COUNTING OCCURRENCES OF 231 IN AN INVOLUTION

Toufik Mansour

Department of Mathematics
University of Haifa, 31905 Haifa, Israel
toufik@math.haifa.ac.il

Sherry H.F. Yan* and Laura L.M. Yang†
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China
**yanhuifang@sina.com*, †*yanglm@hotmail.com*

Abstract

We study the generating function for the number of involutions on n letters containing exactly $r \geqslant 0$ occurrences of 231. It is shown that finding this function for a given r amounts to a routine check of all involutions of length at most $2r+2$.

Nous étudions la fonction génératrice pour le nombre des involutions sur n lettres en comprenent précisément $r \geqslant 0$ apparitions de 231. Nous démontrons q'il est possible a trouver cette fonction pour un nombre r donné par une vérification de routine des toutes les involutions qui ont leur longueur non plus de $2r+2$.

2000 Mathematics Subject Classification: Primary 05A05, 05A15; Secondary 05C90

1. **Introduction**

Permutations. Suppose that S_n is the set of permutations of $[n] = \{1, \dots, n\}$, written in one-line notation. Let $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$ and $\tau = \tau_1\tau_2 \dots \tau_k \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leqslant n$ and $\pi_{i_s} < \pi_{i_t}$ if and only if $\tau_s < \tau_t$ for any $1 \leqslant s, t \leqslant k$. In such a context, τ is usually called a *pattern*. We denote the number of occurrences of τ in π by $\tau(\pi)$.

Starting with 1985, much attention has been paid to the counting problem of the number $S_r^\tau(n)$ of permutations of length n which contain the pattern τ exactly $r \geq 0$ times. Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern (see [1, 2, 3, 6, 13, 16, 17, 18, 19, 20]). For the case $r > 0$ there exist only a few papers, usually restricting themselves to the patterns of length three. Using two simple involutions (reverse and complement) on S_n it is immediate that with respect to being equidistributed, the six patterns of S_3 fall into two classes, namely $\{123, 321\}$ and $\{132, 213, 231, 312\}$. In 1996, Noonan [15] has proved that $S_1^{123}(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to the counting problem was suggested by Noonan and Zeilberger [16]; they gave another proof of Noonan's result, and conjectured that $S_1^{132}(n) = \binom{2n-3}{n-3}$ and

$$S_2^{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)} \binom{2n}{n-4}.$$

The first conjecture was proved by Bóna [5] and the second one was proved by Fulmek [10]. A general conjecture of Noonan and Zeilberger states that the sequence $\{S_r^\tau(n)\}_{n\geqslant 0}$ is P-recursive in n for any r and τ. It was proved by Bóna [4] for $\tau = 132$. However, as stated in [4], a challenging question is to describe $S_r^\tau(n)$, $\tau \in S_3$, explicitly for any given r. In 2002, Mansour and Vainshtein suggested in [14] a new approach to this problem in the case $\tau = 132$, which allows to get an explicit expression for $S_r^{132}(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geqslant 0} S_r^{132}(n)x^n$ for any $r \geqslant 0$. To get the result for a given r, the algorithm performs certain routine checks for each permutation of S_{2r}.

Involutions. An *involution* π is a permutation in S_n such that $\pi = \pi^{-1}$; let $\mathcal{I}_n$ be the set of all the involutions in S_n. We denote $I_{r,n}^\tau$ the number of involutions $\pi \in \mathcal{I}_n$ with $\tau(\pi) = r$, and $I_r^\tau(x)$ the corresponding generating function, that is, $I_r^\tau(x) = \sum_{n\geqslant 0} I_{r,n}^\tau x^n$.

Again, most authors considered the case $r = 0$, namely involutions avoiding a given pattern τ (see [7, 9, 11, 12] and references therein). For the case $r > 0$ there exist only few results. In 2002, Guibert and Mansour [12] gave an explicit expression for $I_{1,n}^{132}$, namely $I_{1,n}^{132} = \binom{n-2}{[(n-3)/2]}$. Egge and Mansour in [8] proved that $I_{1,n}^{231} = (n-1)2^{n-6}$ for $n \geqslant 5$.

In the present paper we suggest a new approach to this problem in the case of $\tau = 231$, which allows to get an explicit expression for $I_{r,n}^{231}$ for any given r. More precisely, we present an algorithm that computes the generating function $I_r^{231}(x) = \sum_{n\geqslant 0} I_{r,n}^{231} x^n$ for any $r \geqslant 0$. To get the result for a given r, the algorithm performs certain routine checks for each element in $\bigcup_{k=1}^{2r+2} I_k$. The algorithm has been implemented in C, and yielded explicit results for $0 \leqslant r \leqslant 7$.

2. Preliminary results

For any involution $\pi \in \mathcal{I}_n$, we can assign a bipartite graph G_π in the following way which is similar to [14].

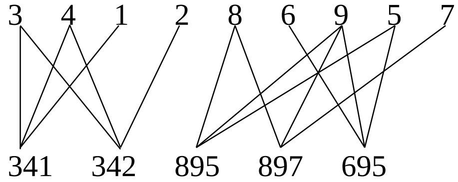

Figure 1. The graph $G_{341286957}$

The vertices in one part of G_π, denoted V_1 are the entries of π, and the vertices of the second part, denoted V_3, are the occurrences of 231 in π. Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_3$ if i enters j. For example, let $\pi = 341286957$, then π contains 5 occurrences of 231, and the graph G_π is presented on Figure 1.

Let $\widetilde{G}$ be an arbitrary connected component of G_π, and let $\widetilde{V}$ be its vertex set. We denote $\widetilde{V}_1 = \widetilde{V} \bigcap V_1$, $\widetilde{V}_3 = \widetilde{V} \bigcap V_3$, $t_1 = |\widetilde{V}_1|$, $t_3 = |\widetilde{V}_3|$. Denote by G_π^n the connected component of G_π containing entry n.

For any $\pi \in \mathcal{I}_n$ with $\pi_j = n$ and $|V_1(G_\pi^n)| > 1$, assume that i_1 is the minimal index such that there exists a subsequence

$$(\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \ldots, i_h, \pi_{i_{h+2}}, i_{h+1}, \ldots, \pi_{i_k}, i_{k-1}, \pi_{i_{k+1}}, i_k, i_{k+1})$$

where $i_1 < i_2 < i_3 < \ldots < i_k < i_{k+1} = j$. we call this subsequence *connected sequence*. For our convenience, we call i_1 the *initial index*. Also, It is obvious that π_{i_1} is the first entry of the subsequence of π contained in G_π^n.

Definition 2.1. *For any $\pi \in \mathcal{I}_n$ and $\pi_j = n$, we define the* 231-tail *by*

$$\chi_\pi = \begin{cases} (n, \pi_{j+1}, \ldots, \pi_{n-1}, j), & \text{if} \quad |V_1(G_\pi^n)| = 1, \\ (\pi_{i_1}, \pi_{i_1+1}, \ldots, \pi_n), & \text{if} \quad |V_1(G_\pi^n)| > 1, \end{cases}$$

where i_1 is the initial index of π.

For example, the 231-tail of the involution 216483957 is 6483957. Denote l_π and c_π the length of π and the number of the occurrences of 231 in π.

In fact, for any $\pi \in \mathcal{I}_n$ with $|V_1(G_\pi^n)| = 1$, the 231-tail χ_π of π can be represented as $\chi_\pi = (n, n-1, \ldots, n-s+1, \lambda)$ where the first entry of λ is not $n-s$. The following lemma holds by the definition of the 231-tail.

Lemma 2.2. *Let $\pi \in \mathcal{I}_n$, the permutation of* 231*-tail of π, χ_π, is an involution, and there exists an involution π' such that $\pi = (\pi', \chi_\pi)$.*

Lemma 2.3. *Let $\pi \in I_n$ with $\chi_\pi = (n, n-1, \ldots, n-s+1, \lambda)$, where λ is nonempty, such that $c_{\chi_\pi} = r$, then $l_{\chi_\pi} \leq 2r+2$. Furthermore, The equality holds if and only if $\chi_\pi = (2r+2, 2r+1, \ldots, r+3, r+1, r+2, r, r-1, \ldots, 1)$.*

Proof. If l_π is maximal, then the first entry of χ_π is l_π. Using Lemma 2.2 we get the last entry of χ_π is 1. By induction we can assume that $\chi_\pi = (n, n-1, \ldots, n-s+1, \mu, s, s-1, \ldots, 1)$ where the first entry of μ is not $n-s$ and μ is nonempty. On the other hand $c_\pi = r$, so $s \leq r$. Hence

$$\chi_\pi = (2r+2, 2r+1, \ldots, r+3, r+1, r+2, r, r-1, \ldots, 1).$$

□

Lemma 2.4. *For any $\pi \in \mathcal{I}_n$ with $|V_1(G_\pi^n)| > 1$, the subsequence of π contained in the connected component G_π^n is just the* 231*-tail χ_π of π.*

Proof. According to the definition of the 231-tail, it is sufficient to prove that the bipartite graph corresponding to χ_π is connected. Assume $\pi_j = n$ and i_1 is the initial index. Suppose that the connected sequence is

$$(\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \ldots, i_h, \pi_{i_{h+2}}, i_{h+1}, \ldots, \pi_{i_k}, i_{k-1}, \pi_{i_{k+1}}, i_k, i_{k+1})$$

where $i_1 < i_2 < i_3 < \ldots < i_k < i_{k+1} = j$. It is obvious that the vertices

$$\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \ldots, i_h, \pi_{i_{h+2}}, i_{h+1}, \ldots, \pi_{i_k}, i_{k-1}, \pi_j, i_k, j$$

are all contained in the connected component G_π^n.

For $i_h < m < i_{h+1}$, if $i_h < \pi_m < \pi_{i_{h+1}}$, then $\pi_m \pi_{i_{h+1}} i_h$ forms a pattern of 231 in π; if $\pi_m > \pi_{i_{h+1}}$, then $\pi_{i_h}\pi_m i_h$ is a subsequence of the pattern 231 in π; if $\pi_m < i_h$, then $m\pi_{i_h}\pi_m$ is a subsequence of the pattern of 231 in π. In these cases, we know that π_m is contained in G_π^n.

For $j < m < n$, if $\pi_m < \pi_{i_k}$, then $\pi_{i_k} n \pi_m$ forms a pattern of 231 in π; if $\pi_m > \pi_{i_k}$, then $\pi_{i_k}\pi_m j$ forms a pattern of 231 in π. In these cases, we know that π_m is contained in G_π^n. □

Studying occurrences of 132 in a permutation which leads to consideration of 231 in a permutation, Mansour and Vainshtein have proved that the relation $t_1 \leq 2t_3 + 1$ in [14]. It is clear that the set of involutions is a subset of permutations. So we have

Lemma 2.5. (see [14, Lemma 2.1]) *For any connected component $\widetilde{G}$ of G_π, one has $t_1 \leq 2t_3 + 1$.*

Remark 2.6. *For any $\pi \in I_n$ with $c_{\chi_\pi} = r$ $(r > 0)$, if $|V_1(G_\pi^n)| = 1$, then $l_{\chi_\pi} \leq 2r + 2$; otherwise $l_{\chi_\pi} \leq 2r + 1$.*

3. Main Theorem and explicit results

Denote by K_t the subset of $\bigcup_{k\leqslant 2t+2} \mathcal{I}_k$ whose elements can be expressed as $(k, k-1, \ldots, k-s+1, \lambda)$ where λ is nonempty, and by H_t be the subset of $\bigcup_{k\leqslant 2t+1} \mathcal{I}_k$ such that the corresponding bipartite graph of each element is connected. It is obvious that $K_t \cap H_t = \emptyset$. Then the main result of this paper can be formulated as follows.

Theorem 3.1. *For any $r \geqslant 1$,*

$$I_r^{231}(x) = \frac{x}{1-x} I_r^{231}(x) + \sum_{\mu \in K_r \cup H_r} x^{l_\mu} I_{r-c_\mu}^{231}(x). \qquad (*)$$

Proof. Denote by $F_r^\mu(x)$ the generating function for the number of involutions $\pi \in \mathcal{I}_n$ such that χ_π is just order-isomorphic to μ. We discuss three cases to find $F_r^\mu(x)$:

- If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (n, n-1, \ldots, n-s+1)$, then $l_\mu = s$ and $\mu = (s, s-1, \ldots, 1)$, so we have
$$F_r^\mu(x) = x^s I_r^{231}(x).$$
- If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (n, n-1, \ldots, n-s+1, \lambda)$ where λ is nonempty, then $\mu \in K_r$ by Lemma 2.3, thus we have
$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$
- If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (\pi_{i_1}, \pi_{i_1+1}, \ldots, \pi_n)$ where i_1 is the initial index of π, then Lemma 2.5 and Lemma 2.4 yield $\mu \in H_r$ and
$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$

Hence, summing over all $\mu \in \{(s, s-1, s-2, \dots, 2, 1) | s \geqslant 1\} \cup K_r \cup H_r$ we get the desired result. $\square$

Theorem 3.1, Lemma 2.3, and Lemma 2.5 provide a finite algorithm for finding $I_r^{231}(x)$ for any given $r > 0$, since we only have to consider all involutions in I_k, where $k \leqslant 2r+2$, and to perform certain routine operations with all 231-tails found so far.

Remark 3.2. *In fact, according to the Lemma 2.3, it is sufficient to check all involutions in I_k, where $k \leq 2r+1$. As a consequence, Formula $(*)$ can be reduced as follows:*

$$I_r^{231}(x) = \frac{x}{1-x} I_r^{231}(x) + x^{2r+2} I_0^{231}(x) + \sum_{\mu \in K_r^* \cup H_r} x^{l_\mu} I_{r-c_\mu}^{231}(x),$$

where K_r^* is the set of all involutions of the form $(n, n-1, \dots, n-s+1, \lambda)$ of in I_k where $k \leq 2t+1$ and λ is nonempty.

Let us start from the case $r = 0$. Observe that $(*)$ remains valid for $r = 0$, provided the left hand side is replaced by $I_0^{231}(x) - 1$; subtracting 1 here accounts for the empty permutation. Note that when $r = 0$, $K_0 \cup H_0$ is empty. Hence we get $I_0^{231}(x) - 1 = \frac{x}{1-x} I_0^{231}(x)$, equivalently

$$I_0^{231}(x) = \frac{1-x}{1-2x}, \tag{$**$}$$

which is the result of Simion and Schmidt (see [17, Proposition 6]).

Let now $r = 1$. Observe that $K_1 \cup H_1 = \{4231\}$. Therefore, $(*)$ amounts to

$$I_1^{231}(x) = \frac{x}{1-x} I_1(x) + x^4 I_0^{231}(x),$$

and we get the following result from Formula $(**)$.

Corollary 3.3. (see Egge and Mansour [8, Theorem 4.3]) *The generating function $I_1^{231}(x)$ for the number of involutions containing exactly one occurrence of the pattern 231 is given by*

$$I_1^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^2};$$

equivalently, for $n \geq 5$,

$$I_{1,n}^{231} = (n-1)2^{n-6}.$$

Let $r = 2$. Exhaustive search adds four new elements to the previous list; these are 653421, 52431, 53241, and 3412, therefore we get

Corollary 3.4. *The generating function $I_2^{231}(x)$ is given by*

$$I_2^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^3} \left(1 - 3x^2 - 2x^3 + x^4 - x^5\right);$$

equivalently, for $n \geqslant 9$,

$$I_{2,n}^{231} = (n^2 + 137n - 234)2^{n-11}.$$

Let $r = 3, 4, 5, 6, 7$; exhaustive search in $\mathcal{I}_8$, $\mathcal{I}_{10}$, $\mathcal{I}_{12}$, $\mathcal{I}_{14}$, and $\mathcal{I}_{16}$ reveals 13, 24, 41, 69, and 103 elements, respectively, and we get

Corollary 3.5. *Let $3 \leqslant r \leqslant 7$, then*

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} Q_r(x),$$

where

$$\begin{aligned}
Q_3(x) &= x^5(4 - 14x + 8x^2 + 11x^3 - 6x^4 - 2x^5 + 2x^6 + 5x^7 - 2x^8 + x^9);\\
Q_4(x) &= x^6(6 - 32x + 49x^2 + 7x^3 - 73x^4 + 40x^5 + 30x^6 - 37x^7 + 2x^8 + 4x^{10}\\
&\quad -9x^{11} + 3x^{12} - x^{13});\\
Q_5(x) &= x^6(8 - 58x + 146x^2 - 120x^3 - 40x^4 - 24x^5 + 290x^6 - 184x^7 - 197x^8\\
&\quad +228x^9 + 30x^{10} - 132x^{11} + 62x^{12} + 13x^{14} - 16x^{15} + 14x^{16} - 4x^{17} + x^{18});\\
Q_6(x) &= x^6(4 - 31x + 80x^2 - 56x^3 + 4x^4 - 384x^5 + 1097x^6 - 830x^7 - 483x^8\\
&\quad +660x^9 + 685x^{10} - 1091x^{11} - 59x^{12} + 722x^{13} - 195x^{14} - 338x^{15}\\
&\quad +285x^{16} - 92x^{17} + 20x^{18} - 45x^{19} + 35x^{20} - 20x^{21} + 5x^{22} - x^{23});\\
Q_7(x) &= x^7(17 - 199x + 969x^2 - 2502x^3 + 3642x^4 - 3274x^5 + 3324x^6 - 4714x^7\\
&\quad +1874x^8 + 6326x^9 - 8262x^{10} - 231x^{11} + 5474x^{12} - 637x^{13} - 4022x^{14}\\
&\quad +1933x^{15} + 1340x^{16} - 1129x^{17} - 518x^{18} + 982x^{19} - 498x^{20} + 166x^{21}\\
&\quad -92x^{22} + 105x^{23} - 62x^{24} + 27x^{25} - 6x^{26} + x^{27}).
\end{aligned}$$

As an easy consequence of Theorem 3.1 we get the following result.

Corollary 3.6. *For any $r \geq 1$ there exist a polynomial $P_{5r-1}(x)$ of degree $5r-1$ with integer coefficients such that*

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} P_{5r-1}(x).$$

Proof. Immediately, by the above cases we have the corollary holds for $1 \leq r \leq 7$. Let us assume by induction that the corollary holds for $1, 2, \dots, r-1$; for r the equation $(*)$ give

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} \sum_{\rho \in K_r \cup H_r} x^{l_\rho} \frac{(1-2x)^r}{1-x} I_{r-c_\rho}^{231}(x).$$

By the induction assumption and $I_0^{231}(x) = \frac{1-x}{1-2x}$ we have that $x^{l_\rho}\frac{(1-2x)^r}{1-x} I_{r-c_\rho}(x)$ is a polynomial with integer coefficients of degree a. So Lemma 2.3 and 2.5 yields

$$a = \max\{b_j | j = 1, \dots, r\},$$

where $b_j = 2j + 2 + r - (r - j + 1) + 1 + 5(r - j) - 1 = 5r - 2j + 1$, which means $a = 5r - 1$, as claimed. $\square$

4. **Further results**

Another direction would be to match the approach of this paper with the previous results on restricted 231-avoiding involutions. Let $\Phi_r(x; k)$ be the generating function for the number of involutions in $\mathcal{I}_n$ containing r occurrences of 231 and avoiding the pattern $12 \dots k \in \mathfrak{S}_k$.

We denote by e_λ the length of the longest increasing subsequence of any involution λ. For example, let $\lambda = 3412$, then $e_\lambda = 2$. We denote by $K_t(k) \cup H_t(k)$ the set of all involutions $\lambda \in K_t \cup H_t$ such that $e_\lambda \leq k-1$.

Theorem 4.1. *For any $r \geqslant 1$ and $k \geqslant 3$,*

$$\Phi_r(x;k) = \frac{x}{1-x}\Phi_r(x;k-1) + \sum_{\mu \in K_r(k)\cup H_r(k)} x^{l_\mu}\Phi^{231}_{r-c_\mu}(x;k-e_\mu).$$

Besides, $\Phi_r(x;1) = \Phi_r(x;2) = 0$, and $\Phi_0(x;1) = 1$ and $\Phi_0(x;2) = \frac{1}{1-x}$.

Similar to the case of $I_r^{231}(x)$, the statement of the above theorem remains valid for $r = 0$, provided the left hand side is replaced by $\Phi_r(x;k) - 1$. This allows to recover known explicit expressions for $\Phi_r(x;k)$ for $r = 0, 1$, as follows.

Corollary 4.2. (see Egge and Mansour [8]) *For all $k \geq 1$,*

$$\Phi_0(x;k) = \sum_{j=0}^{k-1}\left(\tfrac{x}{1-x}\right)^j;$$

$$\Phi_1(x;k) = x^4\sum_{j=0}^{k-3}(j+1)\left(\tfrac{x}{1-x}\right)^j.$$

The final direction would be to match the approach of this note with the previous results on restricted 231-avoiding even or odd involutions. We say π an even (resp; odd) involution if the number of inversion in π, namely $21(\pi)$ is even (resp; odd). We denote by $K_r^+ \cup H_r^+$ the set of all the even involutions $\lambda \in K_r \cup H_r$ and denote by $K_r^- \cup H_r^-$ the set of all the odd involutions $\lambda \in K_r \cup H_r$.

Let $I_r^+(x)$ (resp; $I_r^-(x)$) be the generating function for the number of even (resp; odd) involutions in $\mathcal{I}_n$ containing r occurrences of 231. Our new approach allows to get an explicit expression for $I_r^+(x)$ (or $I_r^-(x)$) for any given $r \geqslant 0$.

Theorem 4.3. *For all $r \geqslant 1$,*

$$I_r^+(x) = \frac{x+x^4}{1-x^4}I_r^+(x) + \frac{x^2+x^3}{1-x^4}I_r^-(x) + \sum_{\mu\in K_r^+\cup H_r^+} x^{l_\mu} I^+_{r-c_\mu}(x) + \sum_{\mu\in K_r^-\cup H_r^-} x^{l_\mu} I^-_{r-c_\mu}(x);$$

$$I_r^-(x) = \frac{x+x^4}{1-x^4}I_r^-(x) + \frac{x^2+x^3}{1-x^4}I_r^+(x) + \sum_{\mu\in K_r^+\cup H_r^+} x^{l_\mu} I^-_{r-c_\mu}(x) + \sum_{\mu\in K_r^-\cup H_r^-} x^{l_\mu} I^+_{r-c_\mu}(x).$$

In particular, we have

$$I_0^+(x) - 1 = \frac{x+x^4}{1-x^4}I_0^+(x) + \frac{x^2+x^3}{1-x^4}I_0^-(x)$$

and

$$I_0^-(x) = \frac{x+x^4}{1-x^4}I_0^-(x) + \frac{x^2+x^3}{1-x^4}I_0^+(x).$$

Proof. Here we only prove the result of $I_r^+(x)$ for any $r \geq 1$. By the same method, we can obtain the formula for $I_r^-(x)$. Denote by $F_r^\mu(x)$ the generating function for the number of even involutions in $\pi \in I_n$ such that χ_π is order-isomorphic to μ.

To find $F_r^\mu(x)$, we recall four four cases. If $\mu = (s, s-1, \ldots, 1)$ and μ is even (that is, $21(\mu) = \frac{(s-1)s}{2} = 2k$ for some positive integer k), then in this case we have

$$F_r^\mu(x) = x^s I_r^+(x).$$

If $\mu = (s, s-1, \ldots, 1)$ and μ is odd (that is, $21(\mu) = \frac{(s-1)s}{2} = 2k-1$ for some positive integer k), then in this case we have

$$F_r^\mu(x) = x^s I_r^-(x).$$

If $\mu \in K_r^+ \cup H_r^+$, then

$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^+(x).$$

If $\mu \in K_r^- \cup H_r^-$, then

$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^-(x).$$

Hence, if summing over all $\mu \in K_r \cup H_r \cup \{(s, s-1, s-2, \ldots, 2, 1) | s \geq 1\}$ then we get the desired result. When $r = 0$, subtracting 1 here accounts for the empty permutation. $\square$

As an example of the above theorem for $r = 0, 1, 2$, we get

Corollary 4.4.

$$I_r^+(x) = \frac{E_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}}, \quad I_r^-(x) = \frac{O_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}};$$

where

$E_0(x) = 1 - 2x + 2x^2 - 2x^3$*;*

$E_1(x) = 2x^6(1 - 2x + 2x^2 - 2x^3)$*;*

$E_2(x) = x^4(1 - 5x + 11x^2 - 15x^3 + 10x^4 + 5x^5 - 11x^6 - 5x^7 + 47x^8 - 94x^9 + 86x^{10} - 62x^{11} + 16x^{12})$;

$O_0(x) = x^2$*;*

$O_1(x) = x^4(1 - 4x + 8x^2 - 12x^3 + 13x^4 - 8x^5 + 4x^6)$*;*

$O_2(x) = x^6(2 - 6x + 6x^2 - 2x^3 - 9x^4 + 4x^5 + 20x^6 - 36x^7 + 53x^8 - 24x^9 + 8x^{10})$.

Again, as an easy consequence of Theorem 4.3 we get the following result.

Corollary 4.5. *For any $r \geqslant 0$, there exists two polynomials $P_{m_r}(x)$ and $P_{n_r}(x)$ of degree m_r and n_r with integer coefficients such that*

$$I_r^+(x) = \frac{P_{m_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}}, \quad I_r^-(x) = \frac{P_{n_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}},$$

where $m_r, n_r \leq \frac{r(r+9)}{2}$.

It can be proved by induction on r as the proof of Corollary 3.6. Here we delete its proof.

As a remark we can derive another results from Theorem 4.3. For example, the generating function for the number of even or odd involution containing exactly r occurrences of the pattern 231 and avoiding $12\dots k$ (or avoiding $k\dots 21$).

References

[1] N. Alon and E. Friedgut, On the number of permutations avoiding a given pattern, *J. Combin. Theory Ser. A* **89** (2000)133–140.

[2] M.D. Atkinson, Restricted permutations, *Discrete Math.* **195** (1999) 27–38.

[3] M. Bóna, Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps, *J. Combin. Theory Ser. A* **80** (1997) 257–272.

[4] M. Bóna, The number of permutations with exactly r 132-subsequences is P-recursive in the size! *Adv. Appl. Math.* **18** (1997) 510–522.

[5] M. Bóna, Permutations with one or two 132-subsequences, *Discrete Math.* **181** (1998) 267–274.

[6] T. Chow and J. West, Forbidden subsequences and Chebyshev polynomials, *Discrete Math.* **204** (1999) 119–128.

[7] E.S. Egge, Restricted 3412-avoiding involutions: Continued fractions, Chebyshev polynomials and enumerations, *Adv. in Appl. Math.*, to appear.

[8] E. Egge and T. Mansour, 231-Avoiding involutions and Fibonacci numbers, *Australasian Journal of Combinatorics*, to appear.

[9] E.S. Egge and T. Mansour, Involutions restricted by 3412, Continued fractions, and Chebyshev polynomials, preprint.

[10] M. Fulmek, Enumeration of permutations containing a presribed number of occurrences of a pattern of length three, *Adv. Appl. Math.* **30** (2003) 607–632.

[11] O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, *PHD-thesis, University Bordeaux 1, France* (1995).

[12] O. Guibert and T. Mansour, Restricted 132-involutions, *Séminaire Lotharingien de Combinatoire* **48** (2002), Article B48a (23pp.).

[13] T. Mansour and A. Vainshtein, Restricted permutations, continued fractions, and Chebyshev polynomials,*Electron. J. Combin.* **7** (2000) #R17.

[14] T. Mansour and A. Vainshtein, Counting occurences of 132 in a permutation, *Adv. Appl. Math.* **28** (2002) 185–195.

[15] J. Noonan, The number of permutations containing exactly one increasing subsequence of length three, *Discrete Math.* **152** (1996) 307–313.

[16] J. Noonan and D. Zeilberger, The enumeration of permutations with a prescribed number of "forbidden" patterns, *Adv. Appl. Math.* **17** (1996) 381–407.

[17] R. Simion, F.W. Schmidt, Restricted Permutations, *Europ. J. of Combin.* **6** (1985) 383–406.

[18] Z. Stankova, Forbidden subsequences, *Discrete Math.* **132** (1994) 291–316.

[19] Z. Stankova, Classification of forbidden subsequences of length 4, *European J. Combin.* **17** (1996) 501–517.

[20] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146** (1995) 247–262.