The Weak Order on Pattern-Avoiding Permutations

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Abstract. We consider a partial order on permutations avoiding a set of patterns. The partial order is induced from the weak order of the symmetric group. Some sets of patterns are shown to give well-known posets, including the Tamari lattice, the Boolean lattice, $J(2{\times}n)$, the integer interval lattice, and the lattice of shifted shapes. In the case of a single pattern, we characterize those patterns which give a lattice.

Résumé. Nous considérons un ordre partiel sur des permutations évitant un ensemble des motifs. L'ordre partiel est induit de l'ordre faible du groupe symétrique. Quelques ensembles des motifs sont montres à donner les posets bien connus, y compris le trellis de Tamari, le trellis de Boole, $J(2 \times n)$, et quelques autres trellis. Dans le cas d'un motif simple, nous caractérisons ces motifs qui donnent un trellis.

1. Introduction

As a partially ordered set, the weak order on a finite Coxeter group is a lattice. In some interesting cases this property is retained when passing to an induced subposet. For example, one can obtain the one-skeleta of generalized associahedra of Fomin and Zelevinski [**5**], and the Cambrian lattices of Reading [**9**] as subposets of the weak order. In type A, some of these subposets can be described using pattern avoidance. The Tamari lattice, the one-skeleton of the associahedron, is isomorphic to the weak order on 312 avoiding permutations. The Boolean lattice is isomorphic to the weak order on 312 and 231 avoiding permutations. See [**9**] for these results and some corresponding type B results.

These results lead to two natural questions. For which sets of patterns T is the weak order on permutations avoiding T a lattice? Also, can any other well-known families of lattices be obtained as the weak order on pattern avoiding permutations?

This paper is organized as follows. Section 2 contains some preliminaries about pattern avoidance, lattices, and the weak order. In section 3, we show that $J(2 \times n) \cup \hat{1}$, the integer interval lattice, and the lattice of shifted shapes can be obtained as the weak order on pattern avoiding permutations. We also give two unnamed lattices which may be of independent interest. In section 4 we outline the proof of the following theorem:

Theorem 1.1. *$S_n(\tau)$ is a lattice for all n if and only if*

τ has exactly one descent, which is of magnitude one or two, or
τ has exactly one ascent, which is of magnitude one or two.

In section 5 we give some related results and a conjecture.

2. Preliminaries

A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n$ in the symmetric group on n elements is said to *contain* a pattern τ if there is a subsequence of π in the same relative order as τ. Otherwise, π is said to *avoid* τ. For example, the permutation $\pi = 1423$ contains the pattern 132 twice, as 142 and 143. The permutation 2134 avoids 132. For a set of patterns T, we will use $S_n(T)$ to denote the permutations of length n which avoid all of the patterns in T. For simplicity of notation, we omit the set brackets for a single pattern. There has been much recent interest in enumerative problems in pattern avoidance. For an overview, see [**15**].

A *descent* in a permutation π occurs in position i if $\pi(i) > \pi(i+1)$. The *magnitude* of a descent is $\pi(i+1) - \pi(i)$. Ascents and their magnitudes are defined similarly. An *inversion* in a permutation π is a pair (i, j) such that $i < j$ and $\pi(j) > \pi(i)$.

The weak order on permutations is as follows. For $\pi, \sigma \in S_n$, we say that σ covers π if there is an adjacent transposition $(i, i+1)$ such that $\pi(i, i+1) = \sigma$, and σ has more inversions than π. The weak order is the transitive closure of this relation. Alternatively, we could describe this as $\pi \leq \sigma \iff \pi$ and σ can be written as products of adjacent transpositions, with the product for π appearing as a prefix of the product for σ. We will let $S_n(T)$ denote the set $S_n(T)$ together with its order relation induced from the weak order. See [**2**] for more information on the weak order of Coxeter groups.

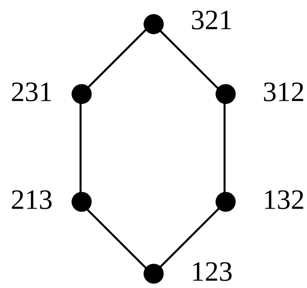

FIGURE 1. The weak order on S_3

A *lattice* L is a partially ordered set with the following property:

For all $x, y \in L$, the set $\{z \in L | z \geq x, z \geq y\}$ has a unique minimal element, called the *join* of x and y and denoted $x \vee y$, and the set $\{z \in L | z \leq x, z \leq y\}$ has a unique maximal element, called the *meet* of x and y and denoted $x \wedge y$.

For background information on lattices, see [**1**] or Chapter 3 of [**11**].

The lattice property of the weak order on S_n was demonstrated in [**7**] and [**16**]. There is a well-known characterization of the weak order which is useful for constructing meets and joins.

LEMMA 1. *For $\sigma \in S_n$, let $I(\sigma) = \{(i, j) | \; i < j, \sigma(j) < \sigma(i)\}$ be the inversion set of σ. Then $\pi \leq \sigma \Leftrightarrow I(\pi) \subseteq I(\sigma)$.*

Let $\pi, \sigma \in S_n$. We will construct the join $\pi \vee \sigma$. First, insert the letter 1. To insert the letter j, insert it immediately to the left of the largest i such that $(i, j) \in I(\sigma) \cup I(\pi)$. If no such i exists, insert j on the right. This gives the unique minimal permutation ω with $I(\pi) \subseteq I(\omega)$ and $I(\sigma) \subseteq I(\omega)$. Since S_n also has a unique minimal element, it is a lattice by the following lemma. This is [**11**], Proposition 3.3.1.

LEMMA 2. *Let P be a finite poset with a unique minimal element. If the join of every pair of elements in P exists, then P is a lattice.*

To consider induced subposets of the weak order, we will use the following easy consequence of Lemma 2.

LEMMA 3. *Let L be a lattice and P an induced subposet. If the following two conditions hold, then P is a lattice:*
1) For all $v \in L \setminus P$, the set $\{v' \in P \mid v' < v\}$ has a unique maximal element, or is the empty set.
2) P has a unique maximal element and a unique minimal element.

We will make use of the following notation. If $\pi \in S_m$ and $\sigma \in S_n$, then $\pi \oplus \sigma$ denotes the permutation in S_{m+n}, where π acts on the first m letters, and σ acts on the last n letters. Similarly, $\pi \ominus \sigma$ denotes the permutation in S_{m+n}, where π acts on the last m letters, and σ acts on the first n letters.

3. Examples of Lattices of Pattern-Avoiding Permutations

To find examples, we tested all sets of patterns including any number of patterns of length three and at most one pattern of length four. Here we give all the resulting lattices, except for the n element chain. In each example, only one representative set of patterns is given.

1. The Tamari Lattice
The Tamari lattice was defined in [**14**] in terms of legal bracketings. See also [**9**]. It can be realized as the weak order on 132 avoiding permutations. This can be seen by using Stanley's representation of the Tamari lattice given in ([**12**], exercise 6.23), and Krattenthaler's bijection [**8**].

2. The Boolean Lattice
The Boolean lattice is isomorphic to the weak order on {132, 213} avoiding permutations. See [**9**].

3. The Lattice of Shifted Shapes
A *shifted shape* is a finite set Q of pairs (i,j), $i < j$, with the following property: If $(i,j) \in Q$, then $(k,j) \in Q \;\forall\; k < i$, and $(i,l) \in Q \;\forall\; l < j$.

Shifted shapes can be thought of as diagrams fitting on top of a staircase (see figure 2). The partial order on shifted shapes with $j \leq n \;\forall\; (i,j) \in Q$ is inclusion of sets. See [**6**] and the references given there.

The lattice of shifted shapes can be realized as $S_n(\{132, 312\})$. A bijection between permutations avoiding 132 and 312 and shifted shapes is given mapping a permutation to its inversion set. The condition σ avoids 132 is equivalent to the first condition for a shifted shape. The condition σ avoids 312 is equivalent to the second.

4. The Integer Interval Lattice
Taking all closed intervals contained in $[1, n]$ with integer endpoints and ordering them by inclusion gives a lattice. This lattice is isomorphic to $S_n(\{231, 312, 2143\})$. Figure 3 shows this lattice for $n = 5$. Permutations avoiding this set of patterns are determined by the pairs $(i, i+1)$ in their inversion sets. Also, for $\sigma \in S_n(\{231, 312, 2143\})$, $(i, i+1), (k, k+1) \in I(\sigma)$ implies $(j, j+1) \in I(\sigma)$ for all $i < j < k$.

5. $J(2 \times n) \cup \hat{1}$
The lattice of order ideals of the poset $2 \times n$ appears often in combinatorics. See [**3**] and [**11**]. This lattice (with an extra maximal element) is isomorphic to $S_n(\{132, 312, 2314\})$. This can

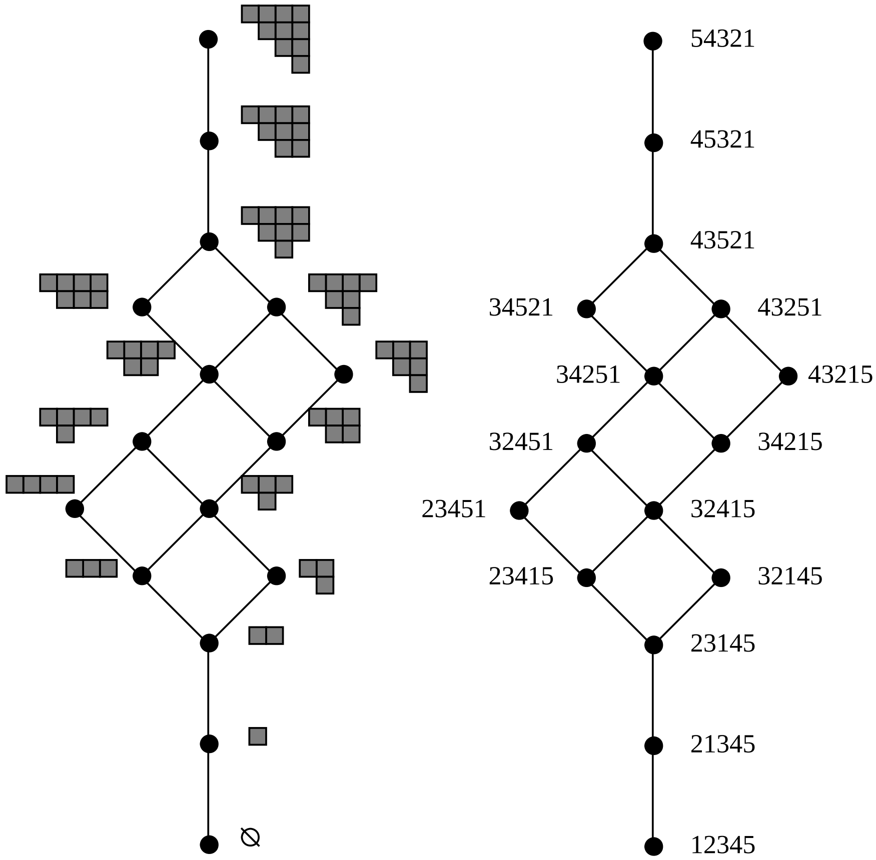

FIGURE 2. $S_n(\{132, 312\})$, the shifted shape lattice

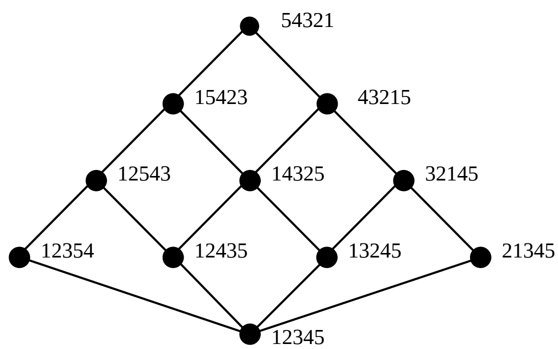

FIGURE 3. $S_n(\{231, 312, 2143\})$, the integer interval lattice

be easily shown by constructing $S_n(\{132, 312, 2314\})$ from $S_{n-1}(\{132, 312, 2314\})$. The same process constructs $J(2 \times n) \cup \hat{1}$ from $J(2 \times (n-1)) \cup \hat{1}$. See figure 4.

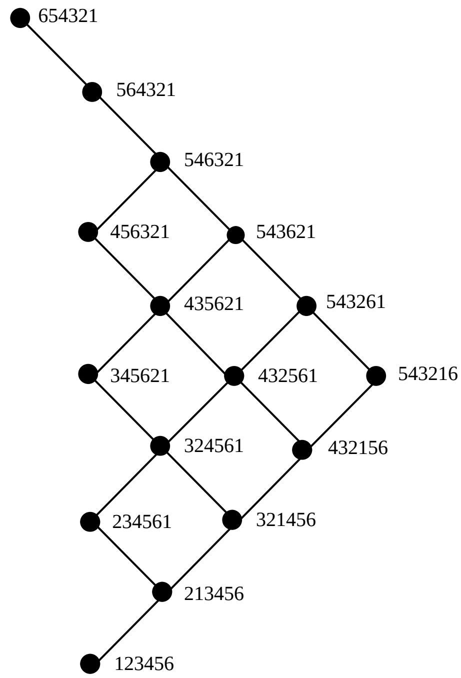

FIGURE 4. $S_n(\{132, 312, 2314\}) \cong J(2 \times n) \cup \hat{1}$

6. A "leaf with ridges" lattice.

The lattice $S_n(\{132, 213, 3421\})$ is graded, and may have some other interesting properties. It has $\binom{n}{2} + 1$ elements. When its Hasse diagram is drawn as in figure 5, it looks somewhat like a leaf with a series of ridges rising out of it.

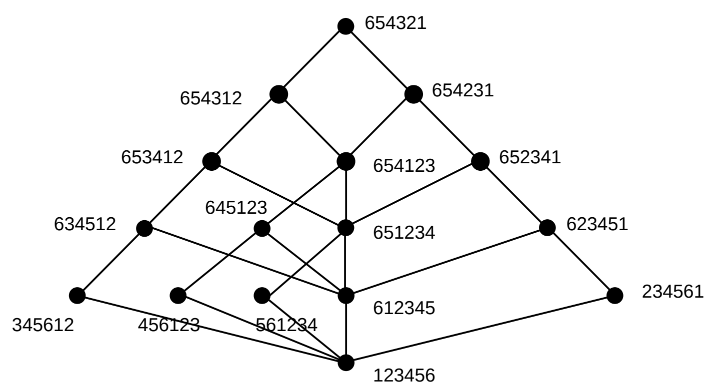

FIGURE 5. $S_n(\{132, 213, 3421\})$

7. A lattice with Fibonacci-many elements.

The lattice $S_n(\{231, 312, 1432\})$ is also graded. It has $F_{n+2} - 1$ elements, where F_n denotes the nth Fibonacci number. Its Hasse diagram is drawn in figure 6 to highlight how the nth lattice in the sequence can be constructed from the $(n-2)$nd and $(n-1)$st. To see this, remove the gray edges.

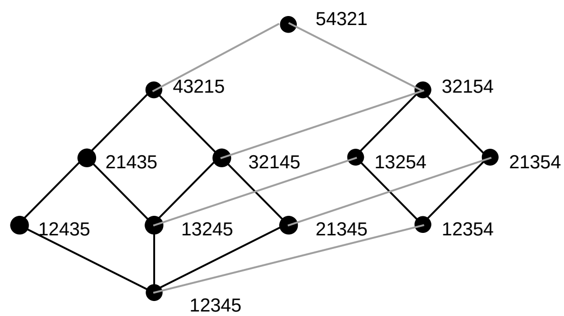

FIGURE 6. $S_n(\{231, 312, 1432\})$

4. Proof Outline of Theorem 1.1

Let τ be a pattern of length k. We show the result in four steps. First, we show that if τ has at least two descents and at least two ascents, then $S_n(\tau)$ is not a lattice for $n \geq k$. To do this, we find the subposet in figure 6, where the edges are covering relations. Then taking the induced subposet in $S_k(\tau)$, shown in figure 7, we find that σ and $\sigma\prime$ do not have a join in $S_k(\tau)$. We can find an isomorphic subposet in S_n for all $n \geq k$.

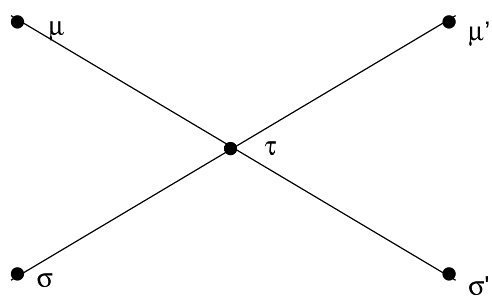

FIGURE 7. Subposet of S_k

Second, we show that if τ has at least two ascents and a descent of magnitude greater than 2, (or vice versa), then $S_n(\tau)$ is not a lattice for $n \geq k+1$. We find similar subposets as in figures 6 and 7, except here the edges are intervals which are chains.

Third, if τ satisfies the conditions of the theorem, we will show that $\pi \in S_n \setminus S_n(\tau)$ implies that the set $\{\pi\prime \leq \pi \mid \pi\prime \in S_n(\tau)\}$ has a maximal element.

Finally we invoke lemma 3 to complete the proof.

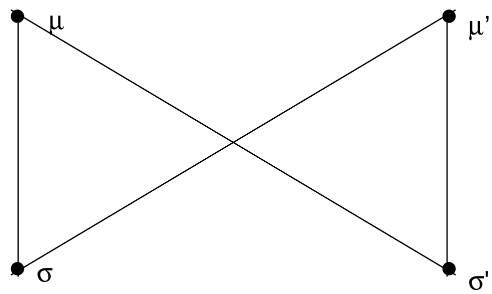

FIGURE 8. Subposet of $S_k(\tau)$

5. Related Results

First let us note that there is an analogous result about meet and join semi-lattices.

THEOREM 5.1. *$S_n(\tau)$ is a meet semi-lattice if and only if τ has at most one descent, which is of magnitude one or two.*

$S_n(\tau)$ is a join semi-lattice if and only if τ has at most one ascent, which is of magnitude one or two.

This implies that the only τ for which $S_n(\tau)$ is a semi-lattice but not a lattice are the strictly increasing and strictly decreasing patterns.

Theorem 1.1 does not generalize immediately to larger sets of patterns. In particular, it is not true that $S_n(\{\tau_1, \tau_2\})$ is a lattice if both $S_n(\tau_1)$ and $S_n(\tau_2)$ are. For example, Stembridge's posets package for Maple [**13**] confirms that $S_5(\{2431, 3124\})$ is not a lattice. Moreover, it is not necessary for both $S_n(\tau_1)$ and $S_n(\tau_2)$ to be lattices in order for $S_n(\{\tau_1, \tau_2\})$ to be a lattice. For example, consider $S_n(\{2134, 2143\})$, which is one case of the following theorem:

THEOREM 5.2. *Let $T = \{21 \oplus \tau \mid \tau \in S_{k-2}\}$. Then $S_n(T)$ is a lattice for all n.*

Proof: Observe that $S_n(T)$ is the set of permutations such that for each descent $\pi(i) > \pi(i+1)$, we have $|\{j \mid j > i, \pi(j) > \pi(i)\}| < k-2$. So if $\pi \in S_n \setminus S_n(T)$, then there is a unique minimal element $\pi\prime$ less than π (in terms of the order on S_n), with $\pi\prime \in S_n(T)$. $\pi\prime$ is obtained by changing all descents which violate the condition above to ascents. Since $12\ldots n \in S_n(T)$ and $n(n-1)\ldots 21 \in S_n(T)$, $S_n(T)$ is a lattice by lemma 3. $\square$

It is probably unfeasable to characterize all sets of patterns T such that $S_n(T)$ is a lattice for all sufficiently large n. However, the following corollary of the proof for Theorem 1.1 might be easier to generalize.

COROLLARY 5.1. *If τ is a pattern of length k, the following are equivalent:*
1) $S_n(\tau)$ is a lattice for all n.
2) $S_{k+1}(\tau)$ is a lattice.

CONJECTURE 1. *There exists an M depending only on the length of the patterns τ_i such that the following are equivalent:*
1) $S_n(\tau_1, \tau_2, \ldots, \tau_r)$ is a lattice for all $n \geq M$.
2) $S_M(\tau_1, \tau_2, \ldots, \tau_r)$ is a lattice.

The Erdös-Szekeres Theorem [**4**] would suggest an M that is roughly the product of the length of the patterns τ_i.

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