ON THE ENUMERATION OF PARKING FUNCTIONS BY LEADING NUMBERS

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ABSTRACT. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of positive integers. An $\mathbf{x}$-parking function is a sequence $(a_1, \ldots, a_n)$ of positive integers whose non-decreasing rearrangement $b_1 \le \cdots \le b_n$ satisfies $b_i \le x_1 + \cdots + x_i$. In this paper we give a combinatorial approach to the enumeration of $(a, b, \ldots, b)$-parking functions by their leading terms, which covers the special cases $\mathbf{x} = (1, \ldots, 1)$, $(a, 1, \ldots, 1)$, and $(b, \ldots, b)$. The approach relies on bijections between the $\mathbf{x}$-parking functions and labeled rooted forests. To serve this purpose, we present a simple method for establishing the required bijections. Some bijective results between certain sets of $\mathbf{x}$-parking functions of distinct leading terms are also given.

RÉSUMÉ. Soit $\mathbf{x} = (x_1, \ldots, x_n)$ un vecteur d'entiers strictement positifs. Une fonction de $\mathbf{x}$-parking est un vecteur $(a_1, \ldots, a_n)$ d'entiers strictement positifs tel que son réordonnement croissant, noté $b_1 \le \cdots \le b_n$ satisfait $b_i \le x_1 + \cdots + x_i$. Dans cet article, nous proposons une approche combinatoire unifiée pour l'énumération des fonctions de $\mathbf{x}$-parking selon leur terme dominant, dans les cas où $\mathbf{x}$ est égal à $(1, \ldots, 1), (a, 1, \ldots, 1), (b, \ldots, b)$, et $(a, b, \ldots, b)$. Cette énumeration s'appuie sur des bijections entre les fonctions de $\mathbf{x}$-parking et les arbres étiquetés et les forêts étiquetées. À cette fin, nou présentons un mécanisme qui simplifie de façon significative l'établissement des bijections. Nous donnons plusieurs résultats bijectifs entre certains ensembles de fonctions de $\mathbf{x}$-parking ayant des termes dominants distincts.

1. INTRODUCTION

A *parking function* of length n is a sequence $\alpha = (a_1, \ldots, a_n)$ of positive integers such that the non-decreasing rearrangement $b_1 \le \cdots \le b_n$ of α satisfies $b_i \le i$. Parking functions were introduced by Konheim and Weiss in [4] when they dealt with a hashing problem in computer science. They derived that the number of parking functions of length n is $(n+1)^{n-1}$, which coincides with the number of labeled trees on $n+1$ vertices by Cayley's formula. Several bijections between the two sets are known (e.g., see [1, 7, 8]). Parking functions have been found in connection to many other combinatorial structures such as acyclic mappings, polytopes, non-crossing partitions, hyperplane arrangements, etc. Refer to [1, 2, 3, 6, 9, 10] for more information. The notion of parking functions were further generalized in [6]. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of positive integers. The sequence $\alpha = (a_1, \ldots, a_n)$ is called an $\mathbf{x}$-*parking function* if the non-decreasing rearrangement $b_1 \le \cdots \le b_n$ of α satisfies $b_i \le x_1 + \cdots + x_i$. Thus the ordinary parking function is the case $\mathbf{x} = (1, \ldots, 1)$. The number of $\mathbf{x}$-parking functions for an arbitrary $\mathbf{x}$ was obtained by Kung and Yan [5] in terms of the determinantal formula of Gončarove polynomials. See also [12, 13, 14] for the explicit formulas and properties for some specified cases of $\mathbf{x}$.

Supported in part by National Science Council, Taiwan, ROC (NSC 93-2115-M-390-005 and 93-2115-M-251-001).

Motivated by the work of Foata and Riordan in [1], we give a combinatorial approach to the enumeration of $(a, b, \ldots, b)$-parking functions by their leading terms in this paper, which covers the special cases $\mathbf{x} = (1, \ldots, 1)$, $(a, 1, \ldots, 1)$, and $(b, \ldots, b)$. An $\mathbf{x}$-parking function $(a_1, \ldots, a_n)$ is said to be k-*leading* if $a_1 = k$. Let $p_{n,k}$ denote the number of k-leading ordinary parking functions of length n. Foata and Riordan [1] derived the generating function for $p_{n,k}$ algebraically,

$$\sum_{k=1}^{n} p_{n,k} x^k = \frac{x}{1-x} \left(2(n+1)^{n-2} - \sum_{k=1}^{n} \binom{n-1}{k-1} k^{k-2} (n-k+1)^{n-k-1} x^k \right). \tag{1}$$

One of our main results is that we obtain some bijective results between certain sets of $(a, 1, \ldots, 1)$-parking functions of distinct leading terms, which lead to the explicit formulas for the number of k-leading $(a, 1, \ldots, 1)$-parking functions. In particular, for the case $a = 1$, we deduce (1) combinatorially. These results rely on a bijection φ between $(a, 1, \ldots, 1)$-parking functions and labeled rooted forests, which is a generalization of the second bijection between acyclic mappings and parking functions of Foata and Riordan [1, Section 3]. The key that opens the way is a simple object, called *triplet-labeled rooted forest*, which not only serves as an intermediate stage of the bijection φ but also enables $(a, 1, \ldots, 1)$-parking functions to be manipulated on forests easily. Furthermore, based on the bijection φ, we establish an immediate bijection between $(a, b, \ldots, b)$-parking functions and labeled rooted forests with edge-colorings, which is equivalent to a bijection given by C. Yan in [14], so as to enumerate $(a, b, \ldots, b)$-parking functions by their leading terms. In the end we propose a structure, by using a generalized triplet-labeled rooted forest, for general $\mathbf{x}$-parking functions.

We organize this paper as follows. The notion of triplet-labeled rooted forests and the bijection φ are given in Section 2. How the bijection φ is applied to enumerate $(a, 1, \ldots, 1)$-parking functions (and hence ordinary parking functions) by the leading terms is given in Section 3. Making use of the notion of labeled rooted forests with edge-colorings, we investigate the cases $\mathbf{x} = (b, \ldots, b)$ and $(a, b, \ldots, b)$ in Section 4. Finally, we propose a structure for general $\mathbf{x}$-parking functions in Section 5.

2. Triplet-labeled rooted forests and the bijection φ

For a rooted forest F and two vertices $u, v \in F$, we say that u is a *descendant* of v if v is contained in the path from u to the root of the component that contains u. If also u and v are adjacent, then u is called a *child* of v, and v is called the *parent* of u. Let $T(u)$ denote the subtree of F consisting of u and the descendants of u, and let $F - T(u)$ denote the remaining part of F when the subtree $T(u)$ and the edge uv are removed. For any two integers $m < n$, we use the notation $[m, n] = \{m, m+1, \ldots, n\}$. In particular, we write $[n] = \{1, \ldots, n\}$.

In this section we consider the case $\mathbf{x} = (a, 1, \ldots, 1)$. We call such an $\mathbf{x}$-parking function α an $(a, \overline{1})$-*parking function*. Note that the non-decreasing rearrangement $b_1 \le \cdots \le b_n$ of α satisfies $b_i \le i + a - 1$. Let $\mathcal{P}_n(a, \overline{1})$ denote the set of $(a, \overline{1})$-parking functions of length n. It is known that $|\mathcal{P}_n(a, \overline{1})| = a(a+n)^{n-1}$ (see [6, 14]). Given an $\alpha = (a_1, \ldots, a_n) \in \mathcal{P}_n(a, \overline{1})$, for $1 \le i \le n$, we define

$$\pi_\alpha(i) = \text{ Card}\{a_j \in \alpha \mid \text{ either } a_j < a_i, \text{ or } a_j = a_i \text{ and } j < i\}. \tag{2}$$

Note that $(\pi_\alpha(1), \ldots, \pi_\alpha(n))$ is a permutation of $[n]$. In fact, $\pi_\alpha(i)$ is the position of the term a_i in the non-decreasing rearrangement of α. Moreover, we define $\tau_\alpha(i) = \pi_\alpha(i) + a - 1$, for $1 \le i \le n$. Note that $a_i \le \tau_\alpha(i)$. We associate α with an a-component forest F_α, called *triplet-labeled rooted* *forest*. The vertex set of F_α is the set

$$\{(i, a_i, \tau_\alpha(i)) |\, a_i \in \alpha\} \cup \{(\rho_i, 0, i) |\, 0 \le i \le a-1\}$$

of *triplets* (i.e., here we identify each vertex with a triplet), where $\rho_i \notin [n]$ is just an artificial label for discriminating the additional triplets. Let $(\rho_0, 0, 0), \dots, (\rho_{a-1}, 0, a-1)$ be the roots of distinct trees of F_α. For any two vertices $v = (x_1, y_1, z_1)$ and $u = (x_2, y_2, z_2)$, u is a child of v whenever $y_2 = z_1 + 1$.

For example, take $a = 2$ and $n = 9$. Consider the $(2, \overline{1})$-parking function $\alpha = (2, 5, 9, 1, 5, 7, 2, 4, 1)$. We have the permutation $(\pi_\alpha(1), \dots, \pi_\alpha(n)) = (3, 6, 9, 1, 7, 8, 4, 5, 2)$ and the sequence $(\tau_\alpha(1), \dots, \tau_\alpha(n)) = (4, 7, 10, 2, 8, 9, 5, 6, 3)$. The triplet-labeled rooted forest associated with α is shown on the left of Figure 1.

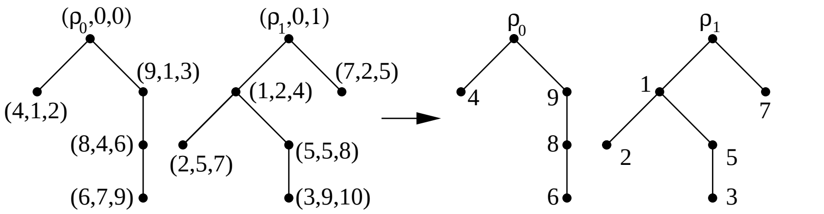

FIGURE 1. the triplet-labeled rooted forest associated with α = (2,5,9,1,5,7,2,4,1) and the corresponding labeled rooted forest $\varphi(\alpha)$ (in the canonical form).

Let $\mathcal{F}_n(a, \overline{1})$ denote the set of a-component rooted forests on the set $\{\rho_0, \dots, \rho_{a-1}\} \cup [n]$ such that the a specified vertices $\rho_0, \dots, \rho_{a-1}$ are the roots of distinct trees. We shall establish a bijection φ between $\mathcal{P}_n(a, \overline{1})$ and $\mathcal{F}_n(a, \overline{1})$ with the triplet-labeled rooted forest as an intermediate stage.

The bijection $\varphi : \mathcal{P}_n(a, \overline{1}) \to \mathcal{F}_n(a, \overline{1})$: Define the mapping φ by carrying $\alpha \in \mathcal{P}_n(a, \overline{1})$ into $\varphi(\alpha) \in \mathcal{F}_n(a, \overline{1})$, where $\varphi(\alpha)$ is the same as the triplet-labeled rooted forest F_α associated with α with vertices labeled by the first entries of the triplets of F_α, i.e., $\varphi(\alpha)$ is obtained from F_α simply by erasing the last two entries of all triplets. As illustrated in Figure 1, the forest on the right is the corresponding forest $\varphi(\alpha)$ of the $(2, \overline{1})$-parking function $\alpha = (2, 5, 9, 1, 5, 7, 2, 4, 1)$.

To describe φ^{-1}, for each $F \in \mathcal{F}_n(a, \overline{1})$, first we express F in a form, called *canonical form*, of a plane rooted forest. We write $F = (T_0, \dots, T_{a-1})$, where T_i denotes the tree of F that is rooted at ρ_i, for $0 \le i \le a-1$. Let $T_0, \dots, T_{a-1}$ be placed from left to right. If a vertex has more than one child then the labels of these children are increasing from left to right. For example, the forest on the right of Figure 1 is in the canonical form. Then we associate the root ρ_i with the triplet $(\rho_i, 0, i)$, for $0 \le i \le a-1$, and associate each non-root vertex $j \in [n]$ with a triplet (j, p_j, q_j), where p_j and q_j are determined by the following algorithm. Here traversing F by a breadth-first search means that we view F as a rooted tree $\widehat{F}$ by connecting the roots of F to a virtual vertex x and then traverse $\widehat{F}$ by a breadth-first search from x.

Algorithm A.

(i) Traverse F by a breadth-first search and label the third entries q_j of the non-root vertices from a to $a + n - 1$.

(ii) For any two vertices $v = (x_1, y_1, z_1)$ and $u = (x_2, y_2, z_2)$, $y_2 = z_1 + 1$ whenever u is a child of v.

As shown in Figure 1, the forest on the left can be recovered from the one on the right by algorithm A. Note that if u is a child of v then $z_2 > z_1$, and hence $y_2 = z_1 + 1 \le z_2$. Sorting the triplets of non-root vertices by the first entries, the sequence $\varphi^{-1}(F) = (p_1, \ldots, p_n)$, which is formed by their second entries, is the required $(a, \overline{1})$-parking function.

Remark: For the special case $a = 1$, φ is a bijection between the set of ordinary parking functions of length n and the set of labeled rooted trees on $[0, n]$ with root $\rho_0 = 0$, which is equivalent to the second bijection between acyclic mappings and parking functions of Foata and Riordan [1, Section 3].

3. Enumerating $(a, \overline{1})$-parking functions by leading terms

Let $\mathcal{P}_{n,k}(a, \overline{1})$ denote the set of k-leading $(a, \overline{1})$-parking functions of length n, and let $p_{n,k}^{(a,\overline{1})} = |\mathcal{P}_{n,k}(a, \overline{1})|$. Let $\mathcal{F}^*_{n,k}(a, \overline{1})$ denote the set of triplet-labeled rooted forests F_α associated with $\alpha \in \mathcal{P}_n(a, \overline{1})$. For each $\alpha = (a_1, \ldots, a_n) \in \mathcal{P}_{n,k}(a, \overline{1})$, we observe that $\pi_\alpha(1) + a - 1 \ge a_1 = k$. With the benefit of triplet-labeled rooted forests, we obtain the following bijective result.

Theorem 3.1. *For $a \le k \le a + n - 2$, there is a bijection between the sets $\mathcal{R}$ and $\mathcal{P}_{n,k+1}(a, \overline{1})$, where $\mathcal{R}$ is the set of k-leading $(a, \overline{1})$-parking functions α of length n that satisfy at least one of the two conditions (i) α has more than one term equal to k, (ii) α has at least $k - a + 1$ terms less than k, and $\mathcal{P}_{n,k+1}(a, \overline{1})$ is the set of $(k + 1)$-leading $(a, \overline{1})$-parking functions of length n.*

Proof: Let $\mathcal{F}^*(\mathcal{R}) \subseteq \mathcal{F}^*_{n,k}(a, \overline{1})$ be the set of forests associated with the parking functions in $\mathcal{R}$. It suffices to establish a bijection $\phi : \mathcal{F}^*(\mathcal{R}) \to \mathcal{F}^*_{n,k+1}(a, \overline{1})$. Given an $F_\alpha \in \mathcal{F}^*(\mathcal{R})$, let $u = (1, k, \tau_\alpha(1)) \in F_\alpha$. Note that the condition (ii) can be rephrased as $\tau_\alpha(1) > k$. Since α satisfies at least one of the two conditions (i) and (ii), there are at least $k + 1$ vertices in the subset $F_\alpha - T(u)$. Traverse $F_\alpha - T(u)$ by a breadth-first search and locate the k-th vertex, say v (we mean that the root of the first tree of $F_\alpha - T(u)$ is the 0-th vertex). The mapping ϕ is defined by carrying F_α into $\phi(F_\alpha)$, where $\phi(F_\alpha)$ is obtained from $F_\alpha - T(u)$ with $T(u)$ attached to v so that u is the first child of v. Updating the second and the third entries of all non-root vertices by algorithm A, the triplet of u becomes $(1, k + 1, \tau(1))$, for some $\tau(1) \ge k + 1$. Hence $\phi(F_\alpha) \in \mathcal{F}^*_{n,k+1}(a, \overline{1})$.

To find ϕ^{-1}, given an $F_\beta \in \mathcal{F}^*_{n,k+1}(a, \overline{1})$ for some $\beta \in \mathcal{P}_{n,k+1}(a, \overline{1})$, let $u = (1, k + 1, \tau_\beta(1)) \in F_\beta$ and let v be the parent of u. In F_β we locate the vertex, say w, the third entry of which is equal to $k - 1$. Then $\phi^{-1}(F_\beta)$ is obtained from $F_\beta - T(u)$ with $T(u)$ attached to w so that u is the first child of w. By algorithm A, the updated triplet of u becomes $(1, k, \tau(1))$. We observe that either $\tau(1) = k$ if v is another child of w, or $\tau(1) > k$ otherwise. Hence $\phi^{-1}(F_\beta) \in \mathcal{F}^*(\mathcal{R})$. □

For example, take $a = 2$ and $n = 9$. Consider the 2-leading $(2, \overline{1})$-parking function $\alpha = (2, 5, 9, 1, 5, 7, 2, 4, 1)$. On the left of Figure 2 is the forest F_α associated with α. Let $u = (1, 2, 4)$. Note that $v = (4, 1, 2)$ is the second vertex of $F_\alpha - T(u)$ that is visited by a breadth-first search. On the right of Figure 2 is the corresponding forest $\phi(F_\alpha)$, which is obtained from $F_\alpha - T(u)$ with $T(u)$ attached to v and with the second and the third entries of the triplets updated. Sorting the triplets of non-root vertices by the first entries, we retrieve the corresponding 3-leading $(2, \overline{1})$-parking function $(3, 6, 9, 1, 6, 7, 2, 4, 1)$ from their second entries.

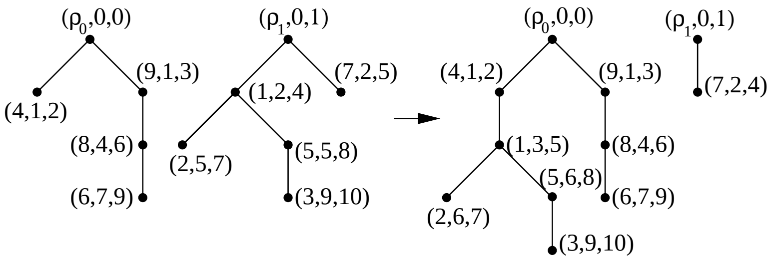

FIGURE 2. the forests associated with the $(2,\bar{1})$-parking functions (2,5,9,1,5,7,2,4,1) and $(3,6,9,1,6,7,2,4,1)$.

The following recurrence relations for $p_{n,k}^{(a,\bar{1})}$ can be derived from Theorem 3.1. Here, unless specified, the labeled rooted forests in $\mathcal{F}_n(a,\bar{1})$ are considered to be in the canonical form. Recall that $|\mathcal{F}_n(a,\bar{1})| = |\mathcal{P}_n(a,\bar{1})| = a(a+n)^{n-1}$.

Theorem 3.2. *For $a \le k \le a+n-2$, we have*

$$p_{n,k}^{(a,\bar{1})} - p_{n,k+1}^{(a,\bar{1})} = \binom{n-1}{k-a} ak^{k-a-1}(n-k+a)^{n-k+a-2}. \tag{3}$$

Proof: By Theorem 3.1, $p_{n,k}^{(a,\bar{1})} - p_{n,k+1}^{(a,\bar{1})}$ is equal to the number of k-leading $(a,\bar{1})$-parking functions α such that the first term of α is the unique term equal to k, and $\tau_\alpha(1) = k$. We shall count the number of forests that are mapped by such parking functions under the mapping φ. Let $F = \varphi(\alpha) \in \mathcal{F}_n(a,\bar{1})$ and let $u = 1 \in F$. We observe that $F - T(u)$ is an a-component labeled forest on k vertices containing the roots $\rho_0, \dots, \rho_{a-1}$, and $T(u)$ is a labeled tree on $n-k+a$ vertices containing u. Since there are $\binom{n-1}{k-a}$ ways to choose $k-a$ numbers from $[2,n]$ for the non-root vertices of $F-T(u)$ and since there are ak^{k-a-1} and $(n-k+a)^{n-k+a-2}$ possibilities for the induced forest $F-T(u)$ and tree $T(u)$, the number of the required forests is $\binom{n-1}{k-a}ak^{k-a-1}(n-k+a)^{n-k+a-2}$. $\square$

In order to evaluate $p_{n,k}^{(a,\bar{1})}$ by Theorem 3.2, we need the following initial conditions (4). Since an $(a,\bar{1})$-parking function of length n with leading term k, $(1 \le k \le a)$ is simply a juxtaposition of k and an $(a+1,\bar{1})$-parking function of length $n-1$, we have

$$p_{n,k}^{(a,\bar{1})} = (a+1)(a+n)^{n-2}, \qquad \text{for } 1 \le k \le a. \tag{4}$$

Now we can derive the explicit formula for $p_{n,k}^{(a,\bar{1})}$ by (3) and (4). In particular, we have $p_{n,a+n-1}^{(a,\bar{1})} = a(a+n-1)^{n-2}$ since an $(a+n-1)$-leading $(a,\bar{1})$-parking function of length n is a juxtaposition of $a+n-1$ and an $(a,\bar{1})$-parking function of length $n-1$. We derive the following enumerator for $(a,\bar{1})$-parking functions by the leading terms.

Theorem 3.3. *If $P^{(a,\bar{1})}(x) = \sum_{k=1}^{a+n-1} p_{n,k}^{(a,\bar{1})} x^k$, then*

$$P^{(a,\bar{1})}(x) = \frac{x}{1-x}\left((a+1)(a+n)^{n-2} - \sum_{k=a}^{a+n-1}\binom{n-1}{k-a} ak^{k-a-1}(n-k+a)^{n-k+a-2}x^k\right).$$

Proof: We have

$$\begin{aligned}\left(\frac{1}{x}-1\right)P^{(a,\bar{1})}(x) &= p_{n,1}^{(a,\bar{1})}-\left(\sum_{k=1}^{a+n-2}\left(p_{n,k}^{(a,\bar{1})}-p_{n,k+1}^{(a,\bar{1})}\right)x^k\right)-p_{n,a+n-1}^{(a,\bar{1})}x^{a+n-1}\\ &= (a+1)(a+n)^{n-2}-\sum_{k=a}^{a+n-1}\binom{n-1}{k-a}ak^{k-a-1}(n-k+a)^{n-k+a-2}x^k,\end{aligned}$$

as required. □

Remark: For the case $a=1$, we deduce that the number $p_{n,k}$ of k-leading ordinary parking functions satisfies the recurrence relations

$$p_{n,k}-p_{n,k+1}=\binom{n-1}{k-1}k^{k-2}(n-k+1)^{n-k-1}, \tag{5}$$

for $1\le k\le n-1$, with the initial condition $p_{n,1}=2(n+1)^{n-2}$. The enumerator (1) for ordinary parking functions by the leading terms is derived anew.

Making use of the bijection φ for the case $a=1$, we derive the following interesting result for ordinary parking functions.

Theorem 3.4. *If n is even, then there is a two-to-one correspondence between the set of 1-leading parking functions of length n and the set of $(\frac{n}{2}+1)$-leading parking functions of length n.*

Proof: Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the set of labeled trees corresponding to the 1-leading (resp. $(\frac{n}{2}+1)$-leading) parking functions of length n under the mapping φ. We shall establish a two-to-one correspondence ϕ between $\mathcal{A}$ and $\mathcal{B}$. For each $T\in\mathcal{A}$, the two vertices $u=1$ and $v=0$ of T are adjacent. Let $T(v)=T-T(u)$. Since T has $n+1$ vertices and n is even, one of $T(v)$ and $T(u)$ contains more than $\frac{n}{2}$ vertices. If $|T(v)|>\frac{n}{2}$, then we traverse $T(v)$ from v by a breadth-first search and locate the $\frac{n}{2}$-th non-root vertex, say w. The tree $\phi(T)\in\mathcal{B}$ is obtained from $T(v)$ with $T(u)$ attached to w so that u becomes the first child of w. Otherwise $|T(u)|>\frac{n}{2}$. Locate the $\frac{n}{2}$-th non-root vertex of $T(u)$, say w', by a breadth-first search. Then $\phi(T)$ is obtained from $T(u)$ with $T(v)$ attached to w' so that v becomes the first child of w' and with a relabeling $u=0$ and $v=1$.

To find ϕ^{-1}, given a tree $T'\in\mathcal{B}$, let $u=1$, $v=0$, and $T(v)=T'-T(u)$. We retrieve two trees of $\mathcal{A}$ from $T(u)$ and $T(v)$. One is obtained by attaching $T(u)$ to v so that u becomes the first child of v, and the other is obtained by attaching $T(v)$ to u so that v becomes the first child of u and relabeling $u=0$ and $v=1$. □

For example, take $n=6$. On the left of Figure 3 are the labeled trees T corresponding to the 1-leading parking functions $(1,4,1,2,4,1)$ and $(1,5,2,1,5,2)$, respectively. Let $u=1$, $v=0$ and $T(v)=T-T(u)$. For the first tree, we observe that $|T(v)|>3$ and the vertex 2 is the third non-root vertex of $T(v)$ that is visited by a breadth-first search. We attach $T(u)$ to 2 to obtain the required tree on the right. For the second tree, we observe that $|T(u)|>3$ and again the vertex 2 is the third non-root vertex of $T(u)$. Likewise, we attach $T(v)$ to 2 and relabel $u=0$ and $v=1$. Note that the tree on the right of Figure 3 corresponds to the 4-leading parking function $(4,3,1,6,3,1)$.

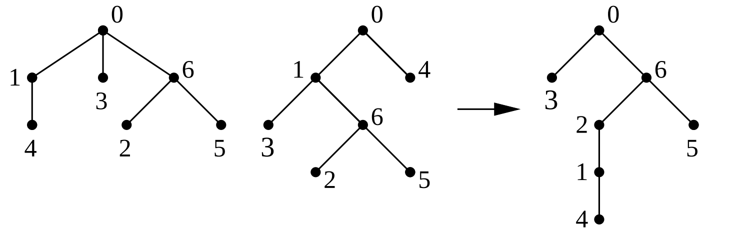

FIGURE 3. an example of the two-to-one correspondence for $n = 6$

4. Edge-colored labeled rooted trees and forests

In this section we shall enumerate $\mathbf{x}$-parking functions by the leading terms for the cases $\mathbf{x} = (b, \dots, b)$ and $(a, b, \dots, b)$. First, we define the required structure for the case $\mathbf{x} = (b, \dots, b)$. Let $\mathcal{T}_n(\bar{b})$ denote the set of labeled trees, called *b-trees*, on the vertex set $[0, n]$, whose edges are colored with the colors $0, 1, \dots, b-1$. There is no further restriction on the colorings of edges. Unless specified, each $T \in \mathcal{T}_n(\bar{b})$ is rooted at 0 and is in the canonical form regarding the vertex-labeling. Let $\kappa(i)$ denote the color of the edge that connects the vertex i and its parent. It is known that $|\mathcal{T}_n(\bar{b})| = b^n(n+1)^{n-1}$.

For the case $\mathbf{x} = (b, \dots, b)$, we call such $\mathbf{x}$-parking functions $(\bar{b})$-*parking functions*. Let $\mathcal{P}_n(\bar{b})$ denote the set of $(\bar{b})$-parking functions of length n. Note that the non-decreasing rearrangement $b_1 \le \cdots \le b_n$ of $\alpha \in \mathcal{P}_n(\bar{b})$ satisfies $b_i \le bi$. We shall establish a bijection $\psi_b : \mathcal{P}_n(\bar{b}) \to \mathcal{T}_n(\bar{b})$ based on the bijection φ for the case $a = 1$. Given an $\alpha = (a_1, \dots, a_n) \in \mathcal{P}_n(\bar{b})$, we associate α with two sequences $\beta = (p_1, \dots, p_n)$ and $\gamma = (r_1, \dots, r_n)$, where $p_i = \lceil \frac{a_i}{b} \rceil$ (the least integer that is greater than or equal to $\frac{a_i}{b}$) and $r_i = bp_i - a_i$, for $1 \le i \le n$. It is easy to see that β is an ordinary parking function of length n and $\gamma \in [0, b-1]^n$, and that α is uniquely determined by such a pair (β, γ), i.e., $a_i = bp_i - r_i$, for $1 \le i \le n$. To establish the mapping ψ_b, we first locate the corresponding labeled tree $\varphi(\beta) \in \mathcal{F}_n(1, \bar{1})$ of β, and then define ψ_b by carrying α into $\varphi(\beta)$ with the edge-coloring $\kappa(i) = r_i$, for $1 \le i \le n$.

For example, take $b = 2$. For the $(\bar{2})$-parking function $\alpha = (2, 7, 4, 18, 1, 9, 2, 9, 8)$, we have the associated ordinary parking function $\beta = (1, 4, 2, 9, 1, 5, 1, 5, 4)$ and the sequence $\gamma = (0, 1, 0, 0, 1, 1, 0, 1, 0)$. On the left of Figure 4 is the triplet-labeled rooted tree associated with β. The 2-tree corresponding to α is shown on the right of Figure 4, where the dotted edges and solid edges represent the colors 0 and 1, respectively.

To find ψ_b^{-1}, given a $T \in \mathcal{T}_n(\bar{b})$, we can retrieve the ordinary parking function $(p_1, \dots, p_n)$ from the vertex-labeling of T, and then derive the require $(\bar{b})$-parking function $\psi_b^{-1}(T) = (a_1, \dots, a_n) \in \mathcal{P}_n(\bar{b})$ by setting $a_i = bp_i - \kappa(i)$, for $1 \le i \le n$.

Proposition 4.1. *The mapping $\psi_b : \mathcal{P}_n(\bar{b}) \to \mathcal{T}_n(\bar{b})$ mentioned above is a bijection.*

Let $p_{n,m}^{(\bar{b})}$ denote the number of m-leading $(\bar{b})$-parking functions of length n. Recall that we have determined the number $p_{n,k}$ of k-leading ordinary parking functions of length n by (5) and an initial condition. We derive $p_{n,m}^{(\bar{b})}$ as follows.

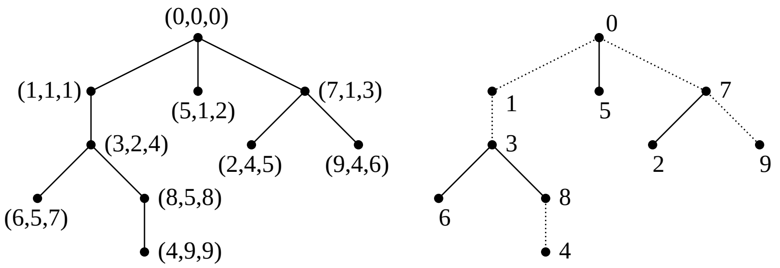

FIGURE 4. the triplet-labeled rooted tree associated with the parking function $(1, 4, 2, 9, 1, 5, 1, 5, 4)$ and the 2-tree corresponding to $(2, 7, 4, 18, 1, 9, 2, 9, 8)$

Proposition 4.2. *For $1 \le k \le n$ and $(k-1)b + 1 \le m \le kb$, we have $p_{n,m}^{(\bar{b})} = b^{n-1} p_{n,k}$.*

Proof: If $(k-1)b+1 \le m \le kb$, then each m-leading $(\bar{b})$-parking function of length n is associated with a pair (β, γ), where β is a k-leading parking function of length n and $\gamma \in [0, b-1]^n$ with the first term equal to $bk - m$. Since there are $b^{n-1}p_{n,k}$ possibilities for such a pair (β, γ), the assertion follows. □

Let us turn to the case $\mathbf{x} = (a, b, \dots, b)$. In [14] C. Yan introduced the notion of sequences of rooted b-forests in order to generalize a bijection of Foata and Riordan [1]. A *rooted b-forest* is a labeled rooted forest with edges colored with the colors $0, \dots, b-1$ (note that each component may be rooted at any vertex). Consider a sequence $(S_0, \dots, S_t)$ of rooted b-forests on $[n]$ such that (i) each S_i is a rooted b-forest, (ii) S_i and S_j are disjoint if $i \ne j$, and (iii) the union of the vertex sets S_i, $(0 \le i \le t)$ is $[n]$. Let $\widehat{S_i}$ denote the rooted tree obtained by connecting the roots of S_i to a new root vertex ρ_i, where the edges incident to ρ_i are not colored with any color, denoted by -1 for such an edge. Let $\mathcal{F}_n(a, \bar{b})$ denote the set of a-component rooted forests of the form $(\widehat{S_0, \dots, S_{a-1}})$, where $(S_0, \dots, S_{a-1})$ is a sequence of rooted b-forest on $[n]$. We call members of $\mathcal{F}_n(a, \bar{b})$ *extended b-forests*. Let $\kappa(i)$ denote the color of the edge that connects the vertex i and its parent. It is known that $|\mathcal{F}_n(a, \bar{b})| = a(a+nb)^{n-1}$ (see [6, 14]).

For $\mathbf{x} = (a, b, \dots, b)$, we call such an $\mathbf{x}$-parking function α an $(a, \bar{b})$-*parking function*. In this case the non-decreasing rearrangement $b_1 \le \cdots \le b_n$ of α satisfies $b_i \le a + (i-1)b$. Let $\mathcal{P}_n(a, \bar{b})$ denote the set of $(a, \bar{b})$-parking functions of length n. We shall establish a bijection $\varphi_b : \mathcal{P}_n(a, \bar{b}) \to \mathcal{F}_n(a, \bar{b})$ based on the bijection $\varphi : \mathcal{P}_n(a, \bar{1}) \to \mathcal{F}_n(a, \bar{1})$ given in Section 2. Given an $\alpha = (a_1, \dots, a_n) \in \mathcal{P}_n(a, \bar{b})$, we associate α with two sequences $\beta = (p_1, \dots, p_n)$ and $\gamma = (r_1, \dots, r_n)$, where

$$(6) \qquad p_i = \begin{cases} a_i & \text{if } a_i \le a, \\ \left\lceil \dfrac{a_i - a}{b} \right\rceil + a & \text{otherwise;} \end{cases} \qquad \text{and} \qquad r_i = \begin{cases} -1 & \text{if } a_i \le a, \\ b(p_i - a) - a_i + a & \text{otherwise.} \end{cases}$$

It is straightforward to verify that β is an $(a, \bar{1})$-parking function of length n and $\gamma \in [-1, b-1]^n$ with $r_i = -1$ whenever $a_i \le a$. Moreover, α is uniquely determined by such a pair (β, γ), i.e., $a_i = p_i$ if $p_i \le a$, and $a_i = b(p_i - a) - r_i + a$ otherwise. To establish the mapping φ_b, we first locate the corresponding labeled forest $\varphi(\beta) \in \mathcal{F}_n(a, \overline{1})$ of β and then define φ_b by carrying α into $\varphi(\beta)$ with the edge-coloring $\kappa(i) = r_i$, for $1 \le i \le n$.

For example, take $a = 2$ and $b = 2$. For the $\alpha = (2, 7, 15, 1, 8, 12, 2, 5, 1)$, we have the associated pair (β, γ), where $\beta = (2, 5, 9, 1, 5, 7, 2, 4, 1)$ and $\gamma = (-1, 1, 1, -1, 0, 0, -1, 1, -1)$. As shown in Figure 1, we have obtained the labeled forest $\varphi(\beta)$. The required extended b-forest $\varphi_b(\alpha)$ is shown in Figture 5, where the arrowed edges are not colored, and the dotted and solid edges represent the colors 0 and 1, respectively.

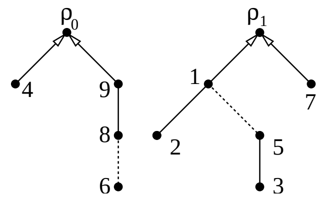

FIGURE 5. the extended 2-forest corresponding to the $(2, \overline{2})$-parking function (2,7,15,1,8,12,2,5,1)

To find φ_b^{-1}, given an $F \in \mathcal{F}_n(a, \overline{b})$, we can retrieve an $(a, \overline{1})$-parking function $(p_1, \ldots, p_n)$ from the vertex-labeling of F and then derive the require $(a, \overline{b})$-parking function $\varphi_b^{-1}(F) = (a_1, \ldots, a_n)$ by setting $a_i = p_i$ if $p_i \le a$, and $a_i = b(p_i - a) - \kappa(i) + a$ otherwise.

Proposition 4.3. *The mapping* $\varphi_b : \mathcal{P}_n(a, \overline{b}) \to \mathcal{F}_n(a, \overline{b})$ *mentioned above is a bijection.*

The bijection φ_b is equivalent to Yan's bijection in [14]. Our method not only simplifies the construction but also provides an approach to the enumeration of $(a, b, \ldots, b)$-parking functions by the leading terms. Let $\mathcal{P}_{n,m}(a, \overline{b})$ denote the set of m-leading $(a, \overline{b})$-parking functions of length n and let $p_{n,m}^{(a,\overline{b})} = |\mathcal{P}_{n,m}(a, \overline{b})|$.

Lemma 4.4. *For* $0 \le k \le n - 2$ *and* $a + kb + 1 \le i, j \le a + (k + 1)b$*, there is a bijection between* $\mathcal{P}_{n,i}(a, \overline{b})$ *and* $\mathcal{P}_{n,j}(a, \overline{b})$*, and hence* $p_{n,i}^{(a,\overline{b})} = p_{n,j}^{(a,\overline{b})}$.

Proof: Given an $\alpha = (a_1, \ldots, a_n) \in \mathcal{P}_{n,i}(a, \overline{b})$, if the first term $(a_1 = i)$ is replaced by j, where $a + kb + 1 \le i, j \le a + (k + 1)b$, then it corresponds to replace the color $\kappa(1) = b(k + 1) - i + a$ of $\varphi_b(\alpha)$ by the color $b(k + 1) - j + a$. Hence there is an immediate bijection between $\mathcal{P}_{n,i}(a, \overline{b})$ and $\mathcal{P}_{n,j}(a, \overline{b})$ by using Proposition 4.3 and an interchange of the color $\kappa(1)$. □

The following result is extended from Theorem 3.1.

Theorem 4.5. *For* $0 \le k \le n-2$*, there is a bijection between the sets* $\mathcal{R}$ *and* $\mathcal{P}_{n,a+kb+1}(a, \overline{b})$*, where* $\mathcal{R}$ *is the set of* $(a + kb)$*-leading* $(a, \overline{b})$*-parking functions* α *of length* n *that satisfy at least one of the two conditions (i)* α *has more than one term belong to the interval* $[a + (k - 1)b + 1, a + kb]$ *(or* α *has more than one term equal to* a *in case* $k = 0$*), (ii)* α *has at least* $k + 1$ *terms less then* $a + kb$*, and* $\mathcal{P}_{n,a+kb+1}(a, \overline{b})$ *is the set of* $(a + kb + 1)$*-leading* $(a, \overline{b})$*-parking functions of length* n*.*

Proof: Given an $\alpha \in \mathcal{R}$, let (β, γ) be the pair associated with α determined by (6). We observe that β is an $(a+k)$-leading $(a, \bar{1})$-parking function with at least one of the two properties (i) β has more than one term equal to $a+k$, (ii) $\pi_\beta(1) = \pi_\alpha(1) > k+1$. By Theorem 3.1, there is a bijection ϕ that carries β into an $(a+k+1)$-leading $(a, \bar{1})$-parking function $\phi(\beta)$ of length n. From the pair $(\phi(\beta), \gamma)$, we retrieve an $(a+(k+1)b)$-leading $(a, \bar{b})$-parking function. The assertion follows from Lemma 4.4. ☐

The following theorem is analogous to Theorems 3.2, and the proof is similar.

Theorem 4.6. *If $0 \le k \le n-2$, then*

$$p^{(a,\bar{b})}_{n,a+kb} - p^{(a,\bar{b})}_{n,a+kb+1} = \binom{n-1}{k} ab^{n-k-1}(a+kb)^{k-1}(n-k)^{n-k-2}. \tag{7}$$

To evaluate $p^{(a,\bar{b})}_{n,m}$ by Lemma 4.4 and Theorem 4.6, we need the following initial conditions (8). Note that an $(a, \bar{b})$-parking function of length n with leading term m, $(1 \le m \le a)$ is a juxtaposition of m and an $(a+b, \bar{b})$-parking function of length $n-1$. Hence

$$p^{(a,\bar{b})}_{n,m} = (a+b)(a+nb)^{n-2}, \qquad \text{for } 1 \le m \le a. \tag{8}$$

By a similar argument of Theorem 3.3, we derive the enumerator for $(a, \bar{b})$-parking functions by the leading terms.

Theorem 4.7. *If $P^{(a,\bar{b})}(x) = \sum_{m=1}^{a+(n-1)b} p^{(a,\bar{b})}_{n,m} x^m$, then*

$$P^{(a,\bar{b})}(x) = \frac{x}{1-x}\left((a+b)(a+nb)^{n-2} - \sum_{k=0}^{n-1}\binom{n-1}{k} ab^{n-k-1}(a+kb)^{k-1}(n-k)^{n-k-2}x^{a+kb}\right).$$

5. A structure for general $\mathbf{x}$-parking functions

In this section we propose a forest structure for general $\mathbf{x}$-parking functions. Given a sequence $\mathbf{x} = (x_1, \dots, x_n)$ of positive integers, let $\mathcal{P}_n(\mathbf{x})$ denote the set of the $\mathbf{x}$-parking functions $\alpha = (a_1, \dots, a_n)$ of length n and let $b_1 \le \cdots \le b_n$ be the non-decreasing rearrangement of α. Likewise, we define the permutation $(\pi_\alpha(1), \dots, \pi_\alpha(n))$ by (2). Let $d_i = x_1 + \cdots + x_i$, for $1 \le i \le n$. We associate α with an x_1-component triplet-labeled rooted forest F_α satisfying the following conditions:

(i) The roots of F_α are the triplets $(0,0,0)$ and $(\rho_{(0,1)}, 0, 1), \dots, (\rho_{(0,x_1-1)}, 0, x_1-1)$.

(ii) For $1 \le j \le n-1$, there are x_{j+1} triplets in F_α associated with b_j. If b_j is the term a_i, for some $i \in [n]$ (i.e., $\pi_\alpha(i) = j$), then F_α contains the triplet (i, a_i, d_j) and the additional $x_{j+1}-1$ triplets $(\rho_{(j,1)}, a_i, d_j+1), \dots, (\rho_{(j,x_{j+1}-1)}, a_i, d_{j+1}-1)$. Finally, there is a triplet (k, a_k, d_n) associated with b_n, where b_n is the term a_k, for some $k \in [n]$ (i.e., $\pi_\alpha(k) = n$).

(iii) For any two vertices $v = (x_1, y_1, z_1)$ and $u = (x_2, y_2, z_2)$, u is a child of v if $y_2 = z_1 + 1$.

Note that the third entries of the triplets are from 0 to $x_1 + \cdots + x_n$. If we erase the second and the third entries of the triplets, then we turn F_α into a rooted forest F with $x_1 + \cdots + x_n + 1$ vertices satisfying the following conditions.

(C.1) The vertex set of F is $[0,n] \cup_{j=0}^{n-1} \{\rho_{(j,1)}, \dots, \rho_{(j,x_{j+1}-1)}\}$.

(C.2) The roots of F are 0 and $\rho_{(0,1)}, \dots, \rho_{(0,x_1-1)}$.

(C.3) The vertices $\rho_{(j,1)}, \ldots, \rho_{(j,x_{j+1}-1)}$ share the same parent with a vertex $u_j \in [n]$, for $1 \leq j \leq n-1$, such that $u_i \neq u_j$ if $i \neq j$.
(C.4) There is an ordering among $\rho_{(j,i)}$, which is created by a breadth-first search in F, $\rho_{(j,i)} < \rho_{(j',i')}$ if $j < j'$, or $j = j'$ and $i < i'$.

Let $\mathcal{F}_n(\mathbf{x})$ denote the set of rooted forests satisfying conditions (C.1)–(C.4). It is easy to see that there is a bijection $\varphi_{\mathbf{x}}$ between $\mathcal{P}_n(\mathbf{x})$ and $\mathcal{F}_n(\mathbf{x})$, with the triplet-labeled rooted forests as the intermediate stage, which is established in a similar manner to the bijection φ given in Section 2.

Let us consider $\mathbf{x}$-parking functions α for a specified case $\mathbf{x} = (1, \ldots, 1, a, 1, \ldots, 1)$, where a occurs at the k-th entry of $\mathbf{x}$ and $k \geq 2$. We call α a *(k,a)-inflating parking function.* Let $\mathcal{Q}_n(k,a)$ denote the set of (k,a)-inflating parking functions of length n. Given an $\alpha = (a_1, \ldots, a_n) \in \mathcal{Q}_n(k,a)$, we observe that the non-decreasing rearrangement $b_1 \leq \cdots \leq b_n$ of α satisfies $b_i \leq i$ if $i \leq k-1$, and $b_i \leq i + a - 1$ otherwise. Define $\tau_\alpha(i) = \pi_\alpha(i)$, for $1 \leq \pi_\alpha(i) \leq k-1$, and $\tau_\alpha(i) = \pi_\alpha(i) + a - 1$, for $k \leq \pi_\alpha(i) \leq n$.

In addition to the triplets $(0,0,0)$ and $\{(i, a_i, \tau_\alpha(i)) |\, 1 \leq i \leq n\}$, we locate the $(k-1)$-th term in the non-decreasing rearrangement of α, say $b_{k-1} = a_r$, and associate it with another $a-1$ triplets $(\rho_1, a_r, k), \ldots, (\rho_{a-1}, a_r, k+a-2)$. The tree T_α associated with α is on the above triplets with the root at $(0,0,0)$. For any two vertices $v = (x_1, y_1, z_1)$ and $u = (x_2, y_2, z_z)$, u is a child of v if $y_2 = z_1 + 1$.

For example, take $k = 4$ and $a = 3$. Consider the $(4,3)$-inflating parking $\alpha = (5,1,4,5,1,10,3,3,7)$. We have the permutation $(\pi_\alpha(1), \ldots, \pi_\alpha(n)) = (6,1,5,7,2,9,3,4,8)$ and $(\tau_\alpha(1), \ldots, \tau_\alpha(n)) = (8,1,7,9,2,11,3,6,10)$. Note that $a_7 = 3$ is the third term in the non-decreasing rearrangement of α. There associate two additional triplets $w_1 = (\rho_1, 3, 4)$ and $w_2 = (\rho_2, 3, 5)$. The tree associated with α is shown on the left of Figure 6.

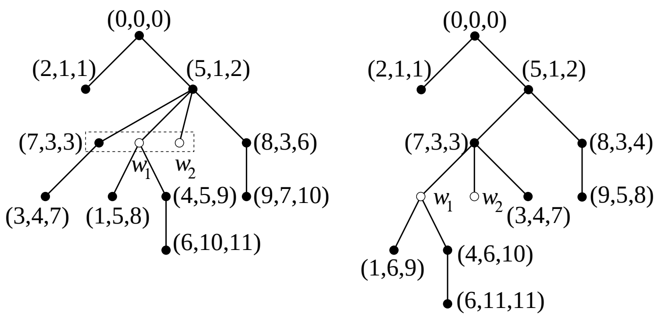

FIGURE 6. the triplet-labeled rooted tree associated with the $(4,3)$-inflating parking function $\alpha =$ (5,1,4,5,1,10,3,3,7) and the tree associated with $\phi(\alpha)$.

Theorem 5.1. *There is a bijection between the set of (k,a)-inflating parking functions of length n and the set of ordinary parking functions of length $n+a-1$ with the initial $a-1$ terms equal to k.*

Proof: We shall establish a bijection ϕ between two sets. Let T_α be the tree associated with α mentioned above. Locate the vertex of T_α with the third entry equal to $k-1$, say $u = (r, a_r, k-1)$, and let $w_j = (\rho_j, a_r, k+j-1) \in T_\alpha$, for $1 \leq j \leq a-1$. We form a new triplet-labeled rooted tree from $T_\alpha - (T(w_1) \cup \cdots \cup T(w_{a-1}))$ by attaching $T(w_1), \ldots, T(w_{a-1})$ to u so that $w_1, \ldots, w_{a-1}$ become the first $a-1$ children of u. Then update the second and the third entries of all triplets by algorithm A. Sorting the triplets by the first entries in the order $\rho_1, \ldots, \rho_{a-1}$ and then $1, \ldots, n$, we obtain the corresponding ordinary parking function $\phi(\alpha) = (a_{\rho_1}, \ldots, a_{\rho_{a-1}}, a_1, \ldots, a_n)$ from their second entries, where $a_{\rho_1} = \cdots = a_{\rho_{a-1}} = k$. □

As illustrated in Figure 6, the triplet-labeled rooted tree associated with $\phi(\alpha)$ is shown on the right, where $w_1 = (\rho_1, 4, 5)$ and $w_2 = (\rho_2, 4, 6)$ and $\phi(\alpha) = (4, 4, 6, 1, 4, 6, 1, 11, 3, 3, 5)$. Note that so far we have solved explicitly the number of (k, a)-inflating parking functions for the case $a = 2$. We are interested to know explicit formulas for other cases.

The forest structure reveals that the number of k-leading $\mathbf{x}$-parking functions is a step function in k.

Theorem 5.2. *For any $\mathbf{x} = (x_1, \ldots, x_n)$, let $d_i = x_1 + \cdots + x_i$, $(1 \le i \le n)$ and $d_0 = 0$. If $d_{k-1} + 1 \le p, q \le d_k$, then the number of p-leading $\mathbf{x}$-parking functions of length n is equal to the number of q-leading $\mathbf{x}$-parking functions of length n.*

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