MAGIC SQUARES, ROOK POLYNOMIALS AND PERMUTATIONS

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Abstract. We study in this paper the vector space of magic squares and their relation with some restricted permutations.
Résumé. Nous étudions dans cet article l'espace vectoriel des carrés magiques et leur relation avec des permutations spéciales.

1. Introduction

The *oldest magic square* $\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$ first appeared in ancient Chinese literature under the name **Lo Shu** two thousands years BC. The reader is likely to have encountered such objects, which following Ehrhart [2] are referred as *historical magic squares*. These are square matrices of order n whose entries are nonnegative integers $\{1, \cdots, n^2\}$ and whose rows and columns and the main diagonals sum up to the same number, which is called *the magic sum*. MacMahon [7] and Stanley [10] defined magic squares in modern combinatorics as square matrices of order n whose entries are nonnegative integers and whose rows and columns sum up to the same number, which is called *the line sum*. In this paper we will study the magic squares following the next definition.

Definition 1.1. A magic square is a square matrix of order n, whose entries are nonnegative integers and the sum of each row, each column and both the main diagonals adds up to the same number, which is called *the magic total*.

Example 1.2. $\begin{pmatrix} 3 & 6 & 0 \\ 0 & 3 & 6 \\ 6 & 0 & 3 \end{pmatrix}$ is a magic square of order 3 and whose magic total is equal to 9.

MacMahon[7] has already enumerated the number of all magic squares of order 3 in 1915, and it was in 2002 that Ahmed et al.[1] could find the number of magic squares of order 4 for a given magic total. The number of magic squares of order $n \geq 5$ with magic total $s \geq 2$ is a challenge ! We will introduce notions on magic permutations which are generators of all magic squares. In 1879, Hertzsprung [5] defined the number of magic permutations as well as the number of permutations without fixed points and without reflected points, well before the development of rook theory ([3],[4],[6],[8],[11]) as a method for enumeration of permutations with restricted positions and it was Riordan [8](1958) and Simpson [9](1995) who recalled these recurrence relations. We will use weighted rook polynomials to generalize the results on generalized restricted permutations and we will give an unexpected relation which relates derangements and restricted permutations. We will denote by MS_n the set (or vector space) of magic squares of order n.

2. Magic permutations

We will recall the following definitions :

Definition 2.1. A permutation σ of order n is a bijection over n objects.

We will denote by $[n]$ the set $\{1, \cdots, n\}$, and by $\mathfrak{S}_n$ the set of all permutations over $[n]$.

Definition 2.2. We say that an integer i is a fixed point for the permutation σ if $\sigma(i) = i$.

Definition 2.3. We say that an integer i is a reflected point for the permutation σ if $\sigma(i) = n - i + 1$.

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We will denote by $Fix(\sigma)$ the set of the fixed points of the permutation σ, and by $Rfl(\sigma)$ the set of its reflected points.

Definition 2.4. We say that an integer i is a pivot point if i is a fixed reflected point.

Remark 2.5. If n is even, all permutation σ of length n does not have a pivot point.

Remark 2.6. The only pivot point of a permutation of length n is the integer $\dfrac{n+1}{2}$ if n is odd.

Example 2.7. For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 5 & 4 & 2 & 7 & 3 \end{pmatrix}$, we have

$$Fix(\sigma) = \{1,\, 4\} \text{ and } Rfl(\sigma) = \{2,\, 3,\, 4\}\,.$$

We will write a permutation σ of length n as a square matrix of order n such that the i-th column is presented by the vector column $e_{\sigma(i)}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \text{ and } e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Example 2.8. If we consider the permutation in Example 2.7, we have :

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark 2.9. The reflected points and the fixed points of a permutation σ are shown in the matrix representation of the permutation σ as the occurrence of the integer 1 on the main diagonals.

Definition 2.10. A magic permutation is a permutation σ whose matrix representation is a magic square of magic total 1.

Proposition 2.11. *A permutation σ is magic if σ has one fixed point and one reflected point.*

Example 2.12. The following permutations σ_1 and σ_2 of length 9 are magic :

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 6 & 8 & 1 & 4 & 9 & 7 \end{pmatrix}$$

and

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 7 & 5 & 1 & 8 & 3 & 6 \end{pmatrix}$$

Proposition 2.13. *There does not exist a magic permutation of length n for $n = 2,\, 3$.*

Proposition 2.14. (1) *All magic squares of order 2 have the form $\left(\begin{smallmatrix} n & n \\ n & n \end{smallmatrix}\right)$, for $n \in \mathbb{N}$.*

(2) *For all magic squares $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & k \end{smallmatrix}\right)$ of magic total s, we have :*

$$\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & k \end{smallmatrix}\right) = (g-f)\left(\begin{smallmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right) + \frac{h}{2}\left(\begin{smallmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{smallmatrix}\right) + \frac{f}{2}\left(\begin{smallmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{smallmatrix}\right),$$

with $2g - f + h = 2/3s$.

Proof. (1) It is easy to verify that if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is magic square, then we have the system of equations $a + b = c + d = a + c = b + d = a + d = b + c$ which involves $a = b = c = d$.

(2) It is left to the reader to prove that these three matrices are independant i.e. if

$$\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix} = \alpha_1\begin{pmatrix} 0&2&1\\ 2&1&0\\ 1&0&2 \end{pmatrix} + \alpha_2\begin{pmatrix} 1&0&2\\ 2&1&0\\ 0&2&1 \end{pmatrix} + \alpha_3\begin{pmatrix} 1&2&0\\ 0&1&2\\ 2&0&1 \end{pmatrix},$$

then $\alpha_1 = \alpha_2 = \alpha_3 = 0$. We leave as exercise to prove that for a given magic square $\begin{pmatrix} a&b&c\\ d&e&f\\ g&h&k \end{pmatrix}$, if we have

$$\begin{pmatrix} a&b&c\\ d&e&f\\ g&h&k \end{pmatrix} = \alpha_1\begin{pmatrix} 0&2&1\\ 2&1&0\\ 1&0&2 \end{pmatrix} + \alpha_2\begin{pmatrix} 1&0&2\\ 2&1&0\\ 0&2&1 \end{pmatrix} + \alpha_3\begin{pmatrix} 1&2&0\\ 0&1&2\\ 2&0&1 \end{pmatrix},$$

then $\alpha_1 = g - f$, $\alpha_2 = \dfrac{h}{2}$, $\alpha_3 = \dfrac{f}{2}$.

□

Corollary 2.15. *$DimMS_2 = 1$ and $DimMS_3 = 3$.*

MacMahon [7] has established the following theorem :

Theorem 2.16. *The number $M_3(s)$ of magic squares of order 3 of magic total s is defined by :*

$$M_3(s) = \begin{cases} \frac{2}{9}s^2 + \frac{2}{3}s + 1 & \text{if } 3 \text{ divides } s,\\ 0 & \text{otherwise.} \end{cases}$$

and their generating function has the closed form :

$$\sum_s M_3(s)t^s = \frac{(1+t^3)^2}{(1-t^3)^2}.$$

Proposition 2.17. *All magic squares of order 4 of magic total s can be written as linear combination of the following seven magic permutations as below :*

$$\begin{aligned} \begin{pmatrix} a&b&c&d\\ e&f&g&h\\ j&k&l&m\\ n&p&q&r \end{pmatrix} = {} & (a-m)\begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix} + m\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix} + (d-f)\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix} + \\ & f\begin{pmatrix} 0&0&0&1\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix} + (b-n)\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix} + c\begin{pmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix} + n\begin{pmatrix} 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 1&0&0&0 \end{pmatrix}, \end{aligned}$$

with $a+b+c+d = s$, $0 \leq a+n,\ b+f,\ d+m,\ d+n \leq s$.

Proof. Similarily to the previous proof, it is left to the reader to prove that those seven magic permutations are independant and for a given magic square $\begin{pmatrix} a&b&c&d\\ e&f&g&h\\ j&k&l&m\\ n&p&q&r \end{pmatrix}$, if we have

$$\begin{aligned} \begin{pmatrix} a&b&c&d\\ e&f&g&h\\ j&k&l&m\\ n&p&q&r \end{pmatrix} = {} & x_1\begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix} + x_2\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix} + x_3\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix} + \\ & x_4\begin{pmatrix} 0&0&0&1\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix} + x_5\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix} + x_6\begin{pmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix} + x_7\begin{pmatrix} 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 1&0&0&0 \end{pmatrix}. \end{aligned},$$

then $x_1 = a-m$, $x_2 = m$, $x_3 = d-f$, $x_4 = f$, $x_5 = b-n$, $x_6 = c$, $x_7 = n$. □

Corollary 2.18. *$DimMS_4 = 7$.*

Ahmed et al. [1] have established the following theorem :

Theorem 2.19. *The generating function of the numbers $M_4(s)$ of magic squares of order 4 of magic total s is defined by :*

$$\sum_s M_4(s)t^s = \frac{t^8 + 4t^7 + 18t^6 + 36t^5 + 50t^4 + 36t^3 + 18t^2 + 4t + 1}{(1-t)^4(1-t^2)^4}.$$

Proposition 2.20. *For all integers $n \geq 4$, we have $DimMS_n = (n-1)^2 - 2$.*

Proof. Since the dimension of the Birkhoff polytope is equal to $(n-1)^2$, this is also the dimension of the vector space of magic squares with Stanley's definition. While adding two equalities for the diagonals, we have $(n-1)^2 - 2 \leq DimMS_n \leq (n-1)^2$. Now, let us consider a magic square $A = (a_{ij})_{1 \leq i,j \leq n}$ of order $n \geq 4$ and of magic total s. We will give the minimal entries that are necessary to create the matrix A. If the first $n-2$ entries of the first $n-1$ columns are given, we deduce the first $n-2$ entries of the last column of the matrix A by the equations

$$\sum_{j=1}^{n} a_{ij} = s, \qquad \text{for all} \quad i = 1..n-2.$$

If we give the entry $a_{n-1\,1}$, we deduce $a_{n\,1}$, $a_{n-1\,2}$ and $a_{n\,2}$. When we give the entries $a_{n-1\,j}$ for $3 \leq j \leq n-3$, we deduce the entries $a_{n\,j}$ for $3 \leq j \leq n-3$, and when we give the entry $a_{n-1\,n-1}$, we deduce the remaining entries $a_{n\,n-1}$, $a_{n\,n}$, $a_{n-1\,n}$, , and $a_{n-1\,n-2}, a_{n\,n-2}$ of the matrix A. It is easy to verify that these given $(n-1)^2 - 2$ entries suffice to create the magic square A and the integer $(n-1)^2 - 2$ is also the dimension of the vector space MS_n. This gives the result. □

Proposition 2.21. *If a permutation σ is magic, then :*

1. *σ^{-1} is magic,*
2. *the reflected permutation σ' of the permutation σ, defined by $\sigma'(i) = n - \sigma(i) + 1$, is magic.*

Proof. Notice that a fixed point of the permutation σ remains a fixed point for σ^{-1} and becomes a reflected point for the reflected permutation σ' and vice-versa. Notice also that if the integer i is a reflected point for the permutation σ, then the integer i is a fixed point for the reflected permutation σ' and the integer $n-i+1$ is a reflected point for σ^{-1} and vice-versa. □

If we denote by a_n and x_n the number of magic permutations and the number of permutations without fixed points and without reflected points of length n respectively, we can find in the following table the first values of these numbers :

n	0	1	2	3	4	5	6	7	8
a_n	0	1	0	0	8	20	96	656	5568
x_n	1	0	0	0	4	16	80	672	4752

Hertzsprung [5] established the following theorem :

Theorem 2.22. *The numbers a_n and x_n satisfy the following reccurrences :*

$$\begin{array}{lcllll}
a_{2n} & = & n(x_{2n} & - & (2n-3)x_{2n-1}) & \\
a_{2n+1} & = & (2n+1)x_{2n} & + & 3nx_{2n-1} & - \quad 2n(n-1)x_{2n-2}, \\
x_n & = & (n-1)x_{n-1} & + & 2(n-d)x_{n-e}, & \\
 & & \multicolumn{4}{l}{\quad \textit{where } (d,e) = (2,4) \textit{ if } n \textit{ is even}, \quad (1,2) \textit{ if } n \textit{ is odd.}}
\end{array}$$

We will generalize the rook polynomial notions to enumerate some restricted permutations.

3. Rook polynomial

We will study in this section the number of permutations $\sigma \in \mathfrak{S}_n$ where for each i, certain values of $\sigma(i)$ are disallowed (namely, $\sigma(i) \neq i$ and $\sigma(i) \neq n-i+1$). We have a board $\mathfrak{B} \subset [n] \times [n]$. Each square s on $\mathfrak{B}$ has a weight ω_s. We define the rook numbers (actually polynomials) of $\mathfrak{B}$ by

$$r_k = \sum_{|A|=k} \prod_{s \in A} \omega_s$$

where the sum is over all subsets $A \subset \mathfrak{B}$ of cardinality k with no two squares on the same row or column. We define the generalized hit numbers h_i :

$$h_k = \sum_{\pi} \omega(\pi)$$

where the sum is over all permutations π of $[n]$ with k hits (values of i such that $(i,\pi(i)) \in \mathfrak{B}$) and the weight $\omega(\pi)$ of π is the product $\prod_{i=1}^{n} \omega_{(i,\pi(i))}$ where $\omega_{(i,\pi(i))}$ is the weight of $(i,\pi(i))$ if $(i,\pi(i)) \in \mathfrak{B}$ and is 1 otherwise. The generalized hit polynomial is

$$H = \sum_k h_k.$$

We can find a relation between H and the rook numbers r_i just as in the usual case. Claim that

$$\sum_k r_k (n-k)! = H^+ \qquad (\star)$$

where H^+ is the result of replacing each weight ω_s for $s \in \mathfrak{B}$ with $\omega_s + 1$. To see this, note that $r_k(n-k)!$ counts pairs (A,π) where A is a rook placement in $\mathfrak{B}$ of size k, and π extends A to a permutation of $[n]$. If we fix π and sum over all possible A, we are summing over all of the subsets of the hits of A and this gives H^+. If we replace each weight ω_s in $(\star)$ by $\omega_s - 1$ we get

$$H = \sum_k r_k^- (n-k)!$$

where r_k^- is the result of replacing each ω_s with $\omega_s - 1$.

Example 3.1. Let $n = 2m$ and let $\mathfrak{B}$ the following board with weights as indicated. (This is the case $n = 6$)

α					β
	α			β	
		α	β		
		β	α		
	β			α	
β					α

By permuting the rows and columns we get

α	β				
β	α				
		α	β		
		β	α		
				α	β
				β	α

We see that

$$\sum_k r_k X^k = \left[1 + (2\alpha + 2\beta)X + (\alpha^2 + \beta^2)X^2\right]^m,$$

so

$$\sum_k r_k^- X^k = \left[1 + (2\alpha + 2\beta - 4)X + ((\alpha-1)^2 + (\beta-1)^2)X^2\right]^m,$$

and therefore $H = \sum_k r_k^- (n-k)!$ with r_k^- as above. For example, to count permutations with no reflected points and no fixed points, we set $\alpha = \beta = 0$ to get

$$\sum_k r_k^- X^k = (1 - 4X + 2X^2)^m$$

and

$$x_{2m} = \sum_k r_k^- (2m-k)!.$$

To count permutations with no reflected points and one fixed point we set $\beta = 0$ and look at the coefficient of α. So we want the coefficient of α in

$$\left[1 + (2\alpha - 4)X + ((\alpha-1)^2 + 1)X^2\right]^m$$

which is easily computed to be

$$2mX(1-X)(1-4X+2X^2)^{m-1},$$

and to count permutations with two fixed points without reflected points, we look at the coefficient of α^2 which is easily computed to be

$$2m(m-1)X^2(1-X)^2(1-4X+2X^2)^{m-2}+mX^2(1-4X+2X^2)^{m-1}.$$

To count permutations with one reflected point and one fixed point we look at the coefficient of $\alpha\beta$, which is

$$4m(m-1)X^2(1-X)^2(1-4X+2X^2)^{m-2}$$

and

$$a_{2m}=\sum_k r_k^-(2m-k)!.$$

and so on. For n odd we take the following board where we have a separate weight for the middle square

α				β
	α		β	
		γ		
	β		α	
β				α

By permuting the rows and columns we get

α	β			
β	α			
		α	β	
		β	α	
				γ

We see that

$$\sum_k r_k X^k=(1+\gamma X)\left[1+(2\alpha+2\beta)X+(\alpha^2+\beta^2)X^2\right]^m,$$

so

$$\sum_k r_k^- X^k=(1+(\gamma-1)X)\left[1+(2\alpha+2\beta-4)X+((\alpha-1)^2+(\beta-1)^2)X^2\right]^m,$$

and therefore $H=\sum_k r_k^-(n-k)!$ with r_k^- as above and $n=2m+1$. For example, to count permutations with no reflected points and no fixed points, we set $\alpha=\beta=\gamma=0$ to get

$$\sum_k r_k^- X^k=(1-X)(1-4X+2X^2)^m$$

and

$$x_{2m+1}=\sum_k r_k^-(2m+1-k)!.$$

To count permutations with no reflected points and one fixed point we set $\beta=\gamma=0$ and look at the coefficient of α. So we want the coefficient of α in

$$(1-X)\left[1+(2\alpha-4)X+((\alpha-1)^2+1)X^2\right]^m$$

which is easily computed to be

$$2mX(1-X)(1-4X+2X^2)^{m-1},$$

and to count permutations with no reflected points and two fixed points, we look at the coefficient of α^2 which is easily computed to be

$$2m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}+mX^2(1-X)(1-4X+2X^2)^{m-1}.$$

To count permutations with a pivot point we set $\alpha=\beta=0$ and look at the coefficient of γ, which is

$$X(1-4X+2X^2)^m.$$

We deduce that the number x_{2m} enumerates also permutations of order $2m+1$ having a pivot point. To count permutations with one fixed point and one reflected point without pivot points, we set $\gamma = 0$ and look at the coefficient of $\alpha\beta$ in

$$(1-X)\left[1+(2\alpha+2\beta-4)X+((\alpha-1)^2+(\beta-1)^2)X^2\right]^m$$

to get

$$4m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}$$

and

$$a_{2m+1} = \sum_k r_k^-(2m+1-k)! + x_{2m}.$$

To count derangements, that is permutations without fixed points, we set $\alpha = \gamma = 0$ and $\beta = 1$ to get

$$(1-X)^n$$

and so on.

Definition 3.2. We say that a subset F of the set $[n]$ is :

(1) semi-reflected if there exists at least one element $i \in F$ such that $n-i+1 \in F$.
(2) self-reflected if $i \in F$ and $n-i+1 \in F$, for all elements i in the subset F.

The proof of the following lemmas is a simple exercise of combinatorics.

Lemma 3.3. *For disjoint subsets F and R of the set $[2n]$ such that $\#F \cup R = 2k$ and $F \cup R$ is self-reflected, the number of pair (F,R) is equal to $2^k\binom{n}{k}$.*

Lemma 3.4. *For disjoint subsets F and R of the set $[2n]$ or $[2n+1]$ such that $\#F \cup R = n$ and $F \cup R$ is not semi-reflected, the number of pair (F,R) is equal to 2^{2n}.*

Theorem 3.5. *The number of permutations of length $2n$ having set of fixed points and reflected points of cardinality $2k$, and which is a self-reflected set, is equal to $\binom{n}{k}2^k x_{2(n-k)}$.*

Proof. We consider the following board with weights as indicated. We illustrate it with the case for $2n = 6$.

α_1					β_1
	α_2			β_2	
		α_3	β_3		
		β_4	α_4		
	β_5			α_5	
β_6					α_6

By permuting the rows and columns we get

α_1	β_1				
β_6	α_6				
		α_2	β_2		
		β_5	α_5		
				α_3	β_3
				β_4	α_4

We see that

$$\sum_k r_k X^k = \prod_{i=1}^{n}[1+(\alpha_i+\alpha_{2n-i+1}+\beta_i+\beta_{2n-i+1})X+(\alpha_i\alpha_{2n-i+1}+\beta_i\beta_{2n-i+1})X^2],$$

so

$$\sum_k r_k^- X^k = \prod_{i=1}^{n}[1+(\alpha_i+\alpha_{2n-i+1}+\beta_i+\beta_{2n-i+1}-4)X+((\alpha_i-1)(\alpha_{2n-i+1}-1)+(\beta_i-1)(\beta_{2n-i+1}-1))X^2],$$

and therefore $H = \sum_k r_k^-(n-k)!$ with r_k^- as above. To count permutations of length $2n$ having set of fixed points and reflected points of cardinality $2k$, and which is a self-reflected set, we look first at the coefficient of $\prod_{s=1}^{p} \mu_{i_s} \prod_{s=p+1}^{k} \nu_{i_s}$ where $\mu_{i_s} = \alpha_{i_s}\alpha_{2n-i_s+1}$ and $\nu_{i_s} = \beta_{i_s}\beta_{2n-i_s+1}$ which is easily computed to be

$$2^k X^{2k}(1-4X+2X^2)^{(n-k)},$$

and we will consider all products of the form $\prod_{s=1}^{p} \mu_{i_s} \prod_{s=p+1}^{k} \nu_{i_s}$ whose number is equal to $\binom{n}{k}$, and this gives the result. □

Theorem 3.6. *The number of permutations of length $2n+1$ having set of fixed points and reflected points of cardinality $2k+1$, and which is a self-reflected set, is equal to $\binom{n}{k} 2^k x_{2(n-k)}$.*

Proof. Notice that if the cardinality of a self-reflected subset of the set $[2n+1]$ is odd, this subset contains the integer $n+1$. We consider the following board with weights as indicated. We illustrate it with the case for $2n+1 = 5$.

α_1				β_1
	α_2		β_2	
		γ		
	β_4		α_4	
β_5				α_5

By permuting the rows and columns we get

α_1	β_1			
β_5	α_5			
		α_2	β_2	
		β_4	α_4	
				γ

We see that

$$\sum_k r_k X^k = (1+\gamma X)\prod_{i=1}^{n}[1+(\alpha_i+\alpha_{2n-i+1}+\beta_i+\beta_{2n-i+1})X+(\alpha_i\alpha_{2n-i+1}+\beta_i\beta_{2n-i+1})X^2],$$

so

$$\begin{aligned}\sum_k r_k^- X^k = \quad & (1+(\gamma-1)X)\textstyle\prod_{i=1}^{n}[1+(\alpha_i+\alpha_{2n-i+1}+\beta_i+\beta_{2n-i+1}-4)X \\ & +((\alpha_i-1)(\alpha_{2n-i+1}-1)+(\beta_i-1)(\beta_{2n-i+1}-1))X^2],\end{aligned}$$

and therefore $H = \sum_k r_k^-(n-k)!$ with r_k^- as above. To count permutations of length $2n+1$ having set of fixed points and reflected points of cardinality $2k$, and which is self-reflected, we set $\gamma = 0$ and we look at the coefficient of $\prod_{s=1}^{p} \mu_{i_s} \prod_{s=p+1}^{k} \nu_{i_s}$ where $\mu_{i_s} = \alpha_{i_s}\alpha_{2n-i_s+1}$ and $\nu_{i_s} = \beta_{i_s}\beta_{2n-i_s+1}$ which is easily computed to be

$$2^k X^{2k}(1-X)(1-4X+2X^2)^{(n-k)},$$

and we will consider all the product of the form $\prod_{s=1}^{p} \mu_{i_s} \prod_{s=p+1}^{k} \nu_{i_s}$ whose number is equal to $\binom{n}{k}$, and this gives the result. □

4. Derangements

We will conclude this paper with an unexpected relation which relates derangements and restricted permutations.

Theorem 4.1. *The number of permutations of length $2n$ having set of fixed points and reflected points of cardinality n, and which is not a semi-reflected set, is equal to $2^{2n} d_n$.*

Proof. We consider again a board as in the proof of Theorem 3.6. To count permutations of length $2n$ having set of fixed points and reflected points of cardinality n, and which is not a semi-reflected set, we look first at the coefficient of $\prod_{s=1}^{p} \alpha_{i_s} \prod_{s=p+1}^{n} \beta_{i_s}$ such that if $\ell \neq m$, then $i_\ell \neq 2n - i_m + 1$ in

$$\prod_{i=1}^{n}[1 + (\alpha_i + \alpha_{2n-i+1} + \beta_i + \beta_{2n-i+1} - 4)X + ((\alpha_i - 1)(\alpha_{2n-i+1} - 1) + (\beta_i - 1)(\beta_{2n-i+1} - 1))X^2]$$

which is easily computed to be

$$X^n(1 - X)^n.$$

If we consider the coefficient of all such products whose number is equal to 2^{2n} by Lemma 3.3, we obtain the result. ☐

Theorem 4.2. *The number of permutations of length $2n + 1$ having set of fixed points and reflected points of cardinality n, and which is not a semi-reflected set, is equal to $2^{2n} d_{n+1}$.*

Proof. We consider again a board as in the proof of Theorem 3.6. To count permutations of length $2n+1$ having set of fixed points and reflected points of cardinality n, and which is not a semi-reflected set, we set $\gamma = 0$ and we look first at the coefficient of $\prod_{s=1}^{p} \alpha_{i_s} \prod_{s=p+1}^{n} \beta_{i_s}$ such that if $\ell \neq m$, then $i_\ell \neq 2n - i_m + 1$ in

$$(1-X)\prod_{i=1}^{n}[1 + (\alpha_i + \alpha_{2n-i+1} + \beta_i + \beta_{2n-i+1} - 4)X + ((\alpha_i - 1)(\alpha_{2n-i+1} - 1) + (\beta_i - 1)(\beta_{2n-i+1} - 1))X^2]$$

which is easily computed to be

$$X^n(1 - X)^{n+1}.$$

If we consider the coefficient of all such products whose number is equal to 2^{2n} by Lemma 3.4, we deduce the result. ☐

Theorem 4.3. *The number of permutations of length $2n + 1$ having set of fixed points and reflected points of cardinality $n + 1$ containing $n + 1$, and which is not a semi-reflected set if the element $n + 1$ is deleted, is equal to $2^{2n} d_n$.*

Proof. We consider again a board as in the proof of Theorem 3.6. To count permutations of length $2n + 1$ having set of fixed points and reflected points of cardinality $n + 1$, and which is not a semi-reflected set if the integer $n + 1$ is deleted, we look first at the coefficient of $\gamma \prod_{s=1}^{p} \alpha_{i_s} \prod_{s=p+1}^{n} \beta_{i_s}$ such that if $\ell \neq m$, then $i_\ell \neq 2n - i_m + 1$ in

$$\prod_{i=1}^{n}[1 + (\alpha_i + \alpha_{2n-i+1} + \beta_i + \beta_{2n-i+1} - 4)X + ((\alpha_i - 1)(\alpha_{2n-i+1} - 1) + (\beta_i - 1)(\beta_{2n-i+1} - 1))X^2]$$

which is easily computed to be

$$X^{n+1}(1 - X)^n.$$

If we add the coefficients of all such products whose number is equal to 2^{2n} by Lemma 3.4, we deduce the result. ☐

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