A Decomposition of the Schur Functions into Non-Symmetric Schur Functions

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Abstract

The Schur functions, $s_\lambda(x)$, form a basis for the vector space of symmetric functions. Recently Haglund, Haiman and Loehr have derived a combinatorial formula for nonsymmetric Macdonald polynomials, which gives a new decomposition of the Macodnald polynomial into nonsymmetric components. Letting $q = t = 0$ in this identity implies $s_\lambda(x) = \sum_\mu NS_\mu(x)$, where the sum is over all rearrangements μ of the partition λ. In this paper, we exhibit a bijection between semi-standard Young tableaux (SSYT) and skyline fillings to give a bijective proof of the formula.

Resumé en Français

Les fonctions de Schur, $s_\lambda(x)$, forment une base de l'espace vectoriel de fonctions symétriques. Les résultats récents de J. Haglund permettent d'introduire un objet nouveau qui est utilisé pour décomposer les fonctions de Schur en fonctions nonsymétriques, $NS_\mu(x)$, numérotées par les compositions au lieu des partitions. Le théorème principal de cet article (qui était conjecturé par J. Haglund) dit que $s_\lambda(x) = \sum_\mu NS_\mu(x)$, sommée sur toutes les transpositions μ de λ. Dans cet article, nous montrons une bijection entre les tableaux de Young semi-standards (SSYT) et les remplissages d'horizon pour démontrer la conjecture.

1 Introduction

A *symmetric function of degree* n over a commutative ring R (with identity) is a formal power series $f(x) = \sum_{\alpha} c_\alpha x^\alpha$, where α ranges over all weak compositions of n (of infinite length), $c_\alpha \in R$, x^α stands for the the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$, and $f(x_{\omega(1)}, x_{\omega(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation ω of the positive integers, $\mathbb{P}$. Many different bases for the vector space of symmetric functions are known. One important basis is the Schur functions.

The *Schur function* $s_\lambda = s_\lambda(x)$ of shape λ in variables $x = (x_1, x_2, \ldots)$ is the formal power series $s_\lambda = \sum_T x^T$, summed over all Semi-Standard Young Tableaux of shape λ. A Semi-Standard Young Tableaux is formed by first placing the parts of λ into rows of squares, where the i^{th} row has λ_i squares, called *cells*. This diagram, called the Young (or Ferrers) diagram, is drawn in the first quadrant, French style, as in [HHL$^+$]. Then each of these cells is assigned a positive integer in such a way that the row entries are weakly increasing and the column entries are strictly increasing. Thus, the values assigned to the cells of λ collectively form the multiset $\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$, for some n where a_i is the number of times i appears in T. Here, $x^T = \prod_{i=1}^n x_i^{a_i}$. See [Sta99] for a more detailed discussion of symmetric functions and the Schur functions in particular.

The Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ are a special class of symmetric functions which contain a vast array of information. Macdonald [Mac88] introduced them and conjectured that their expansion in terms of Schur polynomials should have positive coefficients. A combinatorial formula for the Macdonald polynomials was conjectured by Haglund and proved by Haglund, Haiman, and Loehr [HHL04].

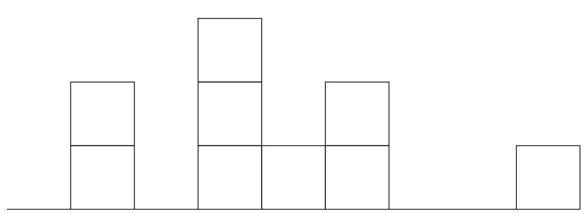
Building on this work, Haglund described [Hag04b] a conjectured combinatorial formula for the non-symmetric Macdonald polynomials. As a consequence of this conjecture he gives a set of objects that decompose the Schur functions into non-symmetric functions indexed by compositions of n instead of partitions of n. They involve statistics generalizing those described in [HHL04]. The weighted sums of these objects are called the *non-symmetric Schur functions*, NS_λ. A composition μ of n is called a *rearrangement* of a partition λ if it consists of n parts such that when the parts are arranged in descending order, the i^{th} part equals λ_i, for all i. Haglund conjectured that the sum of the non-symmetric Schur functions over all rearrangements of a given partition λ is equal to the ordinary Schur function s_λ. In this paper, we prove:

Theorem 1 $\sum_{\lambda'} NS_{\lambda'}(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n)$, *where the sum is over all rearrangements* λ' *of* λ.

This result gave evidence that Haglund's conjectured formula for the non-symmetric Macdonald polynomials was correct. Haglund, Haiman, and Loehr recently proved this formula [HHL05]. Setting $q = t = 0$ in the formula gives a non-bijective proof of the above theorem.

2 Combinatorial Definition of the non-symmetric Schur Functions

A *composition Young diagram of* n is a figure consisting of n *cells* arranged in n columns. A column may contain anywhere from 0 to n cells, and the number of cells in a column is called the *height* of that column. This means that a composition Young diagram is simply the Young diagram of a composition of n into n parts, allowing zeros. The composition 9=0+2+0+3+1+2+0+0+1 is shown in the example below.

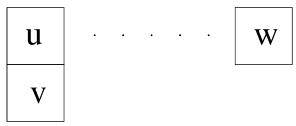


The composition Young diagram for $\lambda = (0, 2, 0, 3, 1, 2, 0, 0, 1)$

A *filling,* σ, of a composition Young diagram, λ, is a function $\sigma : \lambda \to \mathbb{Z}_+$, which we picture as an assignment of positive integer entries to the cells of λ. We consider the 0^{th} row to consist of cells numbered from 1 to n in strictly increasing order. Let $\sigma(i)$ denote the entry in the ith square of the composition Young diagram encountered if we read across rows from left to right, beginning at the highest row and working downwards.

To define the non-symmetric Schur functions, we need the statistics $\mathrm{maj}(\sigma, \lambda)$ and $\mathrm{inv}(\sigma, \lambda)$. As in [Hag04a], a *descent* of σ is a pair of entries $\sigma(u) > \sigma(v)$, where the cell u is directly above v. In other words, $v = (i, j)$ and $u = (i + 1, j)$, where the i^{th} coordinate denotes the height of cell v and the j^{th} coordinate denotes one less than the number of cells to the left of v. Define $\mathrm{Des}(\sigma) = \{u \in \lambda : \sigma(u) > \sigma(v)$ is a descent$\}$.

Three cells $u, v, w \in \lambda$ form a *triple of type A* if they are situated as shown below,

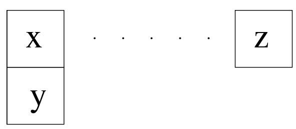


where u and w are in the same row, possibly with cells between them, and the column containing u and v has height greater than or equal to the height of the column containing w.

Define for $x, y \in \mathbb{Z}_+$

$$I(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

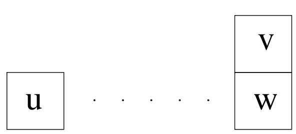
Let σ be a composition filling and let x, y, z be the entries of σ in the cells of a type A triple (u, v, w):



Then the triple (u, v, w) is an *inversion triple of type A* if and only if $I(x, z) + I(z, y) - I(x, y) = 1$.

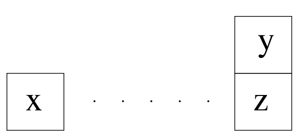
The *reading order* of a filling is an ordering of its cells beginning with the top row and listing the cells from left to right, travelling down, row by row, to the bottom row. Define a filling σ to be *standard* if it is a bijection $\sigma : \mu \cong \{1,...,n\}$. The *standardization* of a composition filling is the unique standard filling ξ such that $\sigma \circ \xi^{-1}$ is weakly increasing, and for each x in the image of σ, the restriction of ξ to $\sigma^{-1}(\{x\})$ is increasing with respect to the reading order. Therefore the triple (u, v, w) is an inversion triple of type A if and only if after standardization, the ordering from smallest to largest of the entries in cells u, v, w induces a counter-clockwise orientation.

Similarly, three cells $u, v, w \in \lambda$ form a *triple of type B* if they are situated as shown below,



Here u and w are in the same row (possibly row 0) and the column containing v and w has greater height than the column containing u.

Let σ be a composition filling and let x, y, z be the entries of σ in the cells of a type B triple (u, v, w):

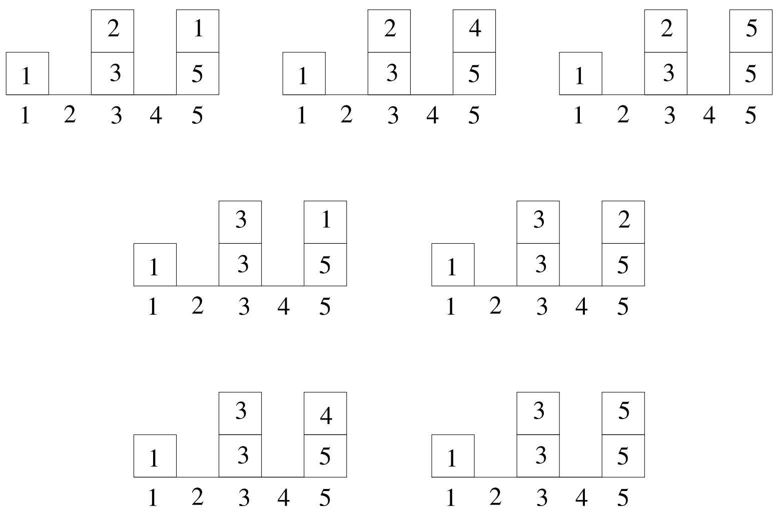


Then the triple (u, v, w) is an *inversion triple of type B* if and only if $I(y, x) + I(x, z) - I(y, z) = 1$. In other words, the triple (u, v, w) is an inversion triple of type B if and only if after standardization, the ordering from smallest to largest of the entries in cells u, v, w induces a clockwise orientation.

Denote by *semi-standard skyline filling* any composition filling K such that $\mathrm{Des}(K) = \emptyset$ and every triple is an inversion triple. These conditions arise by setting $q = t = 0$ in the combinatorial formula for the non-symmetric Macdonald polynomials.

Definition 1 *Let λ be a composition of n into n parts, where some of the parts could be equal to zero. The non-symmetric Schur function $NS_\lambda = NS_\lambda(x)$ in the variables $x = (x_1, x_2, ..., x_n)$ is the formal power series $NS_\lambda(x) = \sum_K x^K$ summed over all semi-standard skyline fillings K of composition λ. Here, $x^K = \prod_{i=1}^n x_{\sigma_i}$ is the weight of σ.*

As an example, take $\lambda = (1, 0, 2, 0, 2)$. The skyline fillings with no descents such that every triple is an inversion triple are as follows:



$\lambda = (1, 0, 2, 0, 2)$

Therefore, $NS_\lambda = x_1^2 x_2 x_3 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_5^2 + x_1^2 x_3^2 x_5 + x_1 x_2 x_3^2 x_5 + x_1 x_3^2 x_4 x_5 + x_1 x_3^2 x_5^2$.

3 Map from SSYTs to Skyline Fillings

The purpose of this paper is to prove that the sum of the non-symmetric Schur functions over all rearrangements of a given partition μ is equal to the ordinary Schur function s_μ. (Here, a *rearrangement* of a partition μ of n is a composition of n into n parts such that when these parts are arranged in decreasing order the partition μ is recovered.) To do this, we must exhibit a bijection between semi-standard young tableaux and skyline fillings which preserves the number of objects with each weight.

Begin with an arbitrary semi-standard young tableau T of shape μ, where $\mu \vdash n$. The cells are labeled by some multiset of positive integers $\{1^{a_1}, 2^{a_2}, ..., n^{a_n}\}$. (Note that some of the a_i might equal 0.) Map the value in each cell to a new value as follows: send α to $n - \alpha + 1$. Call the new filling T'. (Here, a filling of shape μ is a function $\sigma : \mu \to \mathbb{Z}_+$, as defined in [HHL04].) Notice that the column entries are now strictly decreasing and the row entries are weakly decreasing. Let T'_i be the set consisting of the entries of the i^{th} column.

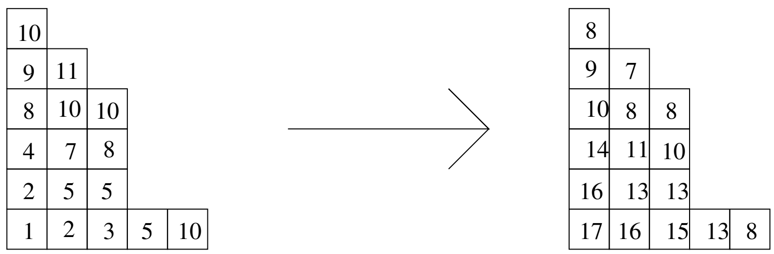
Place the elements of T'_i on top of row $i - 1$ as follows:

Begin with the largest member, α_1, of T'_i. Find the left-most entry of row $i-1$ that is greater than or equal to α_1. We know such an element exists, since in the tableau, the entry to the immediate left of α_1

is greater than or equal to α_1. Place α_1 on top of this element. Next place the second-largest member, α_2 of T_i' in the same way. (Again, an entry greater than α_2 exists because the entry immediately to the left of α_2 in the tableau is greater than or equal to α_2.) Continue in this manner until all the elements of T_i' have been placed. Any remaining cell of row $i-1$ has no cell directly above it.

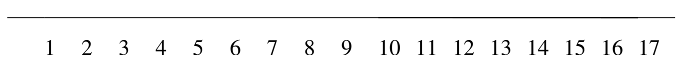
Following this process for each column of T produces a filling of a composition Young diagram, as in the example below:

Example 1 *Begin with a Semi-Standard Young Tableau of shape λ =(5,3,3,3,2,1) (note that $\lambda \vdash$ 17) as pictured below and apply the map described above that sends each of the numbers, α, from 1 through 17, to $17-\alpha+1$.*



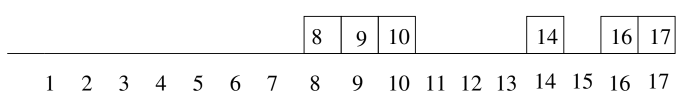
SSYT mapping

Next examine the empty composition filling:



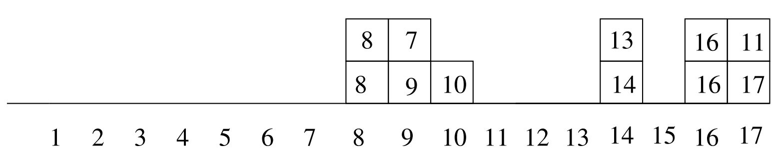
The Empty Composition Filling

We must assign the numbers T_1 ={8,9,10,14,16,17} to the first row of our composition filling according to the map defined above. The figure below shows the placement of these numbers onto the empty composition filling:



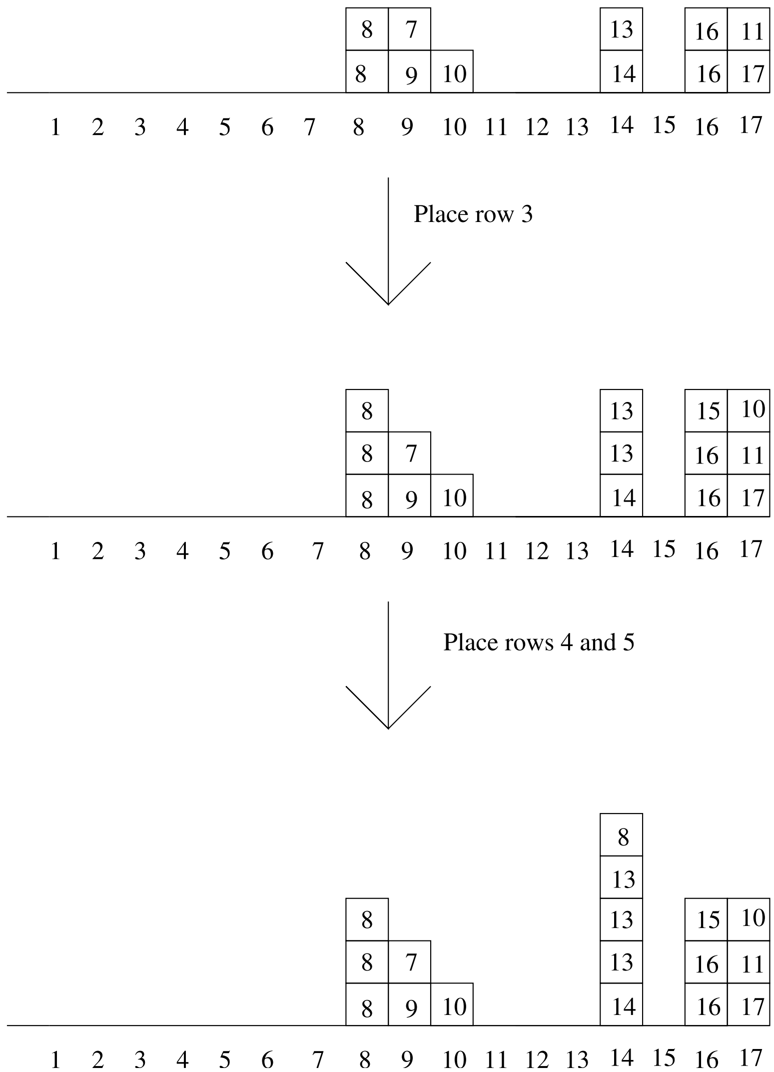
Placement of the first row

The following figure shows the placement of the second row:



Placement of the second row

Next we demonstrate the process of placing each of the additional 3 rows:



Lemma 1 *Once the entries of row $i-1$ have been placed, the arrangement of the elements of T_i' on top of row $i-1$ forms the i^{th} row of a skyline filling, and this placement procedure is the only placement of the elements of T_i' which yields a skyline filling.*

We must show that the following are true:

1. This process yields a skyline filling.

2. This process is the only way to obtain a skyline filling with the given row entries.

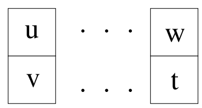
Step 1: This process yields a skyline filling.

Proof. By construction, the filling has no descents. Therefore, we must show that all triples are inversion triples. Recall that we can get an non-inversion triple from either of the following two types of triples of cells:

Type A Type B

In type A, the column containing u and v is weakly taller than the column containing w while in type B, the column containing v and w is taller. After standardization, a type A non-inversion implies a clockwise ordering when the cells are ordered from smallest to largest, and a type B non-nversion implies a counter-clockwise ordering when the cells are ordered from smallest to largest.

First check for type A non-inversion triples. They must look like the cell configuration pictured below, where the column containing u and v has height greater than or equal to the height of the column containing w and t:



Here, we must have $u \leq v$. Therefore, to get a non-inversion triple, we must have $u \leq w \leq v$. Since the elements of T_i' are all distinct, this implies that $u < w$. But then w would have been placed before u. Since $w \leq v$, w would have been placed on top of v or on top of some entry to the left of v. So this configuration would not happen. Therefore, there are no type A non-inversion triples.

Next check for type B non-inversion triples. This can occur in two ways. Either the left cell in the triple has a cell on top of it (Case 1) or the left cell does not have a cell on top of it (Case 2).

Case 1 Case 2

We know that $y \leq z$. Thus, to get an non-inversion triple, we must have $y \leq x \leq z$. Standardization implies that we may assume $y < x < z$.

In Case 1, since y is less than or equal to x and z, y could be placed on top of either. Since w was placed on top of x, w must have been placed before y. So w must be greater than y.

In order for this triple to be a non-inversion triple of type B, the column containing z and y must be taller when we've completed our composition filling. If the w, x column terminated on the next row, a cell, c, would be added on top of y while nothing was added on top of w. However, since $w > y$, c would not be placed on top of y because w is a cell farther to the left on top of which c could be placed without creating any descents. So the column containing w and x can not terminate on the very next row. However, if it does not terminate, an entry must be placed on top of w and an entry must be placed on top of y. Since $w > y$, the entry on top of w will be greater than the entry on top of y. So we will be dealing with the same situation in row $i+1$ as we dealt with in row i. Thus the column containing w and x cannot terminate before the column containing y and z. Therefore Case 1 cannot happen.

In Case 2, since y is less than x and z, we could have placed y on top of x instead. Placing y on top of z means y was not placed as far to the left as it should have been. Thus case 2 cannot happen.

Therefore, our process yields a filling with no descents such that all triples are inversion triples. We conclude that our process yields a skyline filling.

□

Step 2: This process is the only way to obtain a skyline filling from the given row entries $\{T_i'\}$.

Proof. Assume there is another way to get a skyline filling from the same row entries $\{T_i'\}$. Denote by K the skyline filling created by the process above. Denote by K' a different skyline filling whose rows contain the same entries (T_i') as the rows of K but in a different order.

Find the lowest row i of K' whose ordering is not equal to the ordering of row i of K. Consider the largest element of T_i' whose placement in K' does not agree with its placement in K. Call this element u. In K, u was placed in the left-most possible position. Therefore, u must lie in a position further to the right in K'.

Say u lies above the entry v in K and above w in K'. Then this part of the skyline filling looks like the picture below, where x and y might be empty cells:

K K'

Since u is the largest cell of K' to lie in a different place from where it lies in K, y must be less than u. If the column in K' containing y and v were taller than the column containing u and w, then the triple y, u, v would be a non-inversion triple of type A. So the column containing u and w must be taller than the column containing y and v. Then $w < v$, since otherwise u, v, w would be a non-inversion triple of type B.

The 0^{th} row of K' contains the numbers from 1 to n, in increasing order. Therefore, at some row below the row containing v and w, the entry, d, in the column containing v is less than the entry, e, in the column containing w. Find the highest row where this occurs below the row containing v and w. Let f be the entry above d and g be the entry above e. (See Figure 1, below.) Then $g < f < d < e$. So g, d, e is a non-inversion triple of type B.

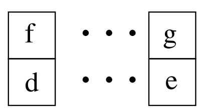


Figure 1

Therefore, regardless of which column is taller, K' contains at least one non-inversion triple. So K' is not a skyline filling. Therefore the skyline filling obtained through the process described at the beginning of this section yields the only possible skyline filling with the given row entries.

□

4 Map from Skyline Fillings to SSYTs

Begin with an arbitrary skyline filling. Select all the entries in the bottom row. Arrange them in a vertical column, sorted into descending order up columns. Then select the entries in the second row, and arrange them in a column immediately to the right of the first column, again in decreasing order. Continue in this manner until there are no more rows left in the composition filling. The shape one gets is clearly a Young diagram, since each column of this figure has height less than or equal to the height of the column to its left.

Lemma 2 *The entries in a column of the Young diagram filling are strictly decreasing as one travels up the column.*

Proof. It is clear by the way we ordered the columns that they are weakly decreasing as one travels up the column. It remains to show that there cannot be two equal entries in a given column. If there were, then in the composition filling there would be two equal entries in a row. (See Figure 2, below).

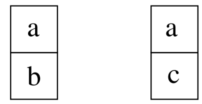


Figure 2

If the column containing b is taller than or equal to the column containing c, then the triple a, a, b would be a type A non-inversion triple. Thus the column containing c has height greater than the column containing b.

If $b \leq c$, then the triple a, b, c (where a is the entry on top of c) is a type B non-inversion triple. So $b > c$. The argument at the end of section 3 demonstrates that this also leads to a non-inversion triple found in lower rows.

□

We just proved that no two entries in the same row of a composition filling can be equal. This implies that all the entries in a column of our young diagram filling must be distinct.

Lemma 3 *Each entry in the Young diagram filling is less than or equal to the entry immediately to its left.*

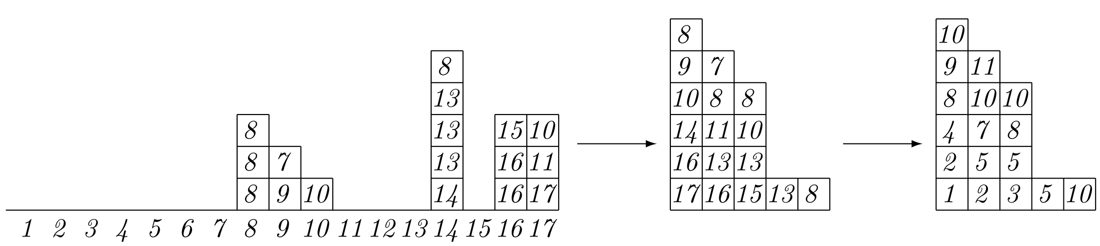
Proof. The entry, j, at height α in the i^{th} column of the Young diagram filling is the α^{th} largest entry in the i^{th} row of the skyline filling. If this value is greater than the value to its left in the Young diagram filling, at most $\alpha - 1$ entries on the $(i-1)^{st}$ row are greater than or equal to j while α entries on the i^{th} row are greater than or equal to j. Then the pigeon-hole principle tells us that at least one entry on the i^{th} row is greater than the entry below it. But then we have a descent and therefore our composition filling is not a skyline filling. Thus, we have a contradiction. So each entry must be less than or equal to the entry immediately to its left.

□

The cells in the Young diagram filling are labelled by the members of the multiset $\{1^{a_1}, 2^{a_2}, ..., k^{a_k}\}$. The total number of cells in the skyline filling, n, is equal to the total number of cells in the Young diagram. Map the value in each cell to a new value by sending α to $n - \alpha + 1$. Before the mapping, the labels were weakly decreasing by column and strictly decreasing by row. Since the map reverses the orders of the labels while preserving the fact that no repeated entries occur within a column, the labels are now weakly increasing by row and strictly increasing by column. Thus, we now have a Semi-Standard Young Tableau.

Say two different composition fillings yield the same SSYT. Then these two composition fillings would have the same set of entries on each row. But we saw in section 3 that once we know the entries on a row, the placement of those entries in a skyline filling is unique. So these two skyline fillings are identical. Thus, our map is injective.

Example 2 *Below we demonstrate the mapping first from a semi-standard skyline filling to a Young diagram filling and then to a semi-standard Young tableau:*



5 The two maps defined above are inverses

Looking back at the two examples, one sees that in this particular case, the two maps are inverses. In fact, this is true in general.

Lemma 4 *The two maps defined in sections 3 and 4 are inverses.*

Proof.

To see this, begin with the map from a SSYT to a skyline filling. This map sends the numbers in a given column to the corresponding row, changing the numbers by mapping α to $n-\alpha+1$, where n is the number which the shape of the SSYT partitions. Then, when we map this skyline filling back to an SSYT, first we take the numbers in each row and place them in the corresponding column in decreasing order. Then we send α to $n-\alpha+1$, which inverts the mapping we did in the first step. So we have the same numbers in each column, arranged in increasing order. Therefore we have the same SSYT that we began with.

Going the other way, we begin with a skyline filling and map each row to a column with the same numbers in decreasing order. We change this shape to a SSYT by mapping α to $n-\alpha+1$. When we map back to a skyline filling, we first send α to $n-\alpha+1$, which inverts the mapping. Next, we enter each column into its corresponding row via the unique map defined in section 3. Since this is the only skyline filling with these particular entries in each row, this is the skyline filling we began with.

$\square$

Thus, the two injective maps are inverses and form a bijection between skyline fillings of rearrangements of μ and SSYT of shape μ. Since the s_λ are symmetric, the number of SSYT of weight $x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ is equal to the number of SSYT of weight $x_{n-1}^{a_1}x_{n-2}^{a_2}...x_1^{a_n}$. Since our map sends each SSYT of weight $x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ to a skyline filling of weight $x_{n-1}^{a_1}x_{n-2}^{a_2}...x_1^{a_n}$, the coefficient of $\prod_{i=1}^n x_i^{\alpha_i}$ in $s_\lambda(\mathrm{x})$ is equal to the coefficient of $\prod_{i=1}^n x_i^{\alpha_i}$ in $\sum_{\lambda'} NS_{\lambda'}(x)$, for all possible multisets $\{\alpha_1,...\alpha_n\}$ with $0\le\alpha_i\le n$, $\forall i$, and $\sum_{i=1}^n \alpha_i = n$.

This proves that the sum of the non-symmetric Schur functions over all rearrangements of a partition, μ, is equal to the Schur function s_μ.

6 A Basis For the Algebra of degree n Polynomials in n variables

Several other bases for symmetric functions have non-symmetric analogues. For instance, the non-smmyetric monomial corresponding to a given composition γ of n into n parts is given by $NM_\gamma = x_1^{\gamma_1}x_2^{\gamma_2}...x_n^{\gamma_n}$. It is clear that the sum over all rearrangements of a given partition μ of the non-symmetric monomials is equal to the monomial symmetric function m_μ. Every polynomial of degree n in n variables can be written as a sum of non-symmetric monomials, so the non-symmetric monomials form a basis for the algebra of polynomials of degree n in n variables.

Definition 2 *The* **reverse dominance order** *on compositions is defined as follows:*
$\mu\le\gamma \iff \sum_{i=k}^n \mu_i \le \sum_{i=k}^n \gamma_i$ *for* $1\le i\le n$.

A semi-standard skyline filling is said to have type α if it contains α_i copies of the number i for each i. If γ and α are compositions of n into n parts, let $NK_{\gamma,\alpha}$ denote the number of semi-standard skyline fillings of shape γ and type α. $NK_{\gamma,\alpha}$ is called a *non-symmetric Kostka number.* The ordinary Kostka numbers are obtained as a sum of non-symmetric Kostka numbers: $K_{\lambda,\alpha} = \sum NK_{\gamma,\alpha}$, where the sum is over all rearrangements γ of λ.

Theorem 2 *Suppose that μ and γ are both compositions of n into n parts and $NK_{\mu,\gamma}\neq 0$. Then $\mu\ge\gamma$ in the dominance order. Moreover, $NK_{\mu,\mu}=1$.*

Proof. Assume that $NK_{\mu,\gamma} \neq 0$. By definition, there exists a semi-standard skyline filling of shape μ and type γ. Assume that a part k appears in one of the first $k-1$ columns. Then this k column would contain a descent, since there is an entry less than k in the column at a lower position than k. Therefore, the parts $k, k+1, ..., n$ all appear in the last $n-k+1$ columns. So $\mu_k + \mu_{k+1} + ... + \mu_n \geq \gamma_k + \gamma_{k+1} + ... + \gamma_n$ for each k, as desired. Moreover, if $\mu = \gamma$, then the i^{th} column must contain only entries with value i, so $NK_{\mu,\mu} = 1$. □

Corollary 1 *The non-symmetric Schur functions form a basis for the algebra of polynomials of degree n in n variables.*

Proof. Theorem 2 is equivalent to the assertion that the transition matrix from the non-symmetric Schur functions to the non-symmetric monomials (with respect to the reverse dominance order) is upper triangular with 1's on the main diagonal. Since this matrix is invertible, the non-symmetric Schur functions of degree n are a basis for polynomials of degree n in n variables. □

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