A Gessel-Viennot-Type Method for Cycle Systems

Christopher R. H. Hanusa

ABSTRACT. We introduce a new determinantal method to count cycle systems in a graph that generalizes Gessel and Viennot's determinantal method on path systems. The presented method gives new insight into the enumeration of domino tilings of Aztec diamonds, Aztec pillows, and other related regions.

1. Introduction

In this article, we present an analogue of the Gessel-Viennot method for counting cycle systems on a type of directed graph we call a hamburger graph. A *hamburger graph* H is made up of two acyclic graphs G_1 and G_2 and a connecting edge set E_3 with the following properties. The graph G_1 has k distinguished vertices $\{v_1, \cdots, v_k\}$ with directed paths from v_i to v_j only if $i < j$. The graph G_2 has k distinguished vertices $\{w_{k+1}, \cdots, w_{2k}\}$ with directed paths from w_i to w_j only if $i > j$. The edge set E_3 connects each vertex v_i to vertex w_{k+i} and vice versa. (See Figure 1 for a visualization.) Hamburger graphs arise naturally in the study of Aztec diamonds, as explained in Section 3.

The Gessel-Viennot method was introduced in [**4, 5**], and has its roots in works by Karlin and McGregor [**7**] and Lindström [**9**]. A nice exposition of the method is given in the article by Aigner [**1**]. The Gessel-Viennot method is a determinantal method to count path systems in an acyclic directed graph G with k sources s_i and k sinks t_j for $1 \leq i, j \leq k$. A *path system* $\mathcal{P}$ is a collection of k vertex-disjoint paths from s_i to $t_{\sigma(i)}$ for some $\sigma \in S_k$ (where S_k is the symmetric group on k elements).

1991 *Mathematics Subject Classification.* Primary 05B45, 05C30; Secondary 05A15, 05B20, 05C38, 05C50, 05C70, 11A51, 11B83, 15A15, 15A36, 52C20.

Key words and phrases. cycle system, path system, Aztec diamond, Aztec pillow, Schröder numbers.

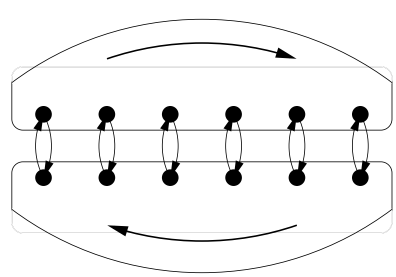

FIGURE 1. A hamburger graph.

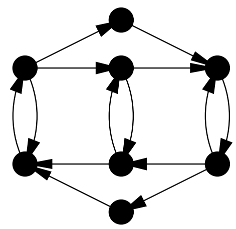

FIGURE 2. A simple hamburger graph H

Call a path system $\mathcal{P}$ *positive* if the sign of this permutation σ satisfies $\text{sgn}(\sigma) = +1$ and *negative* if $\text{sgn}(\sigma) = -1$. Let p^+ be the number of positive path systems and p^- be the number of negative path systems.

Corresponding to this graph G is a $k \times k$ matrix $A = (a_{ij})$ where a_{ij} is the number of paths from s_i to t_j in G. The result of Gessel and Viennot states that $\det A = p^+ - p^-$. For some applications of the Gessel-Viennot method, see [**1, 3, 4, 10**].

This article concerns the following determinantal method for counting cycle systems in a hamburger graph H. A *cycle system* $\mathcal{C}$ is a collection of vertex-disjoint directed cycles in H. Let ℓ be the number of edges in $\mathcal{C}$ that travel from G_2 to G_1 and let m be the number of cycles in $\mathcal{C}$.

Call a cycle system *positive* if $(-1)^{\ell+m} = +1$ and *negative* if $(-1)^{\ell+m} = -1$. Let c^+ be the number of positive cycle systems and c^- be the number of negative cycle systems.

Corresponding to each hamburger graph H is a $2k \times 2k$ block matrix M_H of the form

$$M_H = \begin{bmatrix} A & I_k \\ -I_k & B \end{bmatrix},$$

where the upper triangular matrix $A = (a_{ij})$ represents the number of paths from v_i to v_j in G_1 and the lower triangular matrix $B = (b_{ij})$ represents the number of paths from w_{k+i} to w_{k+j} in G_2. This matrix M_H is referred to as a *hamburger matrix.*

THEOREM 1.1 (The Hamburger Theorem). *If H is a hamburger graph, then* $\det M_H = c^+ - c^-$.

Notice that if the graphs G_1 and G_2 are planar with respect to their embeddings in H, each cycle must use exactly one edge from G_2 to G_1 so that the sign of every cycle system is $+1$. This implies the following corollary.

COROLLARY 1.2. *If H is a hamburger graph such that both G_1 and G_2 are planar,* $\det M_H = c^+$.

The following simple example serves to guide us. Consider the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{w_4, w_5, w_6\}$, $E_1 = \{v_1 \to v_2, v_2 \to v_3, v_1 \to v_3\}$, and $E_2 = \{w_6 \to w_5, w_5 \to w_4, w_6 \to w_4\}$. Our hamburger graph H will be the union of G_1, G_2, and the edge set E_3 consisting of edges $e_i : v_i \to w_{k+i}$ and $e'_i : w_{k+i} \to v_i$. In this example we have that $k = 3$. Figure 2 gives the graphical representation of H.

In terms of this example, the hamburger matrix M_H equals

$$M_H = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \end{bmatrix}.$$

The determinant of M_H is 17, corresponding to the seventeen cycle systems (each with sign $+1$) in Figure 3.

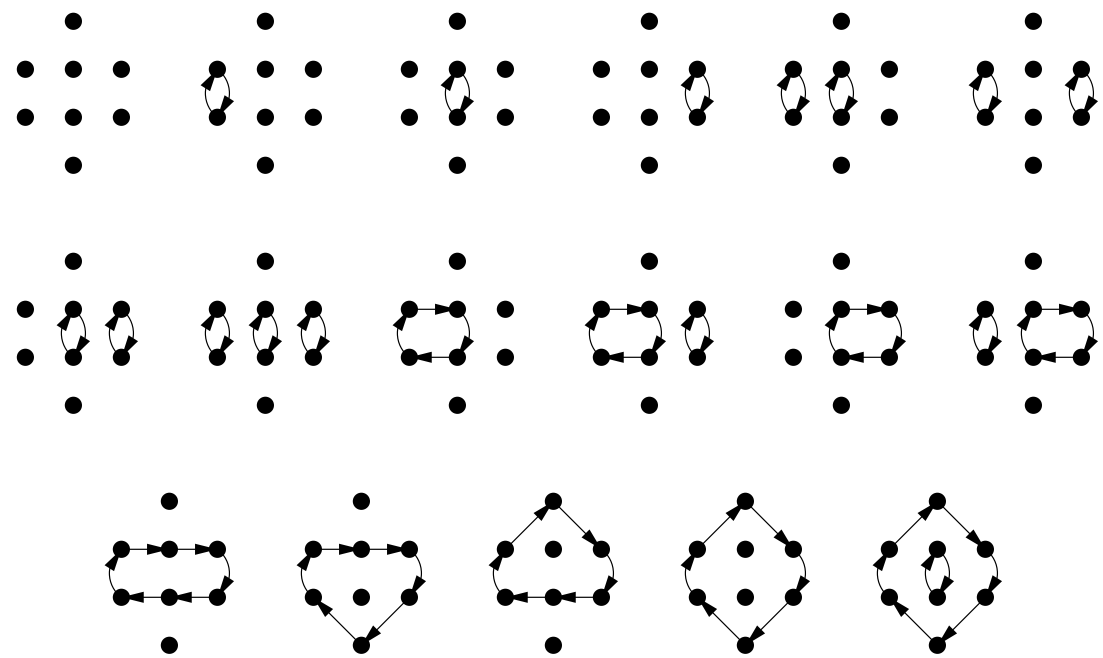

FIGURE 3. The seventeen cycle systems for the hamburger graph in Figure 2

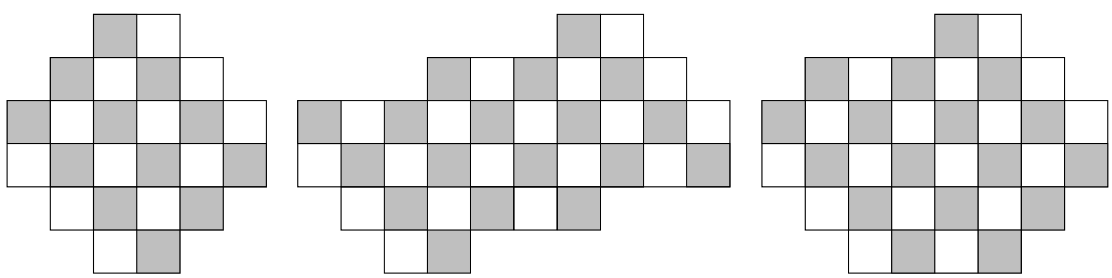

FIGURE 4. Examples of an Aztec diamond, an Aztec pillow, and a generalized Aztec pillow

The graph that inspires the definition of a hamburger graph comes from the work of Brualdi and Kirkland [**2**] in which they give a new proof that the number of tilings of the Aztec diamond is $2^{n(n+1)/2}$.

An *Aztec diamond*, denoted AD_n, is a union of the $2n(n+1)$ unit squares with integral vertices (x, y) such that $|x| + |y| \leq n + 1$. An *Aztec pillow*, as it was initially presented in [**12**], is also a rotationally symmetric region in the plane. On the top left diagonal however, the steps are composed of three squares to the right for every square up. Analogous to an Aztec diamond, we denote the Aztec pillow with $2n$ squares in each of the central rows as AP_n.

We also introduce the idea of a *generalized Aztec pillow*. This will be a horizontally convex and vertically convex region such that each of the steps along each diagonal is composed of an odd number of squares horizontally for every one square vertically. A key fact is that any generalized Aztec pillow can be recovered from a large enough Aztec diamond by the placement of horizontal dominoes. See Figure 4 for examples of an Aztec diamond, an Aztec pillow, and a generalized Aztec pillow.

Brualdi and Kirkland's proof of the Aztec diamond formula starts with the $n(n+1) \times n(n+1)$ Kasteleyn-Percus matrix (see [**8, 11**]) of a particular digraph to enumerate its cycle systems. The

Hamburger Theorem allows us to now enumerate domino tilings by taking the determinant of a $2n \times 2n$ matrix. An analogous reduction in determinant size occurs in all regions to which this theorem applies. In addition, whereas Kasteleyn theory applies only to planar graphs, there is no restriction of planarity for hamburger graphs. For this reason, the Hamburger Theorem introduces a new counting method for cycle systems in some non-planar graphs.

2. Idea of the Proof of the Hamburger Theorem

We now present a sketch of the proof of the Hamburger Theorem. Like the proof of the Gessel-Viennot method, the proof of the Hamburger Theorem hinges on terms canceling in the permutation decomposition of the determinant of M_H. For this proof, we must allow *generic cycle systems*, those unions of directed cycles which may not be vertex-disjoint, since they can appear in the permutation decomposition of the determinant of M_H.

If a cycle system arising from the permutation decomposition of the determinant is M_H is not vertex-disjoint, there are four possibilities. First, the cycle system may contain a cycle that is self-intersecting. Second, the cycle system may have two intersecting cycles, neither of which is a 2-cycle.

LEMMA 2.1. *If a generic cycle system $\mathcal{C}$ contains a cycle that is self-intersecting or contains two intersecting cycles that are not 2-cycles, $\mathcal{C}$ belongs to one well-defined family $\mathcal{F}$ of generic cycle systems of these types that cancel each other in the permutation expansion of the determinant of M_H.*

If either of these first two properties hold, there is some notion of order to determine which holds first. This vertex of first intersection is the basis for a family of cycle systems that all intersect for the first time at this vertex. We have that this family contributes a net zero weight in the determinantal expansion of M_H.

When calculating the number of cycle systems, we notice that the determinantal expansion of M_H has various summands that may represent the same cycle system. Consider the second cycle system in the third row of Figure 3. Since the solitary directed cycle visits vertices v_1, v_2, v_3, w_6, and w_4 in order, this cycle contributes a non-zero weight in the permutation expansion of the determinant corresponding to the permutation cycle (12364) in S_6. Notice that this cycle will also contribute a non-zero weight in the permutation expansion of the determinant corresponding to the permutation cycle (1364) since the cycle follows a path from v_1 to v_3 (by way of v_2), returning to v_1 via w_6 and w_4. Because of this ambiguity, we must introduce the idea of a *minimal permutation cycle* and a *minimal cycle system*, and realize that the determinant of M_H counts minimal cycle systems. The minimal permutation cycle for this second cycle system in the third row of Figure 3 is (1364).

In the case when neither the first nor second properties hold and that the generic cycle system is not vertex-disjoint or not minimal, at least one of two additional properties hold. The third property is that two cycles intersect and one of the cycles is a 2-cycle. The fourth property is that the cycle system may not be minimal.

LEMMA 2.2. *Let $\mathcal{C}$ be a generic cycle system that does not satisfy the conditions of Lemma 2.1. If $\mathcal{C}$ is not minimal or if it contains a cycle that intersects with a 2-cycle, $\mathcal{C}$ belongs to one well-defined family $\mathcal{F}$ of generic cycle systems of these types that cancel each other in the permutation expansion of the determinant of M_H.*

We can determine where the 2-cycle intersections and non-minimalities occur, and corresponding to this set of violations, we create a family of cycle systems each with this set of violations. We have that this family contributes a net zero weight in the determinantal expansion of M_H.

LEMMA 2.3. *The generic cycle system $\mathcal{C}$ is a minimal cycle system with vertex-disjoint cycles if and only if $\mathcal{C}$ does not satisfy the conditions of Lemmas 2.1 and 2.2.*

The cancellation from the above sets of families gives us that only minimal cycle systems contribute to the determinantal expansion of M_H. This contribution is the signed weight of each cycle system, implying that the determinant of M_H exactly equals $c^+ - c^-$. This proves the theorem. •

A weighted version of the Hamburger Theorem also exists. We allow *weights* $\mathrm{wt}(e)$ on edges of the hamburger graph; the simplest weighting which counts of the number of cycle systems assigns $\mathrm{wt}(e) \equiv 1$. We only require that $\mathrm{wt}(e_i)\mathrm{wt}(e_i') = 1$ for all $2 \leq i \leq k-1$, and do not require this condition for $i = 1$ nor for $i = k$. We now define the $2k \times 2k$ *weighted hamburger matrix* M_H to be the block matrix

$$M_H = \begin{bmatrix} A & D_1 \\ -D_2 & B \end{bmatrix}. \tag{2.1}$$

The upper triangular $k \times k$ matrix $A = (a_{ij})$ represents the sum of the products of the weights of edges over all paths from v_i to v_j in G_1 and the lower triangular $k \times k$ matrix $B = (b_{ij})$ represents the sum of the products of the weights of edges over all paths from w_{k+i} to w_{k+j} in G_2. The diagonal $k \times k$ matrix D_1 has as its entries $d_{ii} = \mathrm{wt}(e_i)$ and the diagonal $k \times k$ matrix D_2 has as its entries $d_{ii} = \mathrm{wt}(e_i')$. Note that when the weights of the edges in E_3 are all 1, these matrices satisfy $D_1 = D_2 = I_k$.

In our hamburger graph H, there are two possible types of cycles. There are k *2-cycles*

$$c : v_i \xrightarrow{e_i} w_{k+i} \xrightarrow{e_i'} v_i$$

and many more *general cycles* in H that alternate between G_1 and G_2. We can think of a cycle of this form as a path P_1 in G_1 connected by an edge $e_{1,1} \in E_3$ to a path Q_1 in G_2, which in turn connects to a path P_2 in G_1 by an edge $e_{1,2}'$, continuing in this fashion until arriving at a final path Q_ℓ in G_2 whose terminal vertex is adjacent to the initial vertex of P_1. We write

$$c : P_1 \xrightarrow{e_{1,1}} Q_1 \xrightarrow{e_{1,2}'} P_2 \xrightarrow{e_{2,1}} \cdots \xrightarrow{e_{\ell,1}} P_\ell \xrightarrow{e_{\ell,2}'} Q_\ell.$$

To each cycle c, we define the weight $\mathrm{wt}(c)$ of c to be the product of all the weights of the edges traversed by c:

$$\mathrm{wt}(c) = \prod_{e \in c} \mathrm{wt}(e).$$

We define a *weighted cycle system* to be a collection $\mathcal{C}$ of m vertex-disjoint cycles. We again define the *sign* of a weighted cycle system to be $\mathrm{sgn}(\mathcal{C}) = (-1)^{\ell+m}$, where ℓ is the total number of edges from G_2 to G_1 in $\mathcal{C}$. We say that a weighted cycle system $\mathcal{C}$ is *positive* if $\mathrm{sgn}(\mathcal{C}) = +1$ and *negative* if $\mathrm{sgn}(\mathcal{C}) = -1$.

For a hamburger graph H, let c^+ be the sum of the weights of positive weighted cycle systems, and c^- be the sum of the weights of negative weighted cycle systems.

THEOREM 2.4 (The weighted Hamburger Theorem). *The determinant of the weighted hamburger matrix M_H equals $c^+ - c^-$.*

3. Applications of the Hamburger Theorem

We discuss first the application of the Hamburger Theorem in the case when the region is an Aztec diamond, mirroring results of Brualdi and Kirkland. Then we discuss the results from the case when the region is an Aztec pillow, and lastly we explain how to implement the Hamburger Theorem when our region is any generalized Aztec pillow.

The Hamburger Theorem applies to the enumeration of domino tilings of Aztec diamonds and generalized Aztec pillows. To illustrate this connection, we count domino tilings of the Aztec diamond by enumerating an equivalent quantity, the number of matchings on the dual graph G of the Aztec diamond. The natural matching N of horizontal neighbors in G as exemplified in Figure 5a on AD_4 is a reference point. Given any other matching M on G, such as in Figure 5b, their symmetric difference is a union of cycles in the graph, such as in Figure 5c. When we orient the edges in N from black vertices to white vertices and in M from white vertices to black vertices, the symmetric

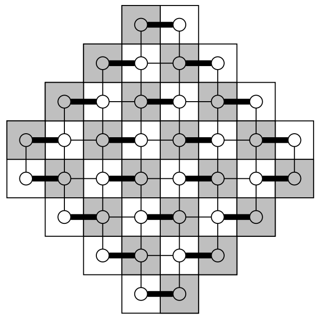

FIGURE 5. The symmetric difference of two matchings gives a cycle system

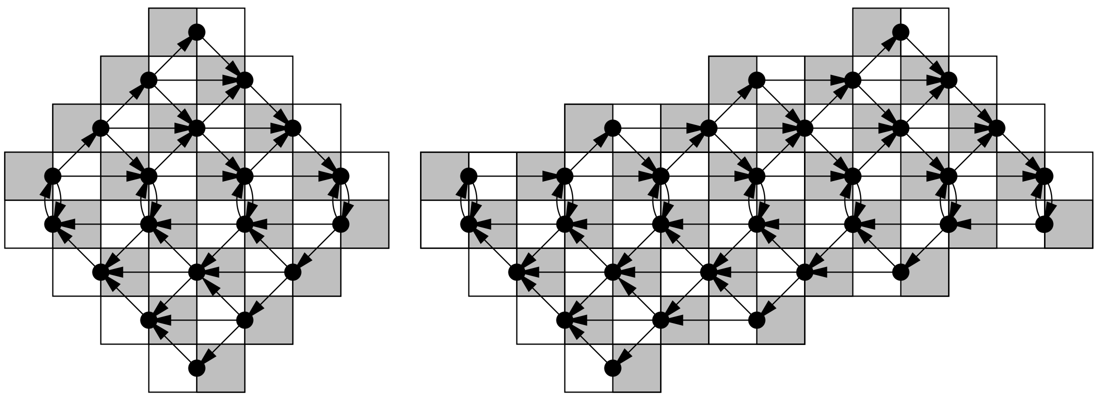

FIGURE 6. The hamburger graph for an Aztec diamond and an Aztec pillow

difference becomes a union of directed cycles. Notice that edges in the upper half of G all go from left to right and the edges in the bottom half of G go from right to left.

Since the edges of N always appear in the cycles, we can contract the edges of N to points and the graph will retain its structure in terms of the cycle systems it produces. This new graph H is of the form in Figure 6a. This argument shows that the number of domino tilings of an Aztec diamond equals the number of cycle systems of this new condensed graph, called the region's *digraph*. In the case of generalized Aztec pillows, the region's digraph is always a hamburger graph.

We wish to concretize this notion of a *digraph* of the Aztec diamond AD_n. Given the natural tiling of an Aztec diamond consisting solely of horizontal dominoes, we place a vertex in the center of every domino. The edges of this digraph are made up of three families of edges. From every vertex in the top half of the diamond, create edges to the east, to the northeast, and to the southeast whenever there is a vertex there. From every vertex in the bottom half of the diamond, form edges to the west, to the southwest, and to the northwest whenever there is a vertex there. Additionally, label the bottom vertices in the top half v_1 through v_n from west to east and the top vertices in the bottom half w_{n+1} through w_{2n}. For all i between 1 and n, create a directed edge from v_i to w_{n+i} and from w_{n+i} to v_i. The result when this construction is applied to AD_4 is a graph of the form in Figure 6a.

THEOREM 3.1. *The digraph of an Aztec diamond is a hamburger graph.*

Since both the upper half of the digraph and the lower half of the digraph are both planar, there are no negative cycle systems. This implies that the determinant of the corresponding hamburger matrix counts exactly the number of cycle systems in the digraph.

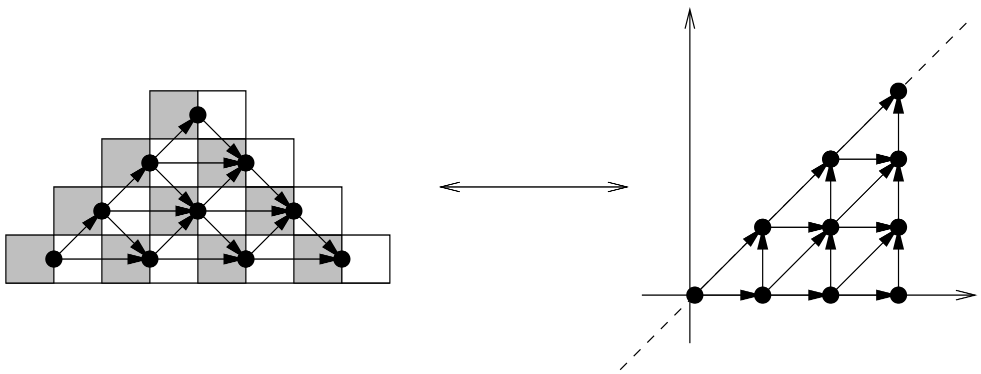

FIGURE 7. The equivalence between paths in D and lattice paths in the first quadrant

To apply Theorem 1.1 to count the number of tilings of AD_n, we need to find the number of paths in the upper half of D from v_i to v_j and the number of paths in the lower half of D from w_{n+j} to w_{n+i}. The key observation is that by the equivalence in Figure 7, we are in effect counting the number of paths from (i, i) to (j, j) using steps of size $(0, 1)$, $(1, 0)$, or $(1, 1)$ that do not pass above the line $y = x$. This is exactly a combinatorial interpretation for the $(j - i)$-th large Schröder number. The large Schröder numbers $s_0, \ldots, s_5$ are $1, 2, 6, 22, 90, 394$, and are referenced as A006318 in the Encyclopedia of Integer Sequences [**13**].

COROLLARY 3.2. *The number of domino tilings of the Aztec diamond AD_n is equal to*

$$\#AD_n = \det \begin{bmatrix} S_n & I_n \\ -I_n & S_n^T \end{bmatrix},$$

where S_n is an upper triangular matrix with the i-th Schröder number on its ith superdiagonal.

For example, when $n = 6$, the matrix S_6 is

$$S_6 = \begin{bmatrix} 1 & 2 & 6 & 22 & 90 & 394 \\ 0 & 1 & 2 & 6 & 22 & 90 \\ 0 & 0 & 1 & 2 & 6 & 22 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Brualdi and Kirkland prove a similar determinant formula for the number of tilings of an Aztec diamond in a matrix-theoretical fashion based on the Kasteleyn matrix of the graph H and a Schur complement calculation. The Hamburger Theorem gives a purely combinatorial way to reduce the calculation of the $n(n+1) \times n(n+1)$ Kasteleyn determinant to the calculation of a $2n \times 2n$ Hamburger determinant. Following the cues from Brualdi and Kirkland, we can reduce this to an $n \times n$ determinant via a Schur complement calculation.

In the case of the block matrix M_H in Equation 2.1, taking the Schur complement of B in M_H gives that

$$\det M_H = \det B \cdot \det(A + D_1 B^{-1} D_2) = \det(A + D_1 B^{-1} D_2), \tag{3.1}$$

since B is an lower triangular matrix with 1's on the diagonal. In this way, every hamburger determinant can be reduced to a smaller determinant of a Schur complement matrix. In the case of a simple hamburger graph where $D_2 = D_1 = I$, the determinant calculation reduces further to $\det(A + B^{-1})$. Lastly, in the case where the hamburger graph is rotationally symmetric, $B = JAJ$, where J is the exchange matrix. This implies we can write the determinant only in terms of the submatrix A, i.e., $\det(A + JA^{-1}J)$. (Note that $J^{-1} = J$.)

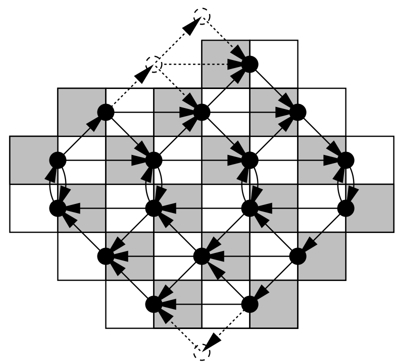

FIGURE 8. The digraph of a generalized Aztec pillow from the digraph of an Aztec diamond

COROLLARY 3.3. *The number of domino tilings of the Aztec diamond AD_n is equal to* $\det(S_n + J_n S_n^{-1} J_n)$*, where J_n is the $n \times n$ exchange matrix.*

In the case of a hamburger graph H, we call this Schur complement $A+B^{-1}$ a *reduced hamburger matrix*. In terms of the Aztec diamond graph example above, we can thus calculate the number of tilings of the Aztec diamond AD_6 as follows. The inverse of S_6 is

$$S_6^{-1} = \begin{bmatrix} 1 & -2 & -2 & -6 & -22 & -90 \\ 0 & 1 & -2 & -2 & -6 & -22 \\ 0 & 0 & 1 & -2 & -2 & -6 \\ 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which implies that the determinant of the reduced hamburger matrix

$$M_6 = \begin{bmatrix} 2 & 2 & 6 & 22 & 90 & 394 \\ -2 & 2 & 2 & 6 & 22 & 90 \\ -2 & -2 & 2 & 2 & 6 & 22 \\ -6 & -2 & -2 & 2 & 2 & 6 \\ -22 & -6 & -2 & -2 & 2 & 2 \\ -90 & -22 & -6 & -2 & -2 & 2 \end{bmatrix}$$

gives the number of tilings of AD_6.

Brualdi and Kirkland were the first to find such a determinantal formula for the number of tilings of an Aztec Diamond [**2**]. They were able to calculate the sequence of determinants $\{M_n\}$ using a J-fraction expansion, which only works when matrices are Toeplitz or Hankel.

Since Aztec pillows and generalized Aztec pillows can be created from Aztec diamonds by placement of dominoes, we define the digraph of a generalized Aztec pillow to be the restriction of the digraph of an Aztec diamond to the vertices that are on the interior of the pillow. For a visualization, see the example of Figure 8. Since the generalized Aztec pillow's digraph is a restriction of the Aztec diamond's digraph, we have the following corollary.

COROLLARY 3.4. *The digraph of an Aztec pillow or a generalized Aztec pillow is a hamburger graph.*

Aztec pillows were introduced in part because of an intriguing conjecture for their number of tilings given by Propp [**12**].

CONJECTURE 3.5 (Propp's Conjecture). *The number of tilings of an Aztec pillow AP_n is a larger number squared times a smaller number. We write $\#AP_n = \ell_n^2 s_n$. In addition, depending* *on the parity of* n*, the smaller number* s_n *satisfies a simple generating function. For* AP_{2m+1}*, the generating function is*

$$\sum_{m=0}^{\infty} s_{2m+1}x^m = (5+6x+3x^2-2x^3)/(1-2x-2x^2-2x^3+x^4).$$

while for AP_{2m+2}*, the generating function is*

$$\sum_{m=0}^{\infty} s_{2m+2}x^m = (5+3x+x^2-x^3)/(1-2x-2x^2-2x^3+x^4),$$

The underlying hope going into the Hamburger Theorem was that it would allow us to prove Propp's Conjecture. We shall see that although we achieve a faster determinantal method to calculate $\#AP_n$, the evaluation of the sequence of said determinants is not able to be calculated explicitly by known methods.

Using the same method as for Aztec diamonds, creating the hamburger graph H for an Aztec pillow gives Figure 6b. Counting the number of paths from v_i to v_j and from w_{k+j} to w_{k+i} in successively larger Aztec pillows gives us the infinite upper-triangular array $S = (s_{i,j})$ of *modified Schröder numbers* defined by the following combinatorial interpretation. Let $s_{i,j}$ be the number of paths from (i,i) to (j,j) using steps of size $(0,1)$, $(1,0)$, and $(1,1)$, not passing above the line $y = x$ nor below the line $y = x/2$. This equivalence is shown in Figure 9. The principal 7×7 minor matrix S_7 of S is

$$S_7 = \begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 57 & 224 \\ 0 & 1 & 2 & 5 & 16 & 57 & 224 \\ 0 & 0 & 1 & 2 & 6 & 21 & 82 \\ 0 & 0 & 0 & 1 & 2 & 6 & 22 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

THEOREM 3.6. *The number of domino tilings of an Aztec pillow of order* n *is equal to*

$$\#AP_n = \det \begin{bmatrix} S_n & I_n \\ -I_n & J_nS_nJ_n \end{bmatrix},$$

where S_n *is the* $n \times n$ *principal submatrix of* S *and* J_n *is the* $n \times n$ *exchange matrix.*

As in the case of Aztec diamonds, we can calculate the reduced hamburger matrix through a Schur calculation. The inverse of S_6 is

$$S_6^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -2 & -5 \\ 0 & 0 & 1 & -2 & -2 & -5 \\ 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the resulting reduced hamburger matrix for AP_6 is

$$M_6 = \begin{bmatrix} 2 & 1 & 2 & 5 & 16 & 57 \\ -2 & 2 & 2 & 5 & 16 & 57 \\ -2 & -2 & 2 & 2 & 6 & 21 \\ -5 & -2 & -2 & 2 & 2 & 6 \\ -5 & -2 & -1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

This gives us a much faster way to calculate the number of domino tilings of an Aztec pillow than was known previously. We have reduced the calculation of the $O(n^2) \times O(n^2)$ Kasteleyn-Percus determinant to an $n \times n$ reduced hamburger matrix. To be fair, the Kasteleyn-Percus matrix has

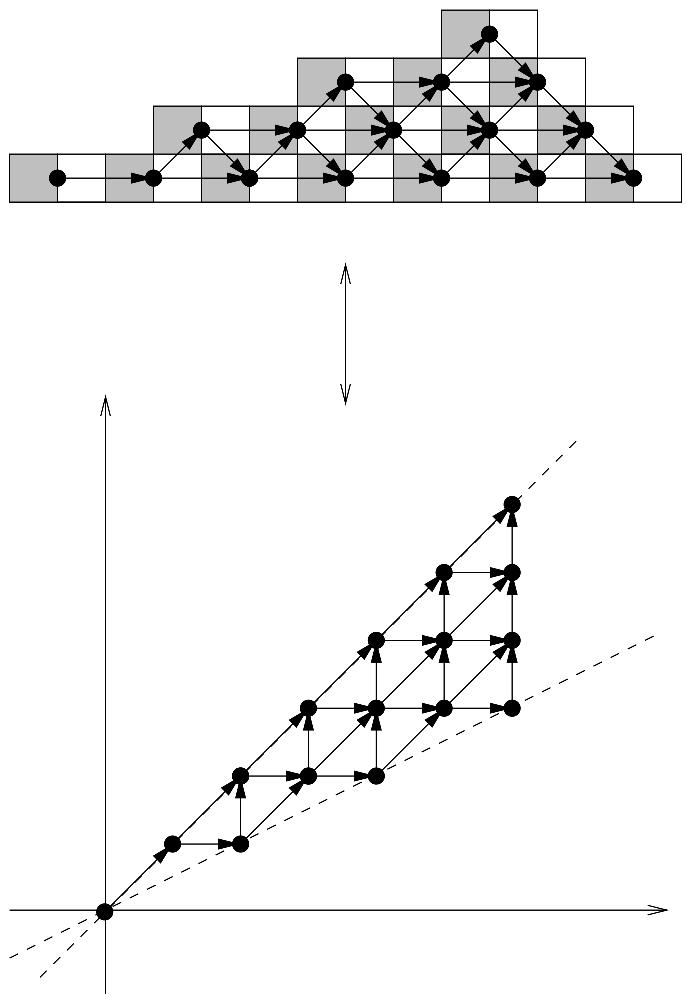

FIGURE 9. The equivalence between paths in D and lattice paths in the first quadrant

-1, 0, and $+1$ entries while the reduced hamburger matrix may have very large entries, which makes running time comparisons difficult theoretically. Experimentally, when calculating the number of domino tilings of AP_{14} using Maple 8.0 on a 447 MHz Pentium III processor, the determinant of the 112×112 Kasteleyn-Percus matrix takes 25.3 seconds while the determinant of the 14×14 reduced hamburger matrix takes less than 0.1 seconds.

Whereas we now have a very understandable determinantal formula for the number of tilings of the region, this does not translate into a proof of Propp's Conjecture because we can not calculate the determinant of the sequence of matrices $\{M_n\}$ explicitly. We can not apply a J-fraction expansion as Brualdi and Kirkland did since the reduced hamburger matrix is not Toeplitz or Hankel.

The Hamburger Theorem allows us to calculate the number of cycle systems in old graphs more quickly and in many new graphs that were inaccessible before, such as non-planar hamburger graphs. Just as Gessel and Viennot's result has found many applications, we hope the Hamburger Theorem to be as useful as well.

References

[1] Martin Aigner. Lattice paths and determinants. In *Computational discrete mathematics*, volume 2122 of *Lecture Notes in Comput. Sci.*, pages 1–12. Springer, Berlin, 2001.

[2] Richard Brualdi and Stephen Kirkland. Aztec diamonds and digraphs, and Hankel determinants of Schröder numbers, available at http://www.math.wisc.edu/~brualdi/aztec2.pdf 2003.

[3] N. Destainville. Entropy and boundary conditions in random rhombus tilings. *J. Phys. A*, 31(29):6123–6139, 1998. arXiv:cond-mat/9804062.

[4] Ira Gessel and Xavier G. Viennot. Binomial determinants, paths, and hook length formulae. *Adv. in Math.*, 58:300–321, 1985.

[5] Ira Gessel and Xavier G. Viennot. Determinants, paths, and plane partitions. 1989. preprint, available at http://www.cs.brandeis.edu/~ira/papers/pp.pdf.

[6] W. Jockusch. Perfect matchings and perfect squares. *J. Combin. Theory Ser. A*, 67(1):100–115, 1994.

[7] Samuel Karlin and James McGregor. Coincidence probabilities. *Pacific J. Math.*, 9:1141–1164, 1959.

[8] P. W. Kasteleyn. The statistics of dimers on a lattice i. the number of dimer arrangements on a quadratic lattice. *Physica*, 27:1209–1225, 1961.

[9] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, 5:85–90, 1973.

[10] M. E. Mays and Jerzy Wojciechowski. A determinant property of Catalan numbers. *Discrete Math.*, 211(1-3):125–133, 2000.

[11] J. Percus. One more technique for the dimer problem. *J. Math. Phys.*, 10:1881–1884, 1969.

[12] James Propp. Enumeration of matchings: problems and progress. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 255–291. Cambridge Univ. Press, Cambridge, 1999. arXiv:math.CO/9904150.

[13] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/.

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA
E-mail address: chanusa@math.washington.edu