$:

Theorem 1. *Let $w \in S_n$. The Schubert variety X_w is Gorenstein if and only if for each descent d of w, $\lambda(\widetilde{v}_d(w))$ has all of its inner corners on the same antidiagonal and w pattern avoids both 31524 and 24153 with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$ and $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$ respectively.*

It is possible to replace the conditions on the descents with an infinite list of patterns to be avoided with certain Bruhat restrictions; this list contains 2 patterns in S_k for each odd $k \geq 5$.

By combining Theorem 1 with the descriptions of the singularities along the "maximal singular locus" of a Schubert variety X_w given in [9, 22], we obtain the following purely geometric corollary.

Corollary 1. *A Schubert variety X_w is Gorenstein if and only if it is Gorenstein along its maximal singular locus.*

The proof follows from classical results characterizing which of the singularities described in [9, 22] are Gorenstein.

In comparing the smoothness characterization of [20] with Theorem 1, the considerations from our description of the Grassmannian case allow one to check that the 1324-pattern avoidance condition of the former implies the "inner corner condition" of the latter. It is also easy to see that the 2143-pattern avoidance condition of the former implies each of the Bruhat-restricted pattern avoidance conditions of the latter. We mention that Fulton [12] has characterized 2143-pattern avoidance in terms of the essential set of a permutation. A similar characterization can be given for the Bruhat-restricted pattern avoidance conditions of Theorem 1.

Example 2. The permutation $w = \underline{5}13\underline{284}6\underline{7} \in S_8$ has descents at positions $1, 3$ and 5 and we have

$$\widetilde{v}_1(w) = 3124, \widetilde{v}_3(w) = 1324, \text{ and } \widetilde{v}_5(w) = 126345.$$

Hence one checks that w satisfies the inner corner condition with

$$\mathfrak{I}(\widetilde{v}_1(w)) = 2,\ \mathfrak{I}(\widetilde{v}_3(w)) = 1,\ \text{and } \mathfrak{I}(\widetilde{v}_5(w)) = 1.$$

The Schubert variety X_w is Gorenstein, since there are no forbidden 31524 and 24153 patterns with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$ or $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$ respectively. Note that the underlined subword of w is a 31524-pattern, but since $w(1 \leftrightarrow 8)$ does not cover w, it does not prevent X_w from being Gorenstein.

We now describe the canonical sheaf of a Gorenstein Schubert variety as a line bundle. Let $T \cong (\mathbb{C}^*)^{n-1}$ be the subgroup of invertible diagonal matrices of determinant 1 in $\mathrm{SL}_n(\mathbb{C})$; the Borel-Weil construction associates to each integral weight $\alpha \in \mathrm{Hom}(T, \mathbb{C}^*)$ a line bundle $\mathcal{L}_\alpha$. Let $\mathcal{L}_\alpha\big|_{X_w}$ denote the restriction of this line bundle to X_w. We will write weights additively in terms of the $\mathbb{Z}$-basis of fundamental weights Λ_r, defined by

$$\Lambda_r \left(\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} \right) = t_1 \cdots t_r.$$

Theorem 2. *If X_w is Gorenstein, then $\omega_{X_w} \cong \mathcal{L}_\alpha\big|_{X_w}$ where $\alpha = \sum_{r=1}^{n-1} \widetilde{\alpha}_r \Lambda_{n-r}$ and*

$$\widetilde{\alpha}_r = \begin{cases} -2 + \mathfrak{I}(\widetilde{v}_r(w)) & \text{if } r \text{ is a descent} \\ -2 & \text{otherwise.} \end{cases} \tag{1}$$

5. Applications and Problems

Theorem 1 extends with no difficulty to Schubert varieties on partial flag varieties. Theorem 2 also extends, though some further calculations are needed. Our results can also be extended to the matrix Schubert varieties originally defined in [12], and thereby used to recover previously known results on Gorensteiness of ladder determinantal varieties [8, 15].

More geometrically, Theorem 2 calculates the sheaf cohomology of some line bundles on Gorenstein Schubert varieties, and gives small hints towards a final theorem on this open problem. Theorem 2 also allows us to characterize which smooth Schubert varieties are Fano, and gives new examples of higher-dimensional Fano varieties.

Further study of the relations between the geometry of Gorensteinness of Schubert varieties and related combinatorics should have potential. The most natural question is:

Problem 1. *Give analogues of Theorems 1 and 2 for generalized flag varieties corresponding to Lie groups other than $\mathrm{SL}_n(\mathbb{C})$.*

We expect that the methods given in this paper will extend to solve Problem 1. For the case of the odd orthogonal groups $\mathrm{SO}_{2n+1}(\mathbb{C})$, the solution for $\mathrm{SL}_n(\mathbb{C})$ leads to an answer which, however, is not entirely in terms of a good generalization of Bruhat-restricted pattern avoidance. The other classical types are not completely understood as of this writing.

In analogy with the determination of the singular loci of singular Schubert varieties [4, 9, 14, 18, 20, 21, 22], it should also be interesting to determine the "non-Gorenstein locus" of a non-Gorenstein Schubert variety; as in the case of singular loci this will for geometric reasons be a union of Schubert subvarieties X_v of our Schubert variety X_w. Therefore, we ask:

Problem 2. *Give a combinatorial characterization for the minimal v in the Bruhat order for which X_w is non-Gorenstein at X_v.*

Presumably, the eventual answer (for $\mathrm{SL}_n(\mathbb{C})$) will have some interesting relationship with the combinatorial characterization of the maximal singular locus. Indeed, in view of Corollary 1, one can hope that the maximal non-Gorenstein locus of X_w is simply the union of those Schubert cells in the maximal singular locus at which X_w is not Gorenstein.

A geometric explanation was recently given in [2] for the appearance of pattern avoidance in characterizations of smooth Schubert varieties. However, this explanation does not have an obvious modification to take into account Bruhat-restrictions. This leads to the following:

Problem 3. *Give a geometric explanation of Bruhat-restricted pattern avoidance which explains its appearance in Theorem 1.*

Finally, for those interested in combinatorial enumeration:

Problem 4. *Give a combinatorial formula (for example, a generating series) computing the number of Gorenstein Schubert varieties in* $\mathrm{Flags}(\mathbb{C}^n)$.

Using the methods of this paper, we computed the number of Gorenstein Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$ for some small values of n (see below). We compare this to the number of smooth Schubert varieties computed using the result of [20] (by the recursive formulas found in [5, 29]).

n	$n! = \#$ Cohen-Macaulay X_w	# Gorenstein X_w	# Smooth X_w
1	1	1	1
2	2	2	2
3	6	6	6
4	24	24	22
5	120	116	88
6	720	636	366
7	5040	3807	1552
8	40320	24314	6652
9	362880	163311	28696

6. ACKNOWLEDGEMENTS

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