A Non-Messing-Up Phenomenon for Posets

Bridget Eileen Tenner

Abstract. We classify finite posets with a particular sorting property, generalizing a result for rectangular arrays. Each poset is covered by two sets of disjoint saturated chains such that, for any original labeling, after sorting the labels along both sets of chains, the labels of the chains in the first set remain sorted. This gives a linear extension of the poset. We also characterize posets with more restrictive sorting properties.

Résumé. Nous classifions les ensembles partiellement ordonnés ayant une certain propriété de triage, généralisant ainsi un résultat connu pour les tables rectangulaires. Chaque ensemble partiellement ordonné est couvert par deux ensembles de chaînes saturées disjointes de telle sorte que, pour tout étiquetage, trier le long des chaînes du premier ensemble, puis celles du second, produit un étiquetage où les étiquettes sont toujours bien ordonnées par rapport au premier ensemble de chaînes. Nous obtenons de cette façon une extension linéaire de l'ensemble partiellement ordonné. Nous caractérisons aussi les ensembles partiellement ordonnés possédant des propriétés de triage plus contraignantes.

1. Introduction

The so-called Non-Messing-Up Theorem is a well known sorting result for rectangular arrays. In [**5**], Donald E. Knuth attributes the result to Hermann Boerner, who mentions it in a footnote in Chapter V, §5 of [**1**]. Later, David Gale and Richard M. Karp include the phenomenon in [**2**] and in [**3**], where they prove more general results about order preservation in sorting procedures. The first use of the term "non-messing-up" seems to be due to Gale and Karp, as suggested in [**4**]. One statement of the result is as follows.

Theorem 1. *Let $A = (a_{ij})$ be an m-by-n array of real numbers. Put each row of A into non-decreasing order. That is, for each $1 \leq i \leq m$, place the values $\{a_{i1}, \ldots, a_{in}\}$ in non-decreasing order (henceforth denoted* row-sort*). This yields the array $A' = (a'_{ij})$. Column-sort A'. Each row in the resulting array is in non-decreasing order.*

Applying the theorem to the transpose of the array A, the sorting can also be done first in the columns, then in the rows, and the columns remain sorted.

Example.

$$\begin{matrix} 4 & 9 & 7 & 8 \\ 12 & 5 & 1 & 10 \\ 2 & 6 & 11 & 3 \end{matrix} \xrightarrow{\text{row-sort}} \begin{matrix} 4 & 7 & 8 & 9 \\ 1 & 5 & 10 & 12 \\ 2 & 3 & 6 & 11 \end{matrix} \xrightarrow{\text{column-sort}} \begin{matrix} 1 & 3 & 6 & 9 \\ 2 & 5 & 8 & 11 \\ 4 & 7 & 10 & 12 \end{matrix}$$

Answering a question posed by Richard P. Stanley, the author's thesis advisor, this paper defines a notion of non-messing-up for posets and Theorem 7 generalizes Theorem 1 by characterizing all posets with this property.

Key words and phrases. Non-messing-up, partially ordered set, sorting, linear extension.

Throughout this paper, we will use standard terminology from the theory of partially ordered sets. A good reference for these terms and other information about posets is Chapter 3 of [**6**].

The rectangular array in Theorem 1 can be viewed as the poset $\boldsymbol{m} \times \boldsymbol{n}$ (where $\boldsymbol{j}$ denotes a j-element chain). The *rows* and *columns* are two different sets of disjoint saturated chains, each covering this poset. Sorting a chain orders the chain's labels so that the minimum element in the chain has the minimum label. Thus, sorting the labels in this manner gives a linear extension of $\boldsymbol{m} \times \boldsymbol{n}$.

Definition. An *edge* in a poset P is a covering relation $x \lessdot y$. Two elements in P are *adjacent* if there is an edge between them.

Definition. A *chain cover* of a poset P is a set of disjoint saturated chains covering the elements of P.

Definition. A finite poset P has the *non-messing-up* property if there exists an unordered pair of chain covers $\{\mathcal{C}_1, \mathcal{C}_2\}$ such that

(1) For any labeling of the elements of P, $\mathcal{C}_i$-sorting and then $\mathcal{C}_{3-i}$-sorting leaves the labels sorted along the chains of $\mathcal{C}_i$, for $i = 1$ and 2; and
(2) Every edge in P is contained in an element of $\mathcal{C}_1$ or $\mathcal{C}_2$.

The set $\mathcal{N}_2$ consists of all posets with the non-messing-up property, where the subscript indicates that an unordered *pair* of chain covers is required. For a non-messing-up poset P with chain covers as defined, write $P \in \mathcal{N}_2$ *via* $\{\mathcal{C}_1, \mathcal{C}_2\}$.

Let us clarify the difference between this result and Gale and Karp's work in [**2**] and [**3**]. Gale and Karp consider a poset P and a partition F of the elements of P. The elements in each block of F are linearly ordered, not necessarily in relation to comparability in P. Given P and F, the authors determine whether each natural labeling of P, sorted within each block of F, yields a labeling that is still natural. In this paper, we do not require that the original labeling be natural. In fact, it is the labelings that are not natural and that do not become natural after the first sort that determine membership in $\mathcal{N}_2$. Additionally, the partition blocks in $\mathcal{N}_2$ are saturated chains, and every covering relation must be in at least one of these chains. The goal of this paper is to determine, for a given poset, when there *exist* chain covers with the non-messing-up property, not if a given pair of chain covers has the property.

It is important to emphasize that $\{\mathcal{C}_1, \mathcal{C}_2\}$ is an unordered pair and that there is a symmetry between the chain covers. We will refer to elements of $\mathcal{C}_1$ and their edges as *red*, and elements of $\mathcal{C}_2$ and their edges as *blue*. If an edge belongs to both chain covers, it is *doubly colored*. The symmetry between the chain covers may be expressed by a statement about red and blue chains and an indication that a *color reversed* version of the statement is also true.

A central object in the classification of $\mathcal{N}_2$ is the following.

Definition. Let $N \geq 3$ be an integer, and consider the poset $P = \boldsymbol{N} \times \boldsymbol{N} = \{(i,j) : 1 \leq i, j \leq N\}$. For integers k_1 and k_2, $3 \leq k_1 \leq k_2 \leq N$, let P' be

$$P \setminus \Big(\{(i,j) : j \geq i + k_1 \text{ or } i \geq j + k_2\} \cup \{(i,j) : j \geq N + k_1 - k_2 + 1\} \Big).$$

Let the poset $\widehat{P}$ be obtained from P' by identifying $(i, k_1+i-1) \sim (k_2+i-1, i)$ for $i = 1, \ldots, N-k_2+1$.

The poset $\widehat{P}$ is $\boldsymbol{N} \times \boldsymbol{N}$ *on the cylinder*. This definition is independent of the values k_1 and k_2.

The classification in Theorem 7 states that $\mathcal{N}_2$ is the set of disjoint unions of connected posets that each can be "reduced" to a convex subposet of $\boldsymbol{N} \times \boldsymbol{N}$ or of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder for some N, subject to a technical constraint. Informally speaking, P reduces to Q if P is formed by replacing particular elements of Q with chains of various lengths. Sample Hasse diagrams for elements of $\mathcal{N}_2$ are shown in Figures 5(a), 6(a), 7(a), and 8(a).

In Section 2 of this paper, we address definitions and preliminary results. The definitions describe the objects and operations needed for the classification, and the results will be the fundamental tools

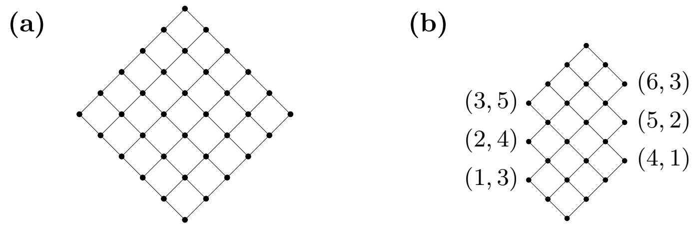

FIGURE 1. (**a**) $P = \mathbf{6} \times \mathbf{6}$. (**b**) P' for $k_1 = 3$ and $k_2 = 4$. To form $\widehat{P}$, identify $(1,3) \sim (4,1)$, $(2,4) \sim (5,2)$ and $(3,5) \sim (6,3)$.

for defining $\mathcal{N}_2$. The main theorem is proved in Section 3 by induction on the size of a connected poset. The final section of the paper discusses further directions for the study of non-messing-up posets, including several open questions.

2. Preliminary results

The definition of a non-messing-up poset requires that every edge be colored. Therefore, as in the case of the product of two chains, $\mathcal{C}_i$-sorting any labeling and then $\mathcal{C}_{3-i}$-sorting yields a linear extension of the poset. The chains of $\mathcal{C}_i$ are disjoint, so each element of a non-messing-up poset is covered by at most two elements, and covers at most two elements.

It is sufficient to consider connected posets, as a poset is in $\mathcal{N}_2$ if and only if each of its connected components is in $\mathcal{N}_2$. Key to determining membership in $\mathcal{N}_2$ is the following fact.

THEOREM 2. *Every convex subposet of an element of $\mathcal{N}_2$ is also in $\mathcal{N}_2$.*

The coloring of a convex subposet Q of $P \in \mathcal{N}_2$ is inherited from the coloring of P in the sense that the chain covers in Q are as in Q when considered as a subposet of P.

LEMMA 3. *If a convex subposet of a non-messing-up poset is a chain, then there is a red chain or a blue chain containing this entire subposet.*

DEFINITION. A *diamond* in a poset is a convex subposet that is the union of distinct (saturated) chains which only intersect at a common minimal element and a common maximal element.

LEMMA 4. *Let Q be a diamond consisting of chains $\boldsymbol{a}$ and $\boldsymbol{b}$ in a non-messing-up poset. Let x be the minimal element in $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted $\min(\boldsymbol{a})$ and $\min(\boldsymbol{b})$, and let $y = \max(\boldsymbol{a}) = \max(\boldsymbol{b})$, with similar notation. Up to color reversal, one of the following is true (where $\boldsymbol{c} \setminus z$ is taken to mean $\boldsymbol{c} \setminus \{z\}$).*

(1) *There exists a red chain containing $\boldsymbol{a} \setminus y$, a blue chain containing $\boldsymbol{b} \setminus y$, a red chain containing $\boldsymbol{b} \setminus x$ and a blue chain containing $\boldsymbol{a} \setminus x$; or*
(2) *There exists a red chain containing $\boldsymbol{a}$ and a blue chain containing $\boldsymbol{b}$.*

Call the former of these a Type I *diamond and the latter a* Type II *diamond.*

DEFINITION. A diamond with *bottom chain* of length k and *top chain* of length l is a convex subposet that is a diamond with minimum x and maximum y, a chain of k elements covered by x, and a chain of l elements covering y, with no other elements or relations among the elements already mentioned.

The technical condition mentioned in the introduction is due to the following requirement.

LEMMA 5. *Let $Q \subseteq P \in \mathcal{N}_2$ be a diamond consisting of chains $\boldsymbol{a}$ and $\boldsymbol{b}$. Suppose there is a coloring of P for which Q has Type I, with bottom and top chains $\boldsymbol{C}$ and $\boldsymbol{D}$. Then there are chains in that coloring such that, up to color reversal, $(\boldsymbol{C} \cup \boldsymbol{a}) \setminus y$ is red, $(\boldsymbol{C} \cup \boldsymbol{b}) \setminus y$ is blue, $(\boldsymbol{a} \cup \boldsymbol{D}) \setminus x$ is blue, and $(\boldsymbol{b} \cup \boldsymbol{D}) \setminus x$ is red. Also, $\max\{|\boldsymbol{C}|, |\boldsymbol{D}|\} < \min\{|\boldsymbol{a}| - 2, |\boldsymbol{b}| - 2\}$.*

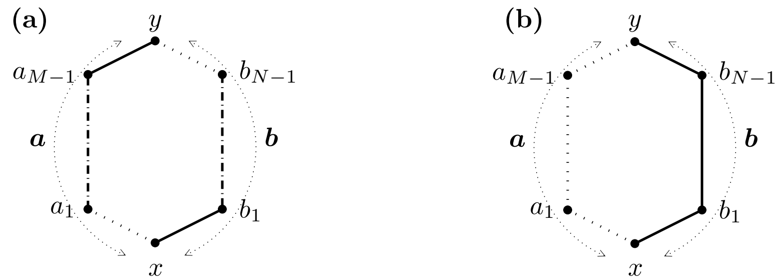

FIGURE 2. (**a**) Type I diamond coloring. (**b**) Type II diamond coloring, where the intervals $\boldsymbol{a} \setminus \{x, y\}$ and $\boldsymbol{b} \setminus \{x, y\}$ may be partially or totally doubly colored.

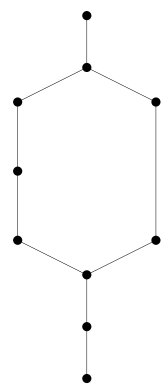

FIGURE 3. A diamond with bottom chain of length 2 and top chain of length 1

If $\max\{|\boldsymbol{C}|, |\boldsymbol{D}|\} = \min\{|\boldsymbol{a}| - 3, |\boldsymbol{b}| - 3\}$, then the described chains have the non-messing-up property, so the bounds in Lemma 5 are sharp.

Recall the definition of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder. As suggested by the main result, this object is crucial in the study of non-messing-up posets.

THEOREM 6. *The poset $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder is in $\mathcal{N}_2$ for all N. The chain covers for this poset are of the same form as the chain covers in Theorem 1.*

Before discussing the main theorem, it remains to rigorously define the notion of reduction.

DEFINITION. The process of *splitting* the element $x \in Q'$ gives a poset Q where

1. $x \in Q'$ is replaced by $\{x_1 \lessdot \cdots \lessdot x_{s(x)}\}$ for some positive integer $s(x)$;
2. All elements and relations in $Q' \setminus x$ are unchanged in Q;
3. If $y \gtrdot x$ in Q', then $y \gtrdot x_{s(x)}$ in Q; and
4. If $y \lessdot x$ in Q', then $y \lessdot x_1$ in Q.

If Q is formed by splitting elements of $\widetilde{Q}$, then Q *reduces* to $\widetilde{Q}$, denoted $Q \rightsquigarrow \widetilde{Q}$.

DEFINITION. Let $P \rightsquigarrow \widetilde{P} \in \mathcal{N}_2$. The coloring of $\widetilde{P}$ *induces* the coloring of P if the edge $\widetilde{u} \lessdot \widetilde{v}$ in $\widetilde{P}$ and its image, the edge $u \lessdot v$ in P, are colored in the same way. Edges in the chain into which an element splits get doubly colored.

3. Characterization of $\mathcal{N}_2$

The classification of the set $\mathcal{N}_2$ is done in two steps. The first direction shows that any poset reducing to a convex subposet of $\boldsymbol{N} \times \boldsymbol{N}$ or of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder, subject to a technical

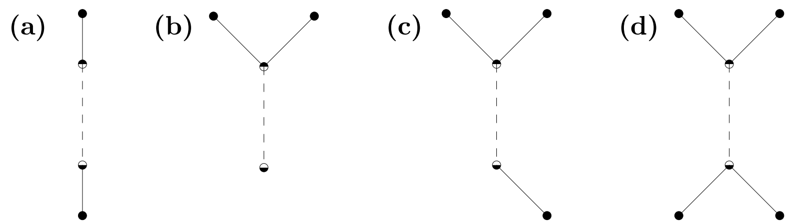

FIGURE 4. How to split a vertex.

constraint imposed by Lemma 5, has the non-messing-up property. The second step shows the reverse inclusion. Both directions are proved by induction on the size of a connected poset.

THEOREM 7. *The collection $\mathcal{N}_2$ is exactly the set of posets each of whose connected components P reduces to $\widetilde{P}$, a convex subposet of $\boldsymbol{N} \times \boldsymbol{N}$ or of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder for some N, given the following stipulation:*

TECHNICAL CONDITION. *For any diamond $\{w \lessdot x, y \lessdot z\}$ in $\widetilde{P}$ that does not realize a generator of the fundamental group of the cylinder,*

$$\max\{s(w), s(z)\} \leq \min\{s(x), s(y)\}.$$

The required coloring of the connected poset $P \in \mathcal{N}_2$ is induced by the coloring of $\widetilde{P}$, which is inherited from the coloring in Theorem 1 or Theorem 6.

Both directions of the proof consider a subposet P' formed by removing either a maximal or a minimal element from P. Thus P' is convex in P, and it is not hard to see that the suppositions for P must hold for P' as well. Each connected component in P' has fewer than $|P|$ elements, so the theorem holds for P' by the inductive assumption.

One case considered in the proof is when a maximal or minimal element of P is adjacent to two other elements but is not in a diamond, and its removal does not disconnect the poset. Observe that this describes a poset P that can only reduce to a poset on the cylinder, while a maximal proper subposet of P reduces to a convex subposet in the plane.

Examples of posets with the non-messing-up property are depicted in Figures 5(a), 6(a), 7(a), and 8(a). The first two of these reduce to convex subposets of $\boldsymbol{N} \times \boldsymbol{N}$, and the last two reduce to convex subposets of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder.

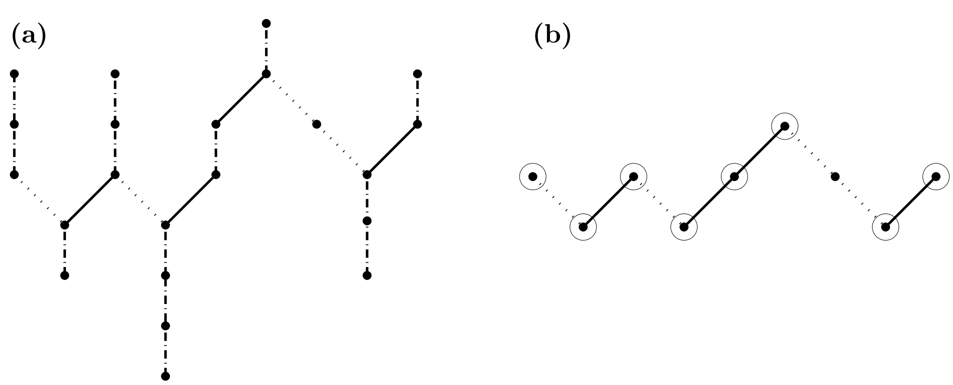

FIGURE 5. (**a**) A poset $P \in \mathcal{N}_2$. (**b**) The reduced poset $\widetilde{P}$, where the elements that split to form P are circled.

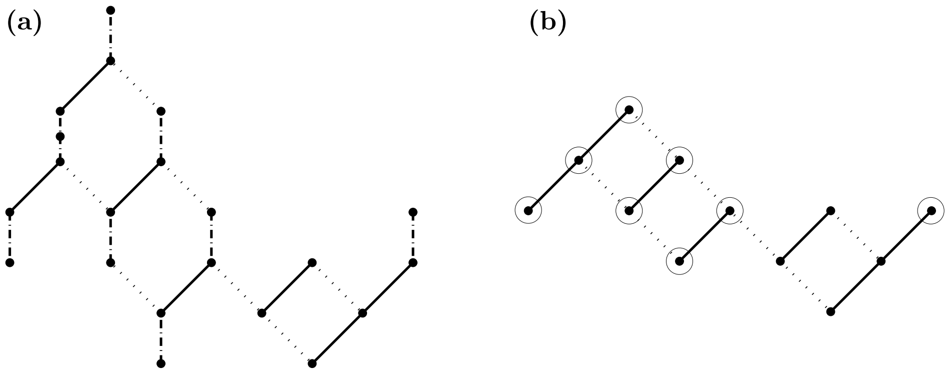

FIGURE 6. (**a**) A poset $P \in \mathcal{N}_2$. (**b**) The reduced poset $\widetilde{P}$, where the elements that split to form P are circled.

Notice that a Type II diamond as described in Lemma 4 occurs only in elements of $\mathcal{N}_2$ that reduce to posets on the cylinder. Moreover, such a diamond must realize a generator of the fundamental group of the cylinder because of the definition of an induced coloring. This explains the technical condition.

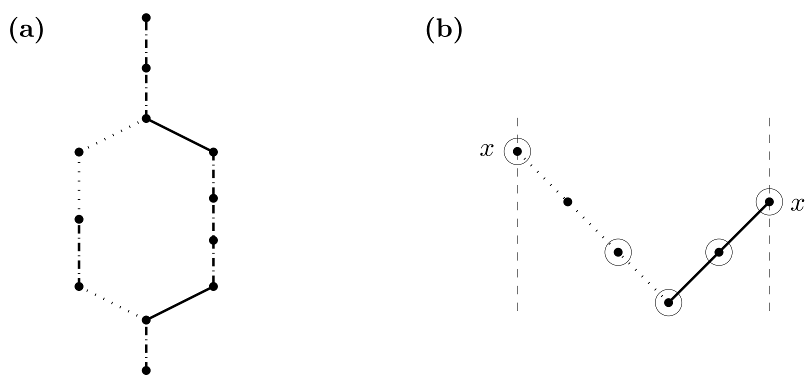

FIGURE 7. (**a**) A poset $P \in \mathcal{N}_2$. (**b**) The reduced poset $\widetilde{P}$ as viewed with identified sides, where the elements that split to form P are circled and the elements that are identified are both labeled x.

The requirement for membership in $\mathcal{N}_2$ is the existence of a pair of chain covers $\{\mathcal{C}_1, \mathcal{C}_2\}$ with particular properties. We might also ask if there are other choices for $\mathcal{C}_i$. A poset of the form depicted in Figure 7(a), that is, a poset consisting of a single diamond and its bottom and top chains, can also be colored so that the diamond has Type I if the bounds of Lemma 5 are satisfied. Otherwise, the only freedom in defining the chain covers arises from the various ways to reduce P due to splits as depicted in Figure 4(a).

4. Further directions

The classification of $\mathcal{N}_2$ prompts further questions relating to the non-messing-up property. In the final section of this paper, we suggest several such questions and provide answers to some.

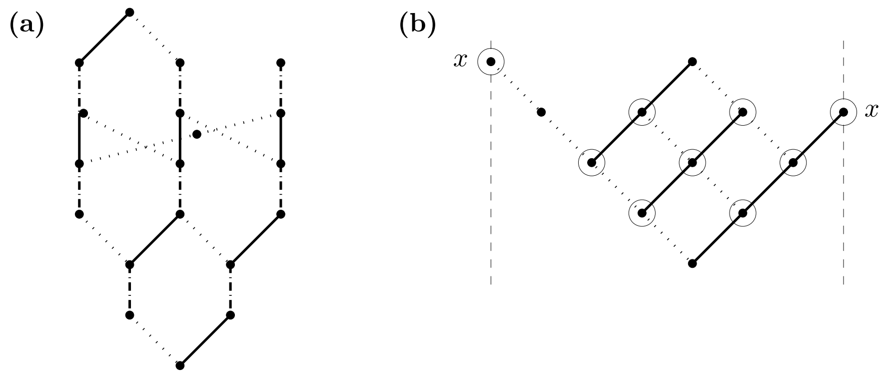

FIGURE 8. (**a**) A poset in $\mathcal{N}_2$. (**b**) The reduced poset $\widetilde{P}$ as viewed with identified sides, where the elements that split to form P are circled and the elements that are identified are both labeled x.

4.1. The set ${\mathcal{N}_2}' \subsetneq \mathcal{N}_2$ with reduced redundancy.

In the classification of $\mathcal{N}_2$, there were instances of a $\mathcal{C}_i$ chain entirely contained in a $\mathcal{C}_{3-i}$ chain. These chain covers have the non-messing-up property, but there is a certain redundancy: this particular $\mathcal{C}_i$ chain adds no information about the relations in the poset since its labels are already ordered after the $\mathcal{C}_{3-i}$-sort.

DEFINITION. The class ${\mathcal{N}_2}'$ consists of all posets $P \in \mathcal{N}_2$ via $\{\mathcal{C}_1, \mathcal{C}_2\}$ such that $\boldsymbol{c}_i \not\subseteq \boldsymbol{c}_{3-i}$ for all $\boldsymbol{c}_1 \in \mathcal{C}_1$ and $\boldsymbol{c}_2 \in \mathcal{C}_2$.

Because the coloring of a non-messing-up poset is induced by its reduced poset, the elements of ${\mathcal{N}_2}'$ can be determined by looking at these reduced posets. Call a chain that shares no covering relation with any diamond a *branch chain* and a maximal such chain a *maximal branch chain.*

THEOREM 8. *The collection ${\mathcal{N}_2}'$ is the set of posets in $\mathcal{N}_2$ where every maximal branch chain in the reduced poset $\widetilde{P}$ consists of exactly two elements, and every element of $\widetilde{P}$ is adjacent to at least two other elements in $\widetilde{P}$.*

4.2. The set ${\mathcal{N}_2}'' \subseteq \mathcal{N}_2$ with reduced redundancy.

In the Non-Messing-Up Theorem as stated in Theorem 1, the rows and columns have minimal redundancy in the sense that for any row $\boldsymbol{r}$ and any column $\boldsymbol{c}$, $\#(\boldsymbol{r} \cap \boldsymbol{c}) = 1$.

DEFINITION. The class ${\mathcal{N}_2}''$ consists of all posets $P \in \mathcal{N}_2$ via $\{\mathcal{C}_1, \mathcal{C}_2\}$ such that $\#(\boldsymbol{c}_1 \cap \boldsymbol{c}_2) \leq 1$ for all $\boldsymbol{c}_i \in \mathcal{C}_i$.

THEOREM 9. *The collection ${\mathcal{N}_2}''$ is the set of posets each of whose connected components is a convex subposet of $\boldsymbol{N} \times \boldsymbol{N}$ or of $\boldsymbol{N} \times \boldsymbol{N}$ on the cylinder.*

4.3. Open questions.

This paper studies finite posets and saturated chains, but interesting results may arise if we relax one or both of these restrictions. Similarly, we could study posets with some variation of the non-messing-up phenomenon. For example, we could consider more than two sets of chains, or expand beyond identities like $S_i S_{3-i} S_i(\mathcal{L}(P)) = S_{3-i} S_i(\mathcal{L}(P))$ for all labelings $\mathcal{L}$ of P and $i \in \{1, 2\}$, where $S_i(\mathcal{L}(P))$ represents $\mathcal{C}_i$-sorting a labeling $\mathcal{L}$ of a poset P.

Additionally, as stated earlier, any labeling of a poset $P \in \mathcal{N}_2$ produces a linear extension of P after performing the two sorts. It would be interesting to understand the distribution of the linear extensions that arise in this way.

References

[1] H. Boerner, Darstellungen von Gruppen, Springer-Verlag, Berlin, 1955, second edition 1967.
[2] D. Gale and R. M. Karp, The Non-Messing-Up Theorem, preprint.
[3] D. Gale and R. M. Karp, A Phenomenon in the Theory of Sorting, J. Comput. System Sci. 6 (1972) 103-115.
[4] D. E. Knuth, private communication, May 2004.
[5] D. E. Knuth, The Art of Computer Programming, vol. 3, Sorting and Searching, Addison-Wesley, Reading, MA, 1973, second edition 1998.
[6] R. P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Studies in Advanced Mathematics, no. 49, Cambridge University Press, Cambridge, 1997.

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: bridget@math.mit.edu