A COMBINATORIAL DERIVATION OF THE PASEP STATIONARY STATE

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ABSTRACT. We give a combinatorial derivation and interpretation of the algebra associated with the stationary distribution of the partially asymmetric exclusion process. Our derivation works by constructing a larger Markov chain on a larger set of generalised configurations. A bijection on this new set of configurations allows us to find the stationary distribution of the new chain. We then show that a *subset* of the generalised configurations is equivalent to the original chain and that the stationary distribution on this subset is simply related to that of the original chain. This derivation allows us to find expressions for the normalisation using both recurrences and path models. These results exhibit classical combinatorial numbers such as $n!$, 2^n and the Catalan numbers.

RESUME. Nous donnons une interprétation combinatoire de l'algèbre associé à la distribution stationnaire du processus d'exclusion partiellement symétrique. Pour cela nous construisons une chaine de Markov qui agit sur les configurations et les configurations marquées. Nous utilisons une bijection sur les configurations marquées pour trouver la distribution stationnaire de la nouvelle chaine qui nous donne aussi la distribution stationnaire de la chaine originelle. Grace à cette construction, nous pouvons calculer dans plusieurs cas particuliers le coefficient de normalisation de la chaine en utilisant des récurrences et/ou des chemins. De nombreux nombres classiques apparaissent: Catalan, factoriel, 2^n...

1. INTRODUCTION

The PASEP (Partially asymmetric exclusion process) is a generalisation of the TASEP model presented last year at FPSAC'04 [13]. This model was introduced by physicists [2, 9, 10, 11, 12, 16]. The TASEP model consists of black particles entering a row of n cells, each of which is occupied by a black particle or vacant. A particle may enter the system from the left hand side, hop to the right and leave the system from the right hand side, with the constraint that a cell contains at most one particle. The particles in the PASEP move in the same way as those in the TASEP, but in addition may enter the system from the right hand side, hop to the left and leave the system from the left hand side. An example is given in Figure 1.
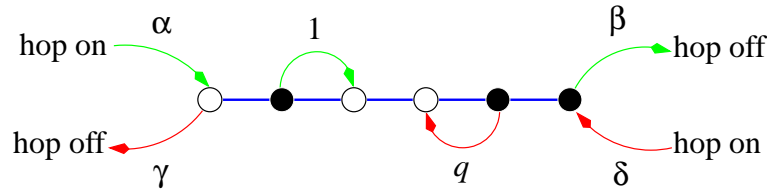

FIGURE 1. The PASEP model. Note that we also use the variables $c = 1/\alpha - 1$ and $d = 1/\beta - 1$.

¿From now on we will say that the empty cells are filled with white particles ∘. A basic configuration is a row of n cells, each containing either a black or a white particle. Let $\mathcal{B}_n$ be the set of basic configurations of n particles. We write these configurations as though they are words of length n in the language $\{\circ, \bullet\}^*$, so that "$\bullet^k$" denotes a string of k black particles and "AB" denotes the configuration made up of the word "A" followed by the word "B". We denote the length of the word A by $|A|$.

The PASEP defines a Markov chain P defined on $\mathcal{B}_n$ with the transition probabilities α, β, γ, δ, η and q. The probability $P_{X,Y}$, of finding the system in state Y at time $t+1$ given that the system is in state X at time t is defined by:

∗ If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then

(1a)
$$P_{X,Y} = \eta/(n+1); \quad P_{Y,X} = q/(n+1)$$

$* $ If $X = \circ B$ and $Y = \bullet B$ then

(1b)
$$P_{X,Y} = \alpha/(n+1); \quad P_{Y,X} = \gamma/(n+1)$$

$* $ If $X = B\bullet$ and $Y = B\circ$ then

(1c)
$$P_{X,Y} = \beta/(n+1); \quad P_{Y,X} = \delta/(n+1)$$

$* $ Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

See an example for $n = 2$ in Figure 2.

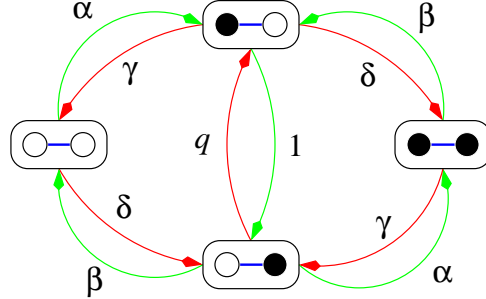
FIGURE 2. The chain P for $n = 2$.

There are many results for the PASEP model. One central question is the computation of the stationary distribution of the chain. This has been most successfully analysed using a matrix product Ansatz [9, 11, 16].

In this paper we give a combinatorial derivation and interpretation of the stationary distribution of the PASEP model. To our knowledge [4, 15] are the only purely combinatorial derivations of the solution to the TASEP model (which is a special case of PASEP). Our derivation works by

[i] constructing a larger Markov chain on both basic configurations and marked basic configurations which we call *marked configurations.*

[ii] using a bijection between marked configurations to find the stationary distribution of the larger chain.

[iii] showing that a *subset* of the configurations is equivalent to the original chain and that the stationary state on this subset is simply related to that of the original chain.

We note that this is similar to the work done in [15] where the authors studied the case $\delta = \gamma = q = 0$. Here we study the stationary distribution of the full model.

The main result of this paper is to give a combinatorial derivation of the following theorem:

Theorem 1. *Let the stationary distribution be denoted*

(2)
$$\mathrm{P}_\infty(X) = \frac{W(X)}{Z_n}$$

where $W(X)$ is the weight *of a basic configuration X and Z_n is the normalisation:*

(3)
$$Z_n = \sum_{X \in \mathcal{B}_n} W(X).$$

The weight of a basic configuration X satisfies the following algebra :

(4a) $\quad W(X) \quad = \quad 1 \quad \text{if } X \in \mathcal{B}_0$

(4b) $\quad \alpha W(\circ X) \quad = \quad W(X) + \gamma W(\bullet X)$

(4c) $\quad \beta W(X\bullet) \quad = \quad W(X) + \delta W(X\circ)$

(4d) $\quad \eta W(A \bullet \circ B) \quad = \quad W(A \circ B) + W(A \bullet B) + q W(A \circ \bullet B)$

where A and B denote configurations of particles.

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We may use this algebra recursively to find expressions for Z_n. Unfortunately we have not yet found a combinatorial derivation of Sasamoto's full five parameter expression for Z_n [16] (one of the six parameters can be set to one without loss of generality), but we are able to find many different specialisations and for example the result of [2]. See Table 1 for a summary of these results.

We also find simple expressions for the stationary distribution of certain configurations. For example:

Proposition 1. *If $\gamma = \delta = 0$ and $q = \alpha = \beta = 1$ then*

$$(5) \qquad \mathrm{P}_\infty(X = \bullet^k A) = \frac{(n-k+1)^k(n-k+1)!}{(n+1)!}$$

where A is any configuration in $\mathcal{B}_{n-k}$.

Another special case is:

Proposition 2. *If $\gamma = \delta = q = 0$ and $\alpha = \beta = 1/2$ then*

$$(6) \qquad \mathrm{P}_\infty(X) = \frac{1}{2^n},$$

independent of the configuration X.

In Section 2 we use the weight algebra to find recurrences for the normalisations and also describe a path model interpretation of the stationary states and normalisations. In Section 3 we define a larger Markov chain whose stationary distribution is related to that of the PASEP chain. In Sections 4 and 5 we give proofs of some of the propositions. A longer version of this paper which includes all the proofs is in preparation [5].

2. Algorithms, numbers and paths

In this section we set[1] $\eta = 1$ and study the normalisation using the weight algebra. By considering the position of first (leftmost) $\circ$ particle the algebra may be used to obtain a recursion to compute Z_n when either γ or δ are zero. Here we consider $\delta = 0$, and we note that similar results may be obtained for $\gamma = 0$.

2.1. **Recursions for Z_n.** Let $W_{n,k}$ be the sum of the weights of configurations in $\mathcal{B}_n$ that start with exactly k $\bullet$s and then at least one $\circ$ or are all black. Similarly let $W_{n,k,j}$ the sum of the weights of basic configurations in $\mathcal{B}_n$ that start with exactly k $\bullet$s then a single $\circ$ and then at least j $\bullet$s. Finally let $Z_{n,k}$ the sum of the weights of basic configurations in $\mathcal{B}_n$ that start with *at least* k $\bullet$s.

Proposition 3. *If $\delta = 0$ then $Z_{n,k}$, $W_{n,k}$ and $W_{n,k,j}$ satisfy the following equations*

$$(7a) \qquad W_{n,k} = \begin{cases} (Z_{n-1,0} + \gamma Z_{n,1})/\alpha & \text{if } k = 0 \\ W_{n-1,k-1} + Z_{n-1,k} + qW_{n,k-1,1} & \text{if } k \in [1, n-1] \\ W_{n-1,n-1}/\beta & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

$$(7b) \qquad W_{n,k,j} = \begin{cases} W_{n,k} & \text{if } j = 0 \\ (Z_{n-1,j} + \gamma Z_{n,j})/\alpha & \text{if } k = 0 \\ 0 & \text{if } k + j > n \\ Z_{n-1,k+j} + W_{n-1,k-1,j} + qW_{n,k+1,j-1} & \text{otherwise} \end{cases}$$

$$(7c) \qquad Z_{n,k} = \begin{cases} W_{n,k} + Z_{n,k+1} & \text{if } k \in [0, n-1] \\ W_{n,n} & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

These follow from application of the relations of the weights equations to basic configurations where the first $\bullet\circ$ pair occurs at position k for $k \in [1, n-1]$. The case $k = n$ corresponds to an all black particle configuration.

Using these recurrences we can compute $Z_n = Z_{n,0}$ for any n. We used this data to guess some of the results for specific α, β, γ, q, and then proved the conjectured forms. In Table 1 we give a list of some results like those in Proposition 1. Note we have used $\alpha = 1/(1+c)$ and $\beta = 1/(1+d)$. The result $q = c = d = 0$ appeared in [4, 13, 11]. The proofs will appear in [5].

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[1]One can show that setting $\eta = 1$ is equivalent to rescaling the other parameters in the stationary state distribution.

set to 0	set to 1	Z_n	$Z_{n,k}$
q, γ	c, d	2^{2n}	2^{2n-k}
q, γ, d, c		$\dfrac{1}{n+2}\dbinom{2n+2}{n+1}$	$\dfrac{k+2}{n+2}\dbinom{2n-k+1}{n-k}$
q, γ, d	c	$\dbinom{2n}{n}$	$\dbinom{2n-k+1}{n-k}$
γ	q	$\displaystyle\prod_{k=1}^{n}(c+d+k+1)$	$(n-k+d+1)^k Z_{n-k}$
	γ, q	$\displaystyle\prod_{i=1}^{n}((c+2)(i+d+1)-1)$	$(n-k+d+1)^k Z_{n-k}$

TABLE 1. Some results for Z_n when $\delta = 0$. Note that $c, d = 0$ implies that $\alpha, \beta = 1$ (respectively) and $c, d = 1$ implies that $\alpha, \beta = 1/2$ (respectively).

2.2. **Path models for $\gamma = \delta = 0$.** Another way to compute Z_n is to give a bijection between basic configurations counted by their weights and a family of weighted lattice paths. A similar result was given in [4] for $\gamma = \delta = q = 0$. In [14] the complete configurations can also be interpreted as paths when $\gamma = \delta = q = 0$. Here we generalise one of the approaches of [4] to get the result for general q.

Definition 1. *A Motzkin path [1] of length n is a sequence of vertices $p = (v_0, v_1, \ldots, v_n)$, with $v_i \in \mathbb{N}^2$ (where $\mathbb{N} = \{0, 1, \ldots\}$), with steps $v_{i+1} - v_i \in \{(1,1), (1,-1), (1,0)\}$ and $v_0 = (0,0)$ and $v_n = (n,0)$. A bicoloured Motzkin path is a Motzkin path in which each step $(1,0)$ is labelled by one of two colours.*

These paths can be mapped to words in the language $\{N, S, \overset{\circ}{E}, \overset{\bullet}{E}\}$ by mapping the steps $(1,1), (1,-1)$ to N and S, respectively and mapping the two different coloured horizontal steps to $\overset{\circ}{E}$ and $\overset{\bullet}{E}$. The height of a step $v_{i+1} - v_i$ is the y-coordinate of the vertex v_i. These heights imply the weights of the paths.

Definition 2. *Let $\mathcal{P}(n)$ be the set of bicoloured Motzkin paths of length n. The weight of the path in $\mathcal{P}(0)$ is 1 and the weight of any other path is the product of the weights of its steps. The weight of a step, p_k, starting at height h is given by:*

$$(8a) \qquad \text{if } p_k = N \quad then \quad w(p_k) = N_h = [h+1]_q$$

$$(8b) \qquad \text{if } p_k = \overset{\circ}{E} \quad then \quad w(p_k) = \overset{\circ}{E}_h = [h+1]_q + q^h c$$

$$(8c) \qquad \text{if } p_k = \overset{\bullet}{E} \quad then \quad w(p_k) = \overset{\bullet}{E}_h = [h+1]_q + q^h d$$

$$(8d) \qquad \text{if } p_k = S \quad then \quad w(p_k) = S_h = [h]_q + q^h(1+c)(1+d) - q^{h-1}cd$$

where $[h]_q = 1 + q + \ldots + q^{h-1}$ for $h > 0$ and $[h]_q = 0$ for $h \leq 0$.

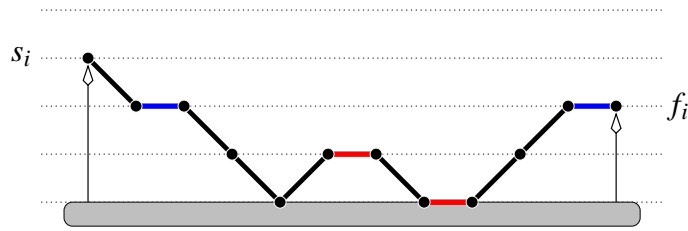An example of a path in $\mathcal{P}(11)$ is given in Figure 3.

FIGURE 3. A path in $\mathcal{P}(11)$ which corresponds to the word $NN\overset{\circ}{E}SSN\overset{\bullet}{E}S\overset{\bullet}{E}NS$.

Let us define a mapping $\theta : \mathcal{P}(n) \mapsto \mathcal{B}_n$ where each bicoloured Motzkin path is mapped to a basic configuration such that each step S or $\overset{\circ}{E}$ is mapped to a white particle and each step N or $\overset{\bullet}{E}$ is mapped to a black particle. This mapping is many-to-one and we denote $\theta^{-1}(X)$ as the set of all paths that map to X.

Theorem 2. *When $\gamma = \delta = 0$ the weight of a basic configuration, X, is given by*

$$(9) \qquad W(X) = \sum_{p \in \theta^{-1}(X)} w(p)$$

and

$$(10) \qquad Z_n = \sum_{p \in \mathcal{P}(n)} w(p)$$

This Theorem gives a combinatorial derivation of the stationary distribution that does not make use of the matrix product Ansatz which was used to obtain the results in [11, 16]. The proof is given in Section 5 and works by showing that the the weight of the paths obeys the same equations as the weight of the basic configurations.

We may specialise the above result to get the corollary:

Corollary 3. *If $c = d = 0$ ($\alpha = \beta = 1$) the paths $\mathcal{P}$ correspond to bicoloured Motzkin Paths of length n where the weight of a step starting at height i is $[i+1]_q$.*

 * *If $q = 0$ then $Z_n = C_{n+1}$, where C_n is the nth Catalan number.*
 * *If $q = 1$ then $Z_n = (n+1)!$ (see [3, 17]).*

This can be linked to well-known results on the q-enumeration of permutations [7, 8]. Also if $c = d = 1$ (i.e. $\alpha = \beta = 1/2$) and $q = 0$, only the paths that are made of east steps have non zero weight. Each such path has weight 2^n. Therefore $W(X) = 2^n$ for any $X \in \mathcal{B}_n$ and $Z_n = 4^n$ in that case — this is Proposition 2.

3. Stationarity and Marked configurations

In this section we define a larger Markov chain, the M-PASEP chain, we which use to study the stationary distribution of the original PASEP chain. In particular we show that the stationary distributions of the M-PASEP and the PASEP chains are simply related.

3.1. **Marked configurations.** We enlarge the state space of the original chain by adding "embellished" configurations which we call "marked configurations" (hence the "M" in M-PASEP).

Definition 3. *A marked configuration (X, i, D) of size n consists of a basic configuration $X \in \mathcal{B}_n$, an integer $i \in [0, n]$ and a "direction" $D \in \{L, R, S, N\}$. The directions are L for "left", R for "right", S for "stable" and N for "nothing". The possible values of D depend on the values of X and i. All triples satisfying the following conditions occur (depending on the structure of X):*

 * *for all X and all $i \in [0, n]$ $D = N$.*
 * *if $X = \circ A$ then $i = 0$ and $D \in \{R, S\}$.*
 * *if $X = \bullet A$ then $i = 0$ and $D \in \{S\}$.*
 * *if $X = A\circ$ then $i = n$ and $D \in \{S\}$.*
 * *if $X = A\bullet$ then $i = n$ and $D \in \{L, S\}$.*
 * *if $X = A \bullet \circ B$ and $|A| \in [0, n-2]$ then $i = |A| + 1$ and $D \in \{L, R, S\}$.*
 * *if $X = A \circ \bullet B$ and $|A| \in [0, n-2]$ then $i = |A| + 1$ and $D \in \{S\}$.*

We define a projection $U(M) = X$ from a marked configuration $M = (X, i, D)$ to the corresponding unmarked configuration, X. We denote the set of all marked configurations of size n by $\mathcal{M}_n$.

Here are all the marked configurations for $n = 2$:

$$\left\{ \begin{array}{llllllll} (\circ\circ, 0, R), & (\circ\circ, 0, S), & (\circ\circ, 0, N), & (\circ\circ, 1, N), & (\circ\circ, 2, S), & (\circ\circ, 2, N), \\ (\circ\bullet, 0, R), & (\circ\bullet, 0, S), & (\circ\bullet, 0, N), & (\circ\bullet, 1, S), & (\circ\bullet, 1, N), & (\circ\bullet, 2, L), & (\circ\bullet, 2, S), & (\circ\bullet, 2, N), \\ (\bullet\circ, 0, S), & (\bullet\circ, 0, N), & (\bullet\circ, 1, L), & (\bullet\circ, 1, R), & (\bullet\circ, 1, S), & (\bullet\circ, 1, N), & (\bullet\circ, 2, S), & (\bullet\circ, 2, N), \\ (\bullet\bullet, 0, S), & (\bullet\bullet, 0, N), & (\bullet\bullet, 1, N), & (\bullet\bullet, 2, L), & (\bullet\bullet, 2, S), & (\bullet\bullet, 2, N) \end{array} \right\}$$

3.2. The M-PASEP chain. We define the M-PASEP chain, C, whose states are both basic and marked configurations. For any $X \in \mathcal{B}_n$ and $M \in \mathcal{M}_n$ the transition probabilities between states in the chain are given by

$$\text{(11a)} \qquad \qquad \text{if} \quad U(M) = X \quad \text{then} \quad C_{X,M} = \frac{W(M)}{(n+1)W(X)}$$

$$\text{(11b)} \qquad \qquad \text{if} \quad U(T(M)) = X \quad \text{then} \quad C_{M,X} = 1$$

$$\text{(11c)} \qquad \qquad \text{otherwise} \qquad C_{M,X} = C_{X,X} = C_{M,M} = 0,$$

where $W(M)$ is the weight of a marked configuration which we define below and $T : \mathcal{M}_n \mapsto \mathcal{M}_n$ is a weight preserving bijection given in Definition 5 below.

Definition 4. *The weight of a marked configuration M is defined in terms of the weights of unmarked configurations as follows:*

$$\text{(12a)} \qquad W(\circ A, 0, R) \ = \ W(A)$$

$$\text{(12b)} \qquad W(\circ A, 0, S) \ = \ W(\bullet A, 0, S) = \gamma W(\bullet A)$$

$$\text{(12c)} \qquad W(\circ A, 0, N) \ = \ (1 - \alpha)W(\circ A)$$

$$\text{(12d)} \qquad W(A\bullet, n, L) \ = \ W(A)$$

$$\text{(12e)} \qquad W(A\bullet, n, S) \ = \ W(A\circ, n, S) = \delta W(A\circ)$$

$$\text{(12f)} \qquad W(A\bullet, n, N) \ = \ (1 - \beta)W(A\bullet)$$

$$\text{(12g)} \qquad W(A \bullet \circ B, i, R) \ = \ W(A \circ B) \quad \text{if} \quad A \in \mathcal{B}_{i-1}$$

$$\text{(12h)} \qquad W(A \bullet \circ B, i, L) \ = \ W(A \bullet B) \quad \text{if} \quad A \in \mathcal{B}_{i-1}$$

$$\text{(12i)} \qquad W(A \bullet \circ B, i, S) \ = \ W(A \circ \bullet B, i, S) = qW(A \circ \bullet B) \quad \text{if} \quad A \in \mathcal{B}_{i-1}$$

$$\text{(12j)} \qquad W(A \bullet \circ B, i, N) \ = \ (1 - \eta)W(A \bullet \circ B) \quad \text{if} \quad A \in \mathcal{B}_{i-1}$$

$$\text{(12k)} \qquad \textit{Otherwise} \ \ W(X, i, N) \ = \ W(X).$$

Note that the chain alternates between marked and basic configurations. The state graph of the chain C for $n = 2$ is shown in Figure 4. The weights of marked and unmarked configurations are simply related and from Definition 4 and Theorem 1, we get:

Lemma 4. *For all $X \in \mathcal{B}_n$ and $i \in [0, n]$:*

$$\text{(13)} \qquad \qquad \sum_D W(X, i, D) = W(X)$$

and

$$\text{(14)} \qquad \qquad \sum_{M \ : \ U(M) = X} W(M) = (n + 1)W(X).$$

The stationary distribution of the PASEP chain is simply related to that of the new chain C:

Proposition 4. *The conditional stationary probability of finding the M-PASEP chain, C, in a state Y given that it is in an unmarked state is related to the stationary distribution of PASEP chain by*

$$\text{(15)} \qquad \qquad \mathrm{P}_\infty^C(Y \ \text{given that} \ Y \in \mathcal{B}_n) = \mathrm{P}_\infty(Y).$$

This proposition relates the stationary distribution of the two chains, but does not tell us what the distributions are. We prove the above proposition in the next section and also expand it to give the proof of Theorem 1.
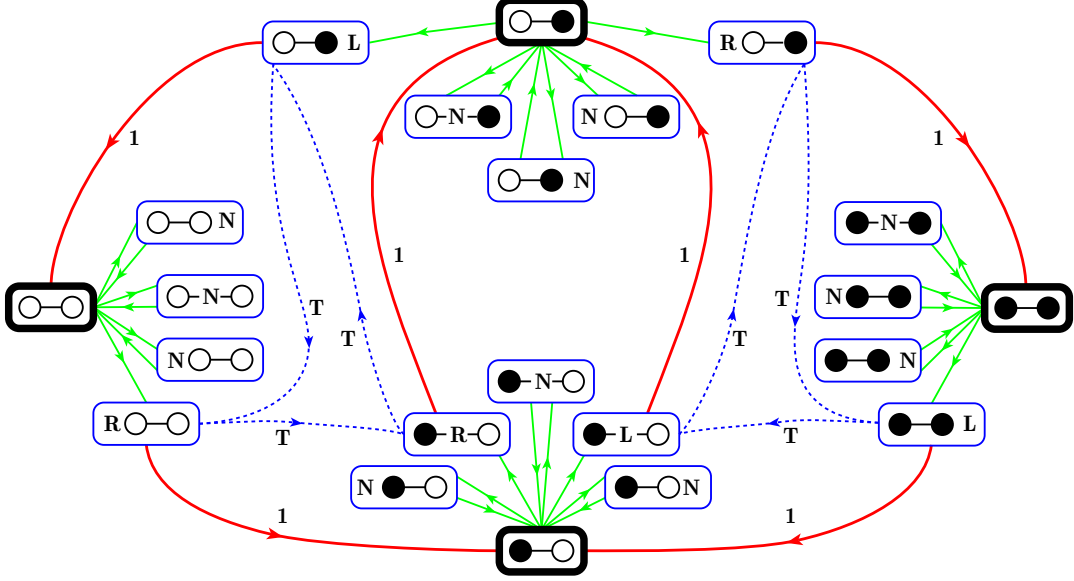
FIGURE 4. The M-PASEP chain C for $n = 2$. Each marked state (X, i, D) is written as the configuration X with its direction D in position i. The short thin arcs have probabilities given by equations (11). The dashed lines show the action of the bijection T.

4. PROOF OF STATIONARITY

To prove stationarity we need two major ingredients. The first is a bijection between states on the M-PASEP chain and the second is lemma which gives conditions under which a Markov chain may be altered while leaving its stationary distribution essentially unchanged.

4.1. The bijection.

Definition 5. *We define the mapping $T : \mathcal{M}_n \to \mathcal{M}_n$. Let $M = (X, i, D) \in \mathcal{M}_n$. We first note that if $D = N$ then $T(M) = M$. Otherwise we define the bijection by the following algorithm:*

* *if $i = 0$ then the colour of the first particle is changed and*
  ◇ *if $D = S$ then i and D are unchanged, or*
  ◇ *if $D = R$ then "shuffle M right".*
  ◇ *Note that $M = (X, 0, L)$ cannot occur.*
* *if $i \in [1, n-1]$ then swap the ith and $i+1^{th}$ particles and*
  ◇ *if $D = S$ then i and D are unchanged, or*
  ◇ *if $D = L$ then "shuffle M left".*
  ◇ *if $D = R$ then "shuffle M right".*
* *if $i = n$ then the colour of the last particle is changed and*
  ◇ *if $D = S$ then i and D are unchanged, or*
  ◇ *if $D = L$ then "shuffle M left".*
  ◇ *Note that $M = (X, n, R)$ cannot occur.*

where "shuffle M right" means

* *choose the minimal $j \in (i, n)$ such that the jth particle is black and the $(j+1)^{th}$ particle is white.*
* *if such a j exists then set $i = j$ and $D = R$*
* *otherwise set $i = n$ and $D = L$.*

and "shuffle M left" means

* *choose the maximal $j \in [0, i)$ such that the jth particle is black and the $(j+1)^{th}$ particle is white.*
* *if such a j exists then set $i = j$ and $D = L$*
* *otherwise set $i = 0$ and $D = R$.*

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Some examples of this mapping are given below

$$
\begin{aligned}
T(\bullet \circ \bullet \bullet \circ \bullet, 0, S) &= (\circ \circ \bullet \bullet \circ \bullet, 0, S) \\
T(\bullet \circ \bullet \bullet \circ \bullet, 4, R) &= (\bullet \circ \bullet \circ \bullet \bullet, 6, L) \\
T(\bullet \circ \bullet \bullet \circ \bullet, 4, L) &= (\bullet \circ \bullet \circ \bullet \bullet, 1, L).
\end{aligned}
$$

The bijection is weight preserving:

Proposition 5. *The mapping T defined in Definition 5 is a bijection from $\mathcal{M}_n$ to itself and $\forall M \in \mathcal{M}_n$*

(16) $\qquad\qquad either \quad T(M) = M \quad or \quad T^2(M) = M \quad or \quad T^{n+1}(M) = M.$

The bijection is also weight-preserving: $W(T(M)) = W(M)$.

Sketch of Proof. It follows directly from the definition of the bijection that if $D = N$ then $T(M) = M$ and that if $D = S$ then $T^2(M) = M$. The weight is invariant in both cases. To show that if $D = L, R$ then $T^{n+1}(M) = M$, we use a one-to-n+1 weight preserving correspondence between the basic configurations in $\mathcal{B}_{n-1}$ and marked configurations in $\mathcal{M}_n$ with $D = L, R$.

Start with a configuration X in $\mathcal{B}_{n-1}$. Suppose that the white particles are located at positions $j_1 < j_2 < \ldots < j_i$ and that the black particles are located at positions $j_{i+2} > j_{i+3} > \ldots > j_n$. Now create $n+1$ marked configuration $M_0, M_1, \ldots, M_n$ as follows :

* $M_0 = (\circ X, 0, R)$.
* $M_l = (X_l, j_l, R)$ where X_l is obtained from X by replacing the $\circ$ located at j_l by $\bullet \circ$ with $1 \le l \le i$.
* $M_{i+1} = (X\bullet, n, L)$
* $M_l = (X_l, j_l, L)$ where X_l is obtained from X by replacing the $\bullet$ located at j_l is replaced by $\bullet \circ$ with $i + 2 \le l \le n$.

One can check that $T(M_i) = M_{i+1}$, $0 \le i \le n - 1$ and $T(M_n) = M_0$. Moreover, the definition of the weight of marked configurations implies that $W(M_i) = W(X)$ for $i = 0 \ldots n$ and so the weight is invariant under T.

To illustrate this consider $X = \circ\bullet$. We obtain the configurations $\{(\circ \circ \bullet, 0, R), (\bullet \circ \bullet, 1, R), (\circ \bullet \bullet, L, 3), (\circ \bullet \circ, L, 2)\}$. One may verify that these form a 4-cycle under T and that all the configurations have weight equal to that of the configuration $\circ\bullet$. $\qquad\square$

4.2. **Enriching a Markov chain.** In order to show that the stationary state of M-PASEP chain C is simply related to that of the PASEP we use the following lemma:

Lemma 5. *Consider a Markov chain C_1 with a transition from a state a to a state b with probability r. We replace the arc $\overrightarrow{ab}$ by the subgraph $\mathcal{H}$ as shown in Figure 5 to create a new chain C_2. Let H be the set of vertices in $\mathcal{H} \setminus \{a, b\}$. If:*

* *the weighted sum over all directed spanning trees in $\mathcal{H}$ rooted at b is equal to r, and*
* *the weighted sum over all directed spanning forests of $\mathcal{H}$ that contain 2 components (one rooted at a and one rooted at b) is equal to 1,*

then $\Pr_\infty^{C_2}(x \text{ given that } x \notin H) = \Pr_\infty^{C_1}(x)$, where $\Pr_\infty^{C_i}(x)$ is the stationary state probability of finding chain C_i in state x.

This follows from the Markov-Tree Theorem [6] and can be proved by applying the Matrix-tree theorem to a transition matrix, though it can also be proved combinatorially (see [5]).
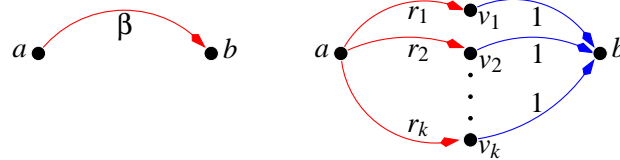

FIGURE 5. Replace the arc from a to b by a subgraph $\mathcal{H}$.

4.3. **The stationary distribution.** Using Lemma 5 and the definitions of the chains C and P, it is then easy to check that Proposition 4 holds. This shows that the two stationary distributions are simply related but does not tell us what they are. The following proposition does this and together with Proposition 4 implies Theorem 1.

Proposition 6. *Let Y be a state in the M-PASEP chain C, then:*

$$(17a) \qquad P_\infty^C(Y \text{ given that } Y \in \mathcal{B}_n) = \frac{W(Y)}{Z_n}, \text{ and}$$

$$(17b) \qquad P_\infty^C(Y \text{ given that } Y \in \mathcal{M}_n) = \frac{W(Y)}{(n+1)Z_n}.$$

where $W(Y)$ is the weight defined by the equations in Theorem 1 and Definition 4.

Proof. To prove stationarity we need to show that the above probabilities are unchanged under the action of the transitions. Denote the probability of finding the chain C in state Y at time t by $\Pr(C(t) = Y)$.

Now suppose that

$$(18a) \qquad \Pr(C(t) = Y \text{ given that } Y \in \mathcal{B}_n) = \frac{W(Y)}{Z_n}, \text{ and}$$

$$(18b) \qquad \Pr(C(t) = Y \text{ given that } Y \in \mathcal{M}_n) = \frac{W(Y)}{(n+1)Z_n}$$

Then because the chain alternates between basic and marked states we have

$$
\begin{aligned}
\Pr(C(t+1) = Y \in \mathcal{B}_n) &= \sum_{M \,:\, U(T(M))=Y} \Pr(C(t) = M) \\
&= \sum_{M \,:\, U(T(M))=Y} \frac{1}{n+1} \frac{W(M)}{Z_n} \\
&= \sum_{M \,:\, U(T(M))=Y} \frac{1}{n+1} \frac{W(T(M))}{Z_n} \\
&= \sum_{M \,:\, U(M)=Y} \frac{1}{n+1} \frac{W(M)}{Z_n} \\
(19) \qquad &= \frac{W(Y)}{Z_n}
\end{aligned}
$$

Above we have used Lemma 4 and the fact that the bijection T is weight preserving, Proposition 5. Similarly we get:

$$
\begin{aligned}
\Pr(C(t+1) = Y \in \mathcal{M}_n) &= \frac{W(M)}{(n+1)W(U(Y))} \Pr(C(t) = U(Y)) \\
(20) \qquad &= \frac{W(Y)}{(n+1)W(U(Y))} \frac{W(U(Y))}{Z_n} = \frac{W(Y)}{(n+1)Z_n}.
\end{aligned}
$$

This completes the proof. $\qquad \square$

5. Weight-equations and lattice paths

In this section we prove Theorem 2 by showing that the weight of the paths in $\mathcal{P}(n)$ and the weight of the basic configurations satisfy the same equations. Let

$$(21) \qquad Q(X) = \sum_{p \in \theta^{-1}(X)} w(p)$$

We work by showing that $Q(X)$ satisfies the same equations as $W(X)$.

Recall that given a path p, $\theta(p)$ is the basic configuration such that each $\overset{\circ}{E}$ and S step is changed to $\circ$ and each $\overset{\bullet}{E}$ and N step is changed to $\bullet$. Also, recall that N_i (resp. S_i, $\overset{\circ}{E_i}$, $\overset{\bullet}{E_i}$) is the weight of a North-East (resp. South East, East $\overset{\circ}{E}$, East $\overset{\bullet}{E}$) step starting at height i.

* If $X \in \mathcal{B}_0$ then $W(X) = 1$ and it follows that $Q(X) = 1$.
* If $X = \circ A$ then $W(X) = (1+c)W(A)$ by Theorem 1. The first step of a path $p \in \theta^{-1}(X)$, must be $\overset{\circ}{E}$ (it cannot start with a South-East step) and its weight is $1+c$. Removing this step gives:

(22)
$$Q(\circ A) = \sum_{\substack{p \in \mathcal{P}(n) \\ \theta(p) = \circ A}} w(p) = (1+c) \sum_{\substack{p' \in \mathcal{P}(n-1) \\ \theta(p') = A}} w(p') = (1+c)Q(A).$$

* Let $X = A\bullet$ then $W(X) = (1+d)W(A)$. The last step of a path $p \in \theta^{-1}(X)$, must be $\overset{\bullet}{E}$ (it cannot end with a North-East step) and its weight is $1+d$. Removing this step gives:

(23)
$$Q(A\bullet) = \sum_{\substack{p \in \mathcal{P}(n) \\ \theta(p) = A\bullet}} w(p) = (1+d) \sum_{\substack{p' \in \mathcal{P}(n-1) \\ \theta(p') = A}} w(p'). = (1+d)Q(A).$$

* Let $X = A \bullet \circ B$ where $A \in \mathcal{B}_{k-1}$, then $W(X) = W(A \bullet B) + W(A \circ B) + qW(A \circ \bullet B)$. There are four possibilities that we must consider — let p be a path such that $\theta(p) = X$ such that $(p_k, p_{k+1}) = (N, S)$, $(\overset{\bullet}{E}, \overset{\circ}{E})$, $(\overset{\bullet}{E}, S)$ or $(N, \overset{\circ}{E})$. Let $U = (p_1, \ldots p_{k-1})$ and $V = (p_{k+2}, \ldots, p_n)$.

  [i] Consider $\hat{p} = UNSV$ and $\bar{p} = U\overset{\bullet}{E}\overset{\circ}{E}V$. Form three new paths $p' = U\overset{\circ}{E}V$, $p'' = U\overset{\bullet}{E}V$ and $p''' = U\overset{\circ}{E}\overset{\bullet}{E}V$. Note that $\theta(p') = A \circ B$, $\theta(p'') = A \bullet B$ and $\theta(p''') = A \circ \bullet B$. The heights of vertices in U and V are the same in all of these paths and hence the weight contribution from U and V is the same in all of these paths and can be factored out. Hence the equation

(24)
$$w(\hat{p}) + w(\bar{p}) = w(p') + w(p'') + qw(p''')$$

  is true providing

(25)
$$N_i S_{i+1} + \overset{\bullet}{E_i}\overset{\circ}{E_i} = \overset{\circ}{E_i} + \overset{\bullet}{E_i} + q(S_i N_{i-1} + \overset{\circ}{E_i}\overset{\bullet}{E_i})$$

  is true. The latter can be verified directly from the definition of the weights.

  [ii] Consider $p = UN\overset{\circ}{E}V$ and form $p' = UNV$ and $p'' = UENV$. Note that $\theta(p') = A \bullet B$ and $\theta(p'') = A \circ \bullet B$. Again the heights of vertices in U and V are the same in all of these paths and so the contribution of U and V to the weights of these paths may be factored out. Hence the equation

(26)
$$w(p) = w(p') + qw(p'')$$

  is true providing

(27)
$$N_i \overset{\circ}{E_i} + 1 = N_i + q\overset{\circ}{E_i}N_i$$

  is true, which can be verified from the definition of the weights.

  [iii] Consider $p = U\overset{\bullet}{E}SV$. This case follows similarly to the previous case.

Taking the above and summing over all A and B gives the following

(28)
$$\sum_{\substack{p \in \mathcal{P}(n) \\ \theta(p) = A\bullet\circ B}} w(p) = \sum_{\substack{p' \in \mathcal{P}(n-1) \\ \theta(p') = A\bullet B}} w(p') + \sum_{\substack{p' \in \mathcal{P}(n-1) \\ \theta(p') = A\circ B}} w(p') + q \sum_{\substack{p' \in \mathcal{P}(n) \\ \theta(p') = A\circ\bullet B}} w(p').$$

Which is $Q(A \bullet \circ B) = Q(A \circ B) + Q(A \bullet B) + qQ(A \circ \bullet B)$.

$\square$

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